In G, s' is a two-sided inverse, since $s's = e$ by definition, and $ss' = ess' = s''s' = e$. This completes the proof that $(G, .)$ is a group. We may note that, in fact, G consists of all the left inverses with respect to e, and that the left inverse of each element (with respect to e) is unique.

The construction of the group G depends upon the choice of the right identity e in S. Other right identities always exist in S unless S is a group, as was shown in Note 59.2. All of these right identities are of the form hh' for some $h \in S$. In fact for any s, $h \in S$, we have

$$s(hh') = (se)hh' = s(eh)h' = sh''h' = se = s,$$

so that hh' is a right identity in S for every $h \in S$. Now this right identity will give rise to a group H by a definition analogous to the definition of the group G. Moreover $h \in H$, since

$$(hh')h = h(h'h) = he = h.$$

Thus for each element of S it is possible to find at least one group like G or H to which it belongs.

We have now shown that S is a union of groups, and our theorem will be complete when we have established that these groups are disjoint. Now the identity in a group is unique, so that if G is a group with identity e and H is a group with identity f, then G and H are distinct if, and only if, $e \neq f$. If G and H are not disjoint, there is an element s in both, such that $es = fs = s$. Then if s' is the inverse of s in G,

$$ess' = fss',$$

so

$$ee = fe,$$

and therefore

$$e = f.$$

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Does it matter?—no, not really

DEAR EDITOR,

In the March 1975 issue Miss Shuard had an interesting article on the doubts and difficulties involved in a very conscientious effort to create a good course on analysis. In June 1973 Miss Shuard had an excellent letter in which she mentioned that calculus was the subject most enjoyed in the sixth form and analysis the hardest subject in first-year undergraduate work. It seems to me that her letter indicates the spirit in which we should view the subtleties discussed in her article.

The first thing to realise is that mathematics has no permanent logical foundation. Mathematics continually grows and expands, with an established centre and ragged edges;

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we continually repair these edges, and while we are doing it the subject expands, and the process has to be repeated. The analysis of any epoch, like a function, has its domain; it provides safeguards for correct operations within that domain.

A famous sneer describes university lecturing as answering questions that have not been asked. In no part of mathematics do we so easily lay ourselves open to this reproach as in analysis. We can justify analysis to a student only by showing him the errors into which he may fall, *at his present mathematical stage*, if he does not take correct precautions.

Clearly then, the analysis taught must depend on the maturity of the student, on the level at which he is working, and on the distance in mathematics he expects to go. If we have to choose between a presentation the student finds understandable and fruitful and one which would be preferred by a research worker at the frontiers of present knowledge, our choice must be for the former. A space suit is an essential piece of equipment for an astronaut; it is not very convenient for walking to the office.

It may seem paradoxical, but the fact is that only by observing the past can we cope with the future. If a student is shown how at each stage of mathematics it has been necessary to bring in new concepts and new safeguards, he can realise that our present equipment will have to be overhauled and refined in another 20 years.

The historical account in fact is extremely instructive. I have spoken to many undergraduates who have heard lectures on the Riemann theory of integration. It is rarely if ever I meet one who knows what problem led Riemann to produce this theory. Riemann was working on Fourier series, and had obtained a function which was discontinuous at every rational number with an odd numerator and even denominator; its graph could not be drawn, and yet, since Fourier series are found by integration, there ought to be a definite area under it. A student can then see the point of the Riemann condition for integrability.

In the same way, one could examine the various times in which distinctions between functions and relations, domains, codomains and ranges were made, and show in what situations such distinctions became important. Usually what happens when people use terms drawn from theories they do not know themselves is that these terms are used incorrectly.

Intuition as the decider

A very interesting principle will be found in the footnote on p. 25 of M. Fréchet's *Les espaces abstraits*. He writes: "If there are two definitions, one vague, intuitive, furnished by experience, the other precise and rigorous, we should prefer the former, if the consequences of the latter should show that it does not limit itself to giving a precise mathematical meaning to the intuitive idea, but distorts it in some essential way."

It seems that this deals completely with the question Miss Shuard discusses on pp. 11 and 12 of her article, as to whether a function defined only for $x > 3$ can define a limit when $x \to 2$. We know (intuitively) that it certainly does not; if a certain definition leads to a contrary view, that definition should be rejected at once.

Situations, not words

In the discussion on pp. 13 and 14, Miss Shuard compares definitions of limit and continuity given by Scott and Tims with those given by Rudin and Apostol. But it is immediately evident, if one graphs the situations in question, that they are all considering the same situation, and that they all agree what happens in it. Scott and Tims use a definition of limit appropriate to a point surrounded by a domain, and have to modify it when the point is to the side of a domain. Rudin and Apostol use a definition that does not call for such modification. Surely a student who understands the situation in question can see that either description will fit it.

A test for terminology

Terminology is valuable when it helps to illuminate complicated or doubtful situations; it is harmful if it is used to obscure and complicate simple situations.

I have considerable doubts whether the distinction between $f:\mathbb{R} \to \mathbb{R}$ with $f(x) = e^x$ and $f: \mathbb{R} \to \mathbb{R}^+$ with $f(x) = e^x$ is helpful to the student at the stage of real variable. The situation is not a complicated one. It is evident from the graph of $y = e^x$ that it lies above $y = 0$, that it could be used to give the natural logarithm of any positive number and that it does not help in any way to derive a logarithm of -1. If you wanted to emphasise that you were interested in the mapping $\mathbb{R} \to \mathbb{R}^+$ I suppose you could cut off the graph paper on or below the x-axis. I would be interested to see a lecturer dealing with a persistent student who wanted to know why such considerations were relevant to mathematics *at this level.*

On the other hand, it would be both useful and instructive to collect examples of mathematics at levels where such distinctions are important and even vital. Such collections would be particularly helpful to future teachers, who need both to understand research mathematicians and also to know when not to copy them.

Yours sincerely,

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Symbols for science

DEAR SIR,

It was most interesting to read Mr. John Bausor's recent article *Symbols and how scientists use them* in the *Gazette*, June 1975 issue. It is becoming clear that letters should be allowed to stand for quantities as well as for numbers. What does not seem to be realised is that there is a simple device by which the needs of the scientists and of the mathematicians could be reconciled, i.e. the use of a symbol to stand for a physical quantity and/or the use of *another* symbol to stand for its numerical value. In the example given in Mr. Bausor's article, the setting out might be as follows:

Find g if $v = u + gt$, $u = 4{\cdot}9 \text{ m s}^{-1}$, $v = 19{\cdot}6 \text{ m s}^{-1}$ and $t = 1{\cdot}5$ s.
Let $g = z \text{ m s}^{-2}$. (Or, let $g/\text{m s}^{-2} = z$.)
Then $19{\cdot}6 \text{ m s}^{-1} = 4{\cdot}9 \text{ m s}^{-1} + z \text{ m s}^{-2} \times 1{\cdot}5$ s,
so that $19{\cdot}6 = 4{\cdot}9 + z \times 1{\cdot}5$.
(Or, straight away, $19{\cdot}6 = 4{\cdot}9 + z \times 1{\cdot}5$.)
Solving, $1{\cdot}5z = 19{\cdot}6 - 4{\cdot}9 = 14{\cdot}7$,

$$z = \frac{14{\cdot}7}{1{\cdot}5} = 9{\cdot}8,$$

and $g = 9{\cdot}8 \text{ m s}^{-2}$.

In this presentation, g represents acceleration and z represents its numerical value. It is, indeed, tedious and possibly confusing to retain physical units throughout the sometimes lengthy solution of an equation; anyone who has tried to calculate a final temperature in elementary calorimetry, while faithfully keeping in all the proper units, will have realised this.

The answer, I believe, is not to be found in the early introduction of transposition of formulae (so that in the above example you would find that $g = (v - u)/t$ and then solve by substitution and calculation). For there *is* an inherent difficulty in solving the literal equation $v = u + gt$, for g, compared with solving the numerical equation $19{\cdot}6 = 4{\cdot}9 + z \times 1{\cdot}5$, for z. Transposition of a formula needs a greater degree of abstraction than substituting in the formula for the known values and then finding the unknown value by solving a numerical equation. The learning process should proceed from the concrete to the abstract; learning how to solve $v = u + gt$, for g, should come *after* learning how to