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DISTANCE FUNCTIONS ON CONVEX BODIES AND SYMPLECTIC TORIC MANIFOLDS

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Abstract. In this paper we discuss three distance functions on the set of convex bodies. In particular we study the convergence of Delzant polytopes, which are fundamental objects in symplectic toric geometry. By using these observations, we derive some convergence theorems for symplectic toric manifolds with respect to the Gromov–Hausdorff distance.

§1. Introduction

Convex polytopes, or more generally convex bodies, are classical and important objects in geometry. There are many results in which structures or properties of convex polytopes are shown to have deep connections with other objects, through algebraic or combinatorial procedures. Among other such results, there is the *Delzant construction* [4], which is well known in symplectic geometry. Using the Delzant construction one obtains a natural bijective correspondence between the set of *Delzant polytopes* and the set of *symplectic toric manifolds*. Under this correspondence, the geometric data of symplectic toric manifolds are encoded as combinatorial or topological properties of their corresponding polytopes. For example, the cohomology ring of symplectic toric manifolds can be recovered completely as the *Stanley–Reisner ring* of the associated polytope. See (e.g., [3]) for more details on this dictionary between Delzant polytopes and symplectic toric manifolds.

The purpose of our project is to further develop aspects of this kind of correspondence from the viewpoint of Riemannian or metric geometry. The present paper contains two parts. Firstly, we establish relationships between three natural distance functions on the set of convex bodies. The first function d^W is defined by the *Wasserstein distance* of probability measures associated with convex bodies. The Wasserstein distance is a quite important tool in recent developments of geometric analysis for metric measure spaces. The second distance d^V is defined by the Lebesgue volume of the symmetric difference of convex bodies. This distance function is natural from the viewpoint of symplectic geometry and is studied in [14] and [6]. The third function d^H is the Hausdorff distance, which is a classical and basic tool in geometry of convex bodies. The main result of the first part of this paper is as follows.

Theorem 1 (Theorem 3.1.3). *The metric topologies on the space of convex bodies determined by the distance functions d^W, d^V, and d^H coincide with each other.*

Secondly, we investigate the relationship between the metric geometry of Delzant polytopes and the Riemannian geometry of symplectic toric manifolds through the Delzant construction. Here we equip each symplectic toric manifold with a Kähler metric which we call the *Guillemin metric* [9], and we regard a symplectic toric manifold as a Riemannian manifold. The main results in the second part of this paper are the following.

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THEOREM 2 (Theorem 5.2.2). *For a sequence of Delzant polytopes $\{P_i\}_i$ in $\mathbb{R}^n$, suppose that $\{P_i\}_i$ converges to a Delzant polytope P in $\mathbb{R}^n$ in the d^H-topology (hence also in the d^W-topology and d^V-topology), and the limit of the numbers of facets of $\{P_i\}_i$ coincides with that of P. Then the sequence of symplectic toric manifolds $\{M_{P_i}\}_i$ converges to M_P with respect to the corresponding Guillemin metrics in the torus-equivariant Gromov–Hausdorff topology.*

As a corollary (Corollary 5.2.3), we also have a torus-equivariant stability theorem in the setting of converging symplectic toric manifolds.

The above result suggests a continuity of the one direction of the Delzant construction, from P to M_P. We also have results concerning the converse direction. The following are their rough statements.

THEOREM 3 (Theorem 5.3.1, Theorem 5.3.2). *For a sequence of Delzant polytopes $\{P_i\}_i$ in $\mathbb{R}^n$ and a Delzant polytope P in $\mathbb{R}^n$, suppose that the corresponding sequence of symplectic toric manifolds $\{M_{P_i}\}_i$ converges to M_P with respect to the corresponding Guillemin metrics in the torus-equivariant measured Gromov–Hausdorff topology. Then we have:*

- *if $\{P_i\}_i$ is contained in a large ball, the fixed point set of M_{P_i} converges to that of M_P. In particular we have the lower semi-continuity of the Euler characteristic, and*
- *we have a sequence of maps $\{\hat{f}_i : P_i \to P\}_i$ such that $\{\overline{\hat{f}_i(P_i)}\}_i$ converges to P in d^H-topology by using the approximation maps for $\{M_{P_i}\}_i$. See Theorem 5.3.2 for the precise statement.*

We emphasize that there are no hypotheses on the curvature in the statement of the above theorem. By incorporating "potential functions"as in [1] we may treat more general torus-invariant Riemannian metrics of symplectic toric manifolds which are not necessarily Guillemin metrics.

In the present paper, we only consider the non-collapsing case. It is surely interesting to attack the same problems under collapsing limit, and we will discuss this in a subsequent paper. In addition, our general setting of convex bodies in the first part of this paper is motivated by the fact that non-Delzant polytopes are increasingly important in the context of toric degenerations of integrable systems or projective varieties as in [10], [13] and so on.

This paper is organized as follows. In Section 2, we introduce three distance functions on the set of convex bodies. In Section 3, we show that the three corresponding metric topologies coincide. Note that the equivalence between the distance function defined by the volume and the Hausdorff distance is classically known, by [15] for example. In [14] Pelayo–Pires–Ratiu–Sabatini studied several properties of the moduli space of Delzant polytopes with respect to the natural action of integral affine transformations. This moduli space arises naturally from an equivalence relation of symplectic toric manifolds known as weak equivalence. We also give comments on the distance function and the associated topology on this moduli space which were studied in [6]. In Section 4, we discuss the definition of Delzant polytopes and the description of Guillemin metric on the corresponding symplectic toric manifolds. In Section 5, we discuss the relation between the convergence of Delzant polytopes and the convergence of symplectic toric manifolds. In Appendix A, we record several facts on probability measures and Wasserstein distance. In Appendix B, we provide a disintegration theorem which is important in the proof of Theorem 5.3.2.

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Notations. For a metric space (X,d), a subset Y of X, a point x in X and a positive real number r we use the following notations.

- $B(x,r) := \{y \in X \mid d(x,y) < r\}$: open ball of radius r centered at x.
- $B(Y,r) := \left\{ y \in X \;\middle|\; \inf_{y' \in Y} d(y,y') < r \right\}$: open r-neighborhood of Y.
- $\mathrm{dist}(x,Y) := \inf\{d(x,y) \mid y \in Y\}$: distance between x and Y.
- $\mathrm{Diam}(Y) := \sup\{d(y,y') \mid y,y' \in Y\}$: diameter of Y.

We use the notation $\|\cdot\|$ (resp. $\langle \cdot,\cdot \rangle$) for the Euclidean norm (resp. inner product) on the Euclidean spaces. We also use the notation $|A|$ for the Lebesgue measure of a Lebesgue measurable subset A.

§2. Three distance functions on the set of convex bodies

Let $\mathcal{C}_n$ be the set of all convex bodies in $\mathbb{R}^n$, that is, $\mathcal{C}_n$ is the set of all bounded closed convex sets obtained as closures of open subsets in $\mathbb{R}^n$.

2.1 L^2-Wasserstein distance

For each $C \in \mathcal{C}_n$ let m_C be the probability measure on $\mathbb{R}^n$ with compact support defined by

$$m_C := \frac{\chi_C}{\mathcal{H}^n(C)} \mathcal{H}^n,$$

where χ_C is the characteristic function of C and $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on $\mathbb{R}^n$. Of course $\mathcal{H}^n$ is equal to the n-dimensional Lebesgue measure $\mathcal{L}^n$, however, since we put on the field of view of collapsing phenomena of convex bodies into lower dimensional objects, we prefer to use the Hausdorff measure.

Definition 2.1.1. Define a function $d^W : \mathcal{C}_n \times \mathcal{C}_n \to \mathbb{R}_{\geq 0}$ by

$$d^W(C_1,C_2) := W_2(m_{C_1}, m_{C_2}),$$

where W_2 is the L^2-Wasserstein distance on the set of all probability measures on $\mathbb{R}^n$ with finite quadratic moment.

See Appendix A for basic definitions and facts on L^2-Wasserstein distance.

Lemma 2.1.2. *d^W is a distance function on $\mathcal{C}_n$.*

Proof. Symmetricity, triangle inequality and non-negativity are clear. The non-degeneracy follows from the equivalence between the conditions $d^W(C_1,C_2) = W_2(m_{C_1}, m_{C_2}) = 0$ and $C_1 = \mathrm{supp}\,(m_{C_1}) = \mathrm{supp}\,(m_{C_2}) = C_2$. ∎

2.2 Lebesgue volume

For $C_1, C_2 \in \mathcal{C}_n$, let $d^V(C_1,C_2)$ be the Lebesgue volume of the symmetric difference $C_1 \triangle C_2 := (C_1 \setminus C_2) \cup (C_2 \setminus C_1)$:

$$d^V(C_1,C_2) := |C_1 \triangle C_2| = \int_{\mathbb{R}^n} \chi_{C_1 \triangle C_2}(x) \mathcal{L}^n(dx).$$

This d^V is indeed a distance function on $\mathcal{C}_n$ and used in a study of convex bodies classically. See [5] or [15] for example.

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2.3 Hausdorff distance

Let d^H be the Hausdorff distance on the set of all compact subsets in $\mathbb{R}^n$.We also denote the restriction of d^H to $\mathcal{C}_n$ by the same letter d^H:

$$d^H(C_1,C_2) := \max\{\max_{x\in C_1}\min_{y\in C_2}\|x-y\|,\ \max_{y\in C_2}\min_{x\in C_1}\|x-y\|\} \quad (C_1,C_2\in\mathcal{C}_n).$$

§3. Relation of distance functions

3.1 Equivalence among d^W, d^V, and d^H

It is known that two distance functions d^V and d^H give the same metric topology. More precisely in [15] it is shown that a sequence $\{P_i\}_i$ in $\mathcal{C}_n$ converges to $Q\in\mathcal{C}_n$ in d^V if and only if it converges to Q in d^H.

LEMMA 3.1.1. *For a sequence $\{P_i\}_i$ in $\mathcal{C}_n$ and $Q\in\mathcal{C}_n$, if $d^V(P_i,Q)\to 0$ $(i\to\infty)$, then we have $d^W(P_i,Q)\to 0$ $(i\to\infty)$.*

Proof. Since $\lim_{i\to\infty} d^V(P_i,Q)=0$ implies $\lim_{i\to\infty} d^H(P_i,Q)=0$ we may assume that $P_i\cap Q\neq\emptyset$,

$$K_i := \mathrm{Diam}(P_i)\le 100K := 100\,\mathrm{Diam}(Q),$$

and $|\log(|P_i|/|Q|)|<\epsilon$ for small $\epsilon>0$ and any i large enough.

If $|Q|\ge|P_i|$, then we have $m_Q(Q\cap P_i)\le m_{P_i}(Q\cap P_i)$ and define new probability measures

$$\nu_0 := \frac{1}{m_Q(Q\setminus P_i)} m_Q|_{Q\setminus P_i},$$
$$\nu_1 := \frac{1}{m_Q(Q\setminus P_i)}\left(m_{P_i}|_{P_i\setminus Q} + \left(1-\frac{m_Q(P_i\cap Q)}{m_{P_i}(P_i\cap Q)}\right) m_{P_i}|_{Q\cap P_i}\right).$$

Since $\nu_0,\nu_1\ll\mathcal{L}^n$, by Theorem A.2.2, one can find a Borel map $T:\mathbb{R}^n\to\mathbb{R}^n$ with $T_*\nu_0=\nu_1$ so that $W_2^2(\nu_0,\nu_1)=\int\|x-T(x)\|^2\,\nu_0(dx)$. By using the map T, we define a coupling $\xi_1\in\mathsf{Cpl}(m_Q,m_{P_i})$ by

$$\xi_1 := (\mathsf{Id},T)_* m_Q|_{Q\setminus P_i} + (\mathsf{Id},\mathsf{Id})_* m_Q|_{Q\cap P_i}.$$

Heuristically, the coupling ξ_1 represents the transportation from m_Q to m_{P_i} that

- the mass on $Q\cap P_i$ measured by m_Q keep staying, and
- the mass on $Q\setminus P_i$ measured by m_Q distributes on P_i along the map T.

Note that since $\mathrm{supp}\,(T_*\nu_0)=\mathrm{supp}\,(\nu_1)\subset P_i$, we have $T(x)\in P_i$ for a.e. $x\in Q\setminus P_i$. It also implies that $\|x-T(x)\|\le\mathrm{Diam}\,Q+\mathrm{Diam}\,P_i\le 101K$ for a.e. $x\in Q\setminus P_i$, and hence, we have

$$\int_{\mathbb{R}^n\times\mathbb{R}^n}\|x-y\|^2\xi_1(dx,dy)=\int_{\mathbb{R}^n}\|x-T(x)\|^2 m_Q|_{Q\setminus P_i}(dx)\le\frac{|Q\setminus P_i|}{|Q|}\cdot(101K)^2.$$

On the other hand, if $|P_i|\ge|Q|$, then for two probability measures

$$\nu_0' := \frac{1}{m_{P_i}(P_i\setminus Q)} m_{P_i}|_{P_i\setminus Q}$$
$$\nu_1' := \frac{1}{m_{P_i}(P_i\setminus Q)}\left(m_Q|_{Q\setminus P_i} + \left(1-\frac{|Q|}{|P_i|}\right) m_Q|_{Q\cap P_i}\right)$$

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we can find a Borel map $S:\mathbb{R}^n\to\mathbb{R}^n$ with $S_*\nu_0' = \nu_1'$ so that $W_2^2(\nu_0',\nu_1') = \int \|x-S(x)\|^2\nu_0'(dx)$. Then we have a coupling $\xi_2\in\mathsf{Cpl}(\nu_0',\nu_1')$ by

$$\xi_2 := (\mathsf{Id},S)_* m_{P_i}|_{P_i\setminus Q} + (\mathsf{Id},\mathsf{Id})_* m_{P_i}|_{Q\cap P_i}.$$

Then we have

$$\begin{aligned} d^W(P_i,Q) &\le \sqrt{\int_{\mathbb{R}^n\times\mathbb{R}^n}\|x-y\|^2\xi_1(dx,dy)} \text{ or } \sqrt{\int_{\mathbb{R}^n\times\mathbb{R}^n}\|x-y\|^2\xi_2(dx,dy)} \\ &\le \sqrt{\frac{|Q\setminus P_i|}{|Q|}\cdot(101K)^2} + \sqrt{\frac{|P_i\setminus Q|}{|P_i|}\cdot(101K)^2} \\ &\le 2\cdot 101K\sqrt{\frac{|Q\triangle P_i|}{\min\{|Q|,|P_i|\}}} \\ &\le 2\cdot 101K\sqrt{\frac{d^V(Q,P_i)}{e^{-\epsilon}|Q|}} \to 0 \text{ (as } i\to\infty). \end{aligned}$$

∎

LEMMA 3.1.2. *For a sequence $\{P_i\}_i$ in $\mathcal{C}_n$ and $Q\in\mathcal{C}_n$, if $d^W(P_i,Q)\to 0$ $(i\to\infty)$, then we have $d^V(P_i,Q)\to 0$ $(i\to\infty)$.*

Proof. Suppose that $d^W(P_i,Q)\to 0$ $(i\to\infty)$. Then, $m_i := m_{P_i}$ converges weakly to $m := m_Q$, in particular, we have

$$m_i(Q) = \frac{|P_i\cap Q|}{|P_i|} \to m(Q) = 1$$

by Theorem A.1.2. Since $|P_i\cap Q|\le |Q|$ we have $|P_i|$ is bounded, and hence,

$$\frac{|P_i|}{|Q|} < c$$

for some $c>0$. Corollary A.2.3 implies that for two probability measures m_i and m there exist a sequence of Borel measurable maps $\{T_i:\mathbb{R}^n\to\mathbb{R}^n\}_i$ such that $(\mathtt{id}\times T_i)_* m\in\mathsf{Opt}(m,m_i)$ for all i and

$$m(\{x\in Q \mid \|x-T_i(x)\|\ge a\}) = m(\{x\in\mathbb{R}^n \mid \|x-T_i(x)\|\ge a\}) \to 0 \ (i\to\infty)$$

for all $a>0$. Let us fix an arbitrary positive number ϵ and set

$$\xi := \frac{\epsilon}{(c+1)(|Q|+1)}.$$

Choose η small enough so that

$$|B(Q,\eta)\setminus Q| < \xi.$$

There exists $N\in\mathbb{N}$ such that

$$m(\{x\in Q \mid \|T_i(x)-x\|\ge\eta\}) < \xi$$

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for all $i \geq N$. Take and fix $i > N$. For $x \in Q$ we put $r_x^i := \|x - T_i(x)\|$. Then we have $Q \subset \bigcup_{x \in Q} B(x, r_x^i)$. We put

$$U^i := \bigcup_{x \in Q, r_x^i \leq \eta} \overline{B(x, r_x^i)}.$$

We have

$$\begin{aligned}
|U^i \setminus Q| &\leq |B(Q,\eta) \setminus Q| < \xi, \\
|Q \setminus U^i| &= |Q| m(Q \setminus U^i) \\
&\leq |Q| m(\{x \in Q \mid \|x - T_i(x)\|) \geq \eta\}) \\
&< |Q|\xi,
\end{aligned}$$

and hence, $|Q \triangle U^i| < (|Q|+1)\xi$. On the other hand we have

$$\begin{aligned}
|P_i \setminus U^i| &= |P_i| m_i(P_i \setminus U^i) \\
&= |P_i| (T_i)_* m(P_i \setminus U^i) \\
&= |P_i| m(T_i^{-1}(P_i) \setminus T_i^{-1}(U^i)).
\end{aligned}$$

Since $(T_i)_* m = m_i$ we have that $T_i^{-1}(P_i) = Q$ (m-a.e.). This fact and $T_i^{-1}(\overline{B(x, r_x^i)}) \ni x$ imply that

$$T_i^{-1}(U^i) \supset \{x \in Q \mid \|x - T_i(x)\| \leq \eta\}.$$

In particular we have

$$|P_i \setminus U^i| \leq |P_i| m(\{x \in Q \mid \|x - T_i(x)\| > \eta\}) \leq |P_i|\xi.$$

Similarly we have

$$\begin{aligned}
|U^i \setminus P_i| &= |P_i| m_i(U^i \setminus P_i) = |P_i| m(T_i^{-1}(U^i) \setminus Q) \\
&\leq |P_i| m(B(Q,\eta) \setminus Q) = \frac{|P_i|}{|Q|} |B(Q,\eta) \setminus Q| \\
&< \frac{|P_i|}{|Q|} \xi \leq c\xi,
\end{aligned}$$

and hence $|U^i \triangle P_i| \leq (|P_i| + c)\xi$. Therefore we have

$$\begin{aligned}
& d^V(P_i, Q) = |Q \triangle P_i| \leq |Q \triangle U^i| + |U^i \triangle P_i| \\
& \leq (|Q| + |P_i| + c + 1)\xi \leq ((1+c)|Q| + c + 1)\xi = (1+c)(|Q|+1)\xi \\
& = \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain the conclusion, that is, $d^V(P_i, Q) \to 0$. ∎

As a corollary of Lemmas 3.1.1 and 3.1.2 we have the following by Kratowski's axiom and the coincidence between the metric topology of d^V and d^H as shown in [15].

THEOREM 3.1.3. *Three metric topologies on $\mathcal{C}_n$ determined by d^W, d^V, and d^H coincide with each other.*

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3.2 Moduli space of convex bodies and its topology

We introduce the moduli space of convex bodies following [6] and [14]. Let $G_n := \mathrm{AGL}(n,\mathbb{Z})$ be the integral affine transformation group. Namely G_n is the direct product $\mathrm{GL}(n,\mathbb{Z})\times\mathbb{R}^n$ as a set and the multiplication on G_n is defined by

$$(A_1,t_1)\cdot(A_2,t_2) := (A_1A_2, A_1t_2+t_1)$$

for each $(A_1,t_1),(A_2,t_2)\in G_n$. This group G_n acts on $\mathcal{C}_n$ in a natural way, and $C\in\mathcal{C}_n$ and $C'\in\mathcal{C}_n$ are called *G_n-congruent* if C and C' are contained in the same G_n-orbit.

DEFINITION 3.2.1. The moduli space of convex bodies $\widetilde{\mathcal{C}}_n$ with respect to the G_n-congruence is defined by the quotient

$$\widetilde{\mathcal{C}}_n := \mathcal{C}_n/G_n.$$

Let π be the natural projection from $\mathcal{C}_n$ to $\widetilde{\mathcal{C}}_n$.

DEFINITION 3.2.2. Define a function $D^V:\widetilde{\mathcal{C}}_n\times\widetilde{\mathcal{C}}_n\to\mathbb{R}$ by

$$D^V(\alpha,\beta) := \inf\{d^V(P_1,P_2) \mid \pi(P_1)=\alpha, \pi(P_2)=\beta\}$$

for $(\alpha,\beta)\in\widetilde{\mathcal{C}}_n\times\widetilde{\mathcal{C}}_n$.

THEOREM 3.2.3 [6]. *D^V is a distance function on $\widetilde{\mathcal{C}}_n$ and its metric topology coincides with the quotient topology induced from π.*

This G_n-action and the moduli space $\widetilde{\mathcal{C}}_n$ arise naturally in the context of the geometry of symplectic toric manifolds. In the subsequent sections we will discuss from such point of view.

REMARK 3.2.4. As it is noted in [6] we can not define a distance function on $\widetilde{\mathcal{C}}_n$ by using the infimum of d^H (or d^W) among all representatives, though, one may hope that by considering infimum of d^H among only "standard"representatives we can define a distance function on $\widetilde{\mathcal{C}}_n$. One possible candidates of "standard"representatives are the minimum variance (or quadratic moment) elements in the following sense.

For each $C\in\mathcal{C}_n$ define its variance by

$$\mathsf{Var}(C) := \frac{1}{|C|}\int_C \|x-\mathsf{b}(C)\|^2\mathcal{L}^n(dx),$$

where $\mathsf{b}(C)$ is the barycenter of C which is determined uniquely by the condition

$$\langle \mathsf{b}(C), y\rangle = \int_{\mathbb{R}^n}\langle x,y\rangle\mathcal{L}^n(dx)$$

for any $y\in\mathbb{R}^n$. See [17] for example. The minimum variance element $C\in\mathcal{C}_n$ is an element of

$$\mathsf{argmin}\{\mathsf{Var}(C') \mid C'\in\mathcal{C}_n \text{ is } G_n\text{-congruent to } C\}.$$

One can see that for any $C\in\mathcal{C}_n$ there exist at least one and finitely many minimum variance elements which have the common barycenter are G_n-congruent to C.

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§4. Delzant polytopes and symplectic toric manifolds.

4.1 Delzant polytopes, symplectic toric manifolds and their moduli space

DEFINITION 4.1.1. A convex polytope P in $\mathbb{R}^n$ is called a *Delzant polytope* if P satisfies the following conditions:

- P is simple, that is, each vertex of P has exactly n edges.
- P is rational, that is, at each vertex all directional vectors of edges can be taken as integral vectors in $\mathbb{Z}^n$.
- P is smooth, that is, at each vertex we can take integral directional vectors of edges as a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ in $\mathbb{R}^n$.

We denote the subset of $\mathcal{C}_n$ consisting of all Delzant polytopes by $\mathcal{D}_n$ and define their moduli space by $\widetilde{\mathcal{D}}_n := \mathcal{D}_n/G_n$.

Recall that the data of a (compact) *symplectic toric manifold* (M,ω,ρ,μ) consists of

- a compact connected symplectic manifold (M,ω) of dimension $2n$,
- a homomorphism ρ from the n-dimensional torus T^n to the group of symplectomorphisms of M which gives a Hamiltonian action of T^n on M and
- a moment map $\mu : M \to \mathbb{R}^n = (\mathrm{Lie}(T^n))^*$.

The famous Delzant construction gives a correspondence between Delzant polytopes and symplectic toric manifolds.

THEOREM 4.1.2 [12]. *The Delzant construction gives a bijective correspondence between $\widetilde{\mathcal{D}}_n$ and the set of all weak isomorphism classes of $2n$-dimensional symplectic toric manifolds.*

Here two symplectic toric manifolds $(M_1,\omega_1,\rho_1,\mu_1)$ and $(M_2,\omega_2,\rho_2,\mu_2)$ are *weakly isomorphic*[1] if there exist a diffeomorphism $f : M_1 \to M_2$ and a group isomorphism $\phi : T^n \to T^n$ such that

$$f^*\omega_2 = \omega_1 \text{ and } \rho_1(g)(x) = \rho_2(\phi(g))(f(x)) \text{ for all } (g,x) \in T^n \times M_1.$$

Based on the above fact the moduli space $\widetilde{\mathcal{D}}_n$ is also called the *moduli space of toric manifolds* in [14]. In [14] they show that $(\mathcal{D}_n, d^V)$ is neither complete nor locally compact and $\widetilde{\mathcal{D}}_2$ is path connected.

4.2 Brief review on the Delzant construction

For later convenience we give a brief review on the Delzant construction here.

Let P be an n-dimensional Delzant polytope and

$$l^{(r)}(x) := \langle x, \nu^{(r)} \rangle - \lambda^{(r)} = 0 \quad (r = 1, \dots, N) \tag{4.1}$$

a system of defining affine equations on $\mathbb{R}^n$ of facets of P, each $\nu^{(r)}$ being inward pointing normal vector of rth facet and N is the number of facets of P. In other words P can be described as

$$P = \bigcap_{r=1}^{N} \{x \in \mathbb{R}^n \mid l^{(r)}(x) \geq 0\},$$

[1] In [12] the equivalence relation "weakly isomorphism" is called just "equivalent". In this paper we follow the terminology in [14].

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and we do not allow redundant inequalities. We may assume that each $\nu^{(r)}$ is primitive[2] and they form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Consider the standard Hamiltonian action of the N-dimensional torus T^N on $\mathbb{C}^N$ with the moment map

$$\tilde{\mu} : \mathbb{C}^N \to (\mathbb{R}^N)^* = \mathrm{Lie}(T^N)^*,\ (z_1, \ldots, z_N) \mapsto -\frac{1}{2}(|z_1|^2, \ldots, |z_N|^2) + (\lambda^{(1)}, \ldots, \lambda^{(N)}).$$

Let $\tilde{\pi} : \mathbb{R}^N \to \mathbb{R}^n$ be the linear map defined by $e_r \mapsto \nu^{(r)}$, where e_r $(r = 1, \ldots, N)$ is the rth standard basis of $\mathbb{R}^N$. Note that $\tilde{\pi}$ induces a surjection $\tilde{\pi} : \mathbb{Z}^N \to \mathbb{Z}^n$ between the standard lattices by the last condition in Definition 4.1.1, and hence it induces a surjective homomorphism between tori, still denoted by $\tilde{\pi}$,

$$\tilde{\pi} : T^N = \mathbb{R}^N/\mathbb{Z}^N \to T^n = \mathbb{R}^n/\mathbb{Z}^n.$$

Let H be the kernel of $\tilde{\pi}$ which is an $(N-n)$-dimensional subtorus of T^N and $\mathfrak{h}$ its Lie algebra. We have exact sequences

$$1 \to H \xrightarrow{\iota} T^N \xrightarrow{\tilde{\pi}} T^n \to 1,$$
$$0 \to \mathfrak{h} \xrightarrow{\iota} \mathbb{R}^N \xrightarrow{\tilde{\pi}} \mathbb{R}^n \to 0$$

and its dual

$$0 \to (\mathbb{R}^n)^* \xrightarrow{\tilde{\pi}^*} (\mathbb{R}^N)^* \xrightarrow{\iota^*} \mathfrak{h}^* \to 0,$$

where ι is the inclusion map. Then the composition $\iota^* \circ \tilde{\mu} : \mathbb{C}^N \to \mathfrak{h}^*$ is the associated moment map of the action of H on $\mathbb{C}^N$. It is known that $(\iota^* \circ \tilde{\mu})^{-1}(0)$ is a compact submanifold of $\mathbb{C}^N$ and H acts freely on it. We obtain the desired symplectic manifold $M_P := (\iota^* \circ \tilde{\mu})^{-1}(0)/H$ equipped with a natural Hamiltonian $T^N/H = T^n$-action. Note that the standard flat Kähler structure on $\mathbb{C}^N$ induces a Kähler structure on M_P[3]. We call the associated Riemannian metric the *Guillemin metric*.

There exists an explicit description of the Guillemin metric. We give the description following [1]. Consider a smooth function

$$g_P := \frac{1}{2}\sum_{r=1}^{N} l^{(r)} \log l^{(r)} : P^\circ \to \mathbb{R}, \tag{4.2}$$

where P° is the interior of P. It is known that $M_P^\circ := \mu_P^{-1}(P^\circ)$ is an open dense subset of M_P on which T^n acts freely and there exists a diffeomorphism $M_P^\circ \cong P^\circ \times T^n$. Under this identification $\omega_P|_{M_P^\circ}$ can be described as

$$\omega_P|_{M_P^\circ} = dx \wedge dy = \sum_{i=1}^{n} dx_i \wedge dy_i$$

using the standard coordinate[4] $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in P^\circ \times T^n$. The coordinate on M_P° induced from $(x, y) \in P^\circ \times T^n$ is called the *symplectic coordinate* on M_P.

---

2 An integral vector u in $\mathbb{R}^n$ is called *primitive* if u cannot be described as $u = ku'$ for another integral vector u' and $k \in \mathbb{Z}$ with $|k| > 1$.

3 In [1] this Kähler structure is called the *canonical toric Kähler structure*.

4 Here we regard $T = T^n = (S^1)^n$ and $S^1 = \mathbb{R}/\mathbb{Z}$.

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THEOREM 4.2.1 [9]. *Under the symplectic coordinates* $(x,y) \in P^\circ \times T^n \cong M_P^\circ \subset M_P$*, the Guillemin metric can be described as*

$$\begin{pmatrix} G_P & 0 \\ 0 & G_P^{-1} \end{pmatrix},$$

where $G_P := \mathrm{Hess}_x(g_P) = \left(\frac{\partial^2 g_P}{\partial x_k \partial x_l}\right)_{k,l=1,\dots,n}$ *is the Hessian of* g_P*.*

REMARK 4.2.2. If P and P' in $\mathcal{D}_n$ are G_n-congruent, then the corresponding Riemannian manifolds M_P and $M_{P'}$ are isometric to each other. In fact as it is noted in [1, Section 3.3], for $\varphi \in G_n$ we have an isomorphism between M_P and $M_{\varphi(P)}$ as Kähler manifolds. The isomorphism is induced by the map $P \times T \to \varphi(P) \times T$, $(x,t) \mapsto (\varphi(x), ((\varphi_*)^{-1})^T(t))$, where $(\ )^T$ is the transpose and φ_* is the automorphism of T which is induced by φ.

EXAMPLE 4.2.3. We demonstrate the Delzant construction in dimension 1. For $\alpha \geq 1$ consider the inequalities

$$\xi \geq 0,\ 2\alpha - \xi \geq 0$$

on $\mathbb{R}$. These inequalities determine a 1-dimensional Delzant polytope $P_\alpha = [0, 2\alpha]$.

Let $\tilde{\mu} : \mathbb{C}^2 \to \mathbb{R}^2$ be the moment map defined by

$$\tilde{\mu}(z_1, z_2) := \left(-\frac{1}{2}|z_1|^2, -\frac{1}{2}|z_2|^2 + 2\alpha\right).$$

The inequalities determines a linear map $\tilde{\pi} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\tilde{\pi}(e_1) = 1,\ \tilde{\pi}(e_2) = -1.$$

Let H be the kernel of the induced homomorphism $\tilde{\pi} : T^2 \to T^1$, which is given by

$$H = \{(t,t) \in T^2 \mid t \in U(1)\} \cong S^1.$$

Let $\mu_H : \mathbb{C}^2 \to (\mathrm{Lie}(H))^* \cong \mathbb{R}$ be the induced moment map with respect to the H-action, which is given by

$$\mu_H(z_1, z_2) = -\frac{1}{2}(|z_1|^2 + |z_2|^2) + 2\alpha.$$

One can see that $0 \in \mathbb{R}$ is a regular value of μ_H and the induced action of H on $Z := \mu_H^{-1}(0)$ is free, and hence, the quotient $M_\alpha := \mu_H^{-1}(0)/H$ has a structure of compact 2-dimensional symplectic manifold equipped with Hamiltonian $T := T^2/H \cong S^1$-action.

On the other hand the Hessian of the function

$$g_{P_\alpha}(\xi) := \frac{1}{2}\left(\xi \log \xi + (2\alpha - \xi)\log(2\alpha - \xi)\right)$$

gives the Guillemin metric described as

$$\begin{pmatrix} \frac{1}{\xi(2\alpha - \xi)} & 0 \\ 0 & \xi(2\alpha - \xi) \end{pmatrix}$$

on $P_\alpha^\circ \times T$. By the direct computation we have that the (Gauss) curvature of this metric is constant $\frac{1}{\alpha}$. It turns out that M is isomorphic to the unit sphere with the standard S^1-action and the round metric.

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By taking the limit $\alpha \to 1$, we see that P_α converges to $P_1 = [0,2]$. On the other hand the curvature of M_α converges to the constant 1. In fact by Theorem 5.2.2 M_α converges to the unit sphere M_1 in the T-equivariant Gromov–Hausdorff topology.

§5. Convergence of polytopes and symplectic toric manifolds

Hereafter we do not often distinguish a sequence itself and a subsequence of it.

5.1 Convergence of polytopes and related quantities

For a convex polytope P in $\mathbb{R}^n$ let $N_k(P)$ be the number of k-dimensional faces of P. We denote the set of all k-dimensional faces of P by

$$\{F_k^{(r)}(P) \mid r\} = \{F_k^{(r)}(P) \mid r = 1, \dots, N_k(P)\}.$$

We often omit the superscript r for simplicity and denote each face by $F_k(P)$ for example.

REMARK 5.1.1. Since the Hausdorff distance between two convex bodies is equal to that between the boundaries of them, the following holds :

For a sequence $\{P_i\}_i \subset \mathcal{D}_n$ suppose that $d^H(P_i, P) \to 0$ $(i \to \infty)$ for $P \in \mathcal{D}_n$. Then for any $x \in F_{n-1}^{(r)}(P)$ there exists a sequence $\{x_i \in F_{n-1}^{(r_i)}(P_i)\}_i$ such that $x_i \to x$ $(i \to \infty)$.

This fact implies the following corollaries.

COROLLARY 5.1.2. *For a sequence $\{P_i\}_i \subset \mathcal{D}_n$ suppose that $d^H(P_i, P) \to 0$ $(i \to \infty)$ for $P \in \mathcal{D}_n$. For any $k = 0, 1, \dots, n-1$ and a point $x \in F_k^{(r)}(P)$ there exists a sequence $\{x_i \in F_k^{(r_i)}(P_i)\}_i$ such that $x_i \to x$ $(i \to \infty)$.*

Proof. For any $x \in F_{n-2}^{(r)}(P)$ let $F_{n-1}^{(r')}(P)$ be a facet of P which contains $x \in F_{n-2}^{(r)}(P)$. By the fact in Remark 5.1.1 $F_{n-1}^{(r')}(P)$ can be described as a limit of a union of facets of P_i. It also implies that $F_{n-2}^{(r)}(P)$ can be described as a limit of $(n-2)$-dimensional faces of P_i. One can prove the claim in an inductive way. ∎

COROLLARY 5.1.3. *For a sequence $\{P_i\}_i \subset \mathcal{D}_n$ suppose that $d^H(P_i, P) \to 0$ $(i \to \infty)$ for $P \in \mathcal{D}_n$. Then the number of k-dimensional faces is lower semi-continuous for any k:*

$$N_k(P) \le \liminf_{i \to \infty} N_k(P_i).$$

COROLLARY 5.1.4. *For a sequence $\{P_i\}_i \subset \mathcal{D}_n$ suppose that $d^H(P_i, P) \to 0$ $(i \to \infty)$ for $P \in \mathcal{D}_n$. For any facet $F_{n-1}^{(r)}(P)$ there exists a sequence of facets $\{F_{n-1}^{(r_i)}(P_i)\}_i$ such that the corresponding defining affine functions converge to that of $F_{n-1}^{(r)}(P)$, that is, $l_i^{(r_i)} \to l^{(r)}$ $(i \to \infty)$.*

Proof. For any $x \in F_{n-1}^{(r)}(P)$ of P, one can take a sequence $\{x_i \in F_{n-1}^{(r_i)}(P_i)\}_i$ of P_i which converges to x. We may assume that the sequence of unit normal vectors of $F_{n-1}^{(r_i)}(P_i)$ converges to that of $F_{n-1}^{(r)}(P)$. It implies that the corresponding defining affine functions $l_i^{(r_i)}$ converge to $l^{(r)}$. ∎

We say a sequence of k-dimensional faces $\{F_k^{(r_i)}(P_i)\}_i$ of a sequence $\{P_i\}_i$ in $\mathcal{D}_n$ *converges essentially to a k-dimensional face $F_k^{(r)}(P)$ of $P \in \mathcal{D}_n$* if

$$\lim_{i \to \infty} \mathcal{H}^k(F_k^{(r_i)}(P_i)) > 0$$

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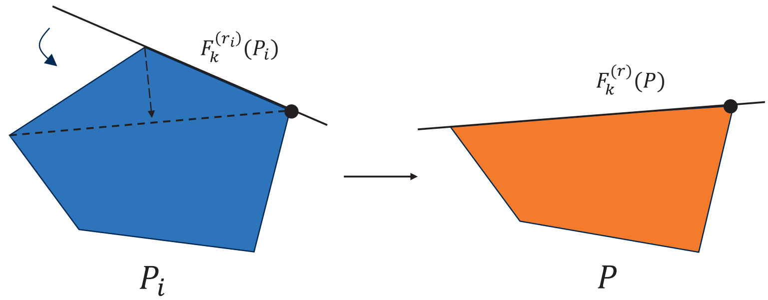

Figure 1.
A sequence of polytopes which has facets converging essentially.

and

$$\lim_{i\to\infty} d^H(F_k^{(r_i)}(P_i), F) = 0$$

for a closed subset F of $F_k^{(r)}(P)$ with respect to the relative topology, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure on $\mathbb{R}^n$. The following Figure 1 gives an example of a sequence of facets which converges essentially to a facet. On the other hand the sequence of slanting facets of the pentagon in Figure 3 converges in a non-essential way.

Next we consider the 2-dimensional case $\mathcal{D}_2$.

THEOREM 5.1.5. *For a sequence $\{P_i\}_i \subset \mathcal{D}_2$ suppose that $d^H(P_i, P) \to 0$ $(i \to \infty)$ for some $P \in \mathcal{D}_2$. If $\sup_i N_1(P_i) < \infty$, then for each facet $F_1^{(r)}(P)$ of P and its primitive normal vector $\nu^{(r)}$, there exists $r_i \in \{1, \cdots, N_1(P_i)\}$ such that a subsequence of primitive normal vectors $\{\nu_i^{(r_i)}\}_i$ of $F_1^{(r_i)}(P_i)$ such that $\nu_i^{(r_i)} \to \nu^{(r)}$ $(i \to \infty)$.*

Proof. We may assume that $r = 1$. By Corollary 5.1.3 and the semi-continuity of the Hausdorff measure in the non-collapsing limit we may assume that for each facet (=edge) $F_1^{(r)}(P)$ there exists a sequence $\{F_1^{(r_i)}(P_i)\}_i$ of facets of $\{P_i\}_i$ which converges essentially to $F_1^{(r)}(P)$. We rearrange the indices so that $r_i = 1$ for all i and may assume that the facets are numbered in a counterclockwise way.

Since $\{F_1^{(1)}(P_i)\}_i$ converges essentially to $F_1^{(1)}(P)$ the sequence of inward unit normal vectors converges:

$$\frac{\nu_i^{(1)}}{\|\nu_i^{(1)}\|} \to \frac{\nu^{(1)}}{\|\nu^{(1)}\|} \quad (i \to \infty).$$

Since $\{\nu_i^{(1)}\}_i$ is a sequence of integral vectors if $\{\|\nu_i^{(1)}\|\}_i$ is a bounded sequence, then $\nu_i^{(1)} = \nu^{(1)}$ for sufficiently large i, and hence, we have the required subsequence.

We consider the case that $\{\nu_i^{(1)}\}_i$ is unbounded. By taking a subsequence we have

$$\left|\det\left(\frac{\nu_i^{(1)}}{\|\nu_i^{(1)}\|}, \frac{\nu_i^{(2)}}{\|\nu_i^{(2)}\|}\right)\right| = \frac{1}{\|\nu_i^{(1)}\|\|\nu_i^{(2)}\|}|\det(\nu_i^{(1)}, \nu_i^{(2)})| \le \frac{1}{\|\nu_i^{(1)}\|} \to 0 \quad (i \to \infty), \tag{5.1}$$

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and hence, we have

$$\frac{\nu_i^{(2)}}{\|\nu_i^{(2)}\|} \to \pm \frac{\nu^{(1)}}{\|\nu^{(1)}\|} \quad (i \to \infty). \tag{5.2}$$

We first show the following claim.

CLAIM. There exists a subsequence of $\{F_1^{(2)}(P_i)\}_i$ which converges to a point or a segment in $F_1^{(1)}(P)$. □

Proof of the claim. Let $A = \lim_{i\to\infty} F_1^{(1)}(P_i)$. If $\lim_{i\to\infty} \operatorname{diam}(F_1^{(2)}(P_i)) = 0$, then $F_1^{(2)}(P_i)$ converges to a point. In this case $F_1^{(1)}(P_i) \cup F_1^{(2)}(P_i)$ converges to A, and hence $F_1^{(2)}(P_i)$ converges to a point in A. If $\limsup_{i\to\infty} \operatorname{diam}(F_1^{(2)}(P_i)) > 0$, then a subsequence of $F_1^{(2)}(P_i)$ converges to an interval B with positive length. Suppose that $\frac{\nu_i^{(2)}}{\|\nu_i^{(2)}\|}$ converges to $-\frac{\nu^{(1)}}{\|\nu^{(1)}\|}$. If so then (5.1) implies that for any $\varepsilon > 0$ the interior angle between $F_1^{(1)}(P_i)$ and $F_1^{(2)}(P_i)$ is smaller than ε for any $i \gg 0$. Then we have $|P_i| < (\operatorname{diam}(P_i))\varepsilon$, which contradicts to $P_i \to P$ in d_H-topology, and hence, we have $\frac{\nu_i^{(2)}}{\|\nu_i^{(2)}\|} \to \frac{\nu^{(1)}}{\|\nu^{(1)}\|}$. In particular the interior angle between $F_1^{(1)}(P_i)$ and $F_1^{(2)}(P_i)$ converges to π, and hence, B is contained in the line which contains A. It implies the claim, $A \cup B \subset F_1^{(1)}(P)$. □

By taking a subsequence we may assume that $N_1(P_i)$ is constant for $i \gg 1$, say $\sup_i N_1(P_i)$. We can take the smallest number s so that $2 \le s \le \sup_i N_1(P_i)$ and $\{\nu_i^{(s)}\}_i$ is bounded. In fact, if not, then by repeating the argument in the proof of the above claim inductively we see that P_i converges to a subset of $F_1^{(1)}(P)$, which is a contradiction. By using the minimality of s and the argument (5.1) repeatedly we have

$$\frac{\nu_i^{(s)}}{\|\nu_i^{(s)}\|} \to \pm \frac{\nu^{(1)}}{\|\nu^{(1)}\|} \quad (i \to \infty).$$

Again, by using the argument in the proof of the above claim repeatedly, we see that by taking a subsequence,

$$\bigcup_{1\le t\le s-1} F_1^{(t)}(P_i)$$

converges to a segment in $F_1^{(1)}(P)$ and $\frac{\nu_i^{(s)}}{\|\nu_i^{(s)}\|}$ converges to $\frac{\nu^{(1)}}{\|\nu^{(1)}\|}$ as $i \to \infty$. The boundedness of $\{\nu_i^{(s)}\}_i$ implies that this is the required subsequence.

REMARK 5.1.6. In Theorem 5.1.5 the boundedness of each sequence of primitive normal vectors $\{\nu_i^{(r_i)}\}_i$ implies that it contains a constant subsequence. In other words, only the constant terms vary in the defining equations of the (sub)sequence $\{P_i\}_i$. See Figure 2.

By the same argument we have the following convergence in the higher dimensional non-degenerate case.

THEOREM 5.1.7. *For a sequence $\{P_i\}_i \subset \mathcal{D}_n$ suppose that $d^H(P_i, P) \to 0$ $(i \to \infty)$ for some $P \in \mathcal{D}_n$ and $N_{n-1}(P) = \lim_{i\to\infty} N_{n-1}(P_i)$. For each facet $F^{(r)}(P)$ of P and its primitive*

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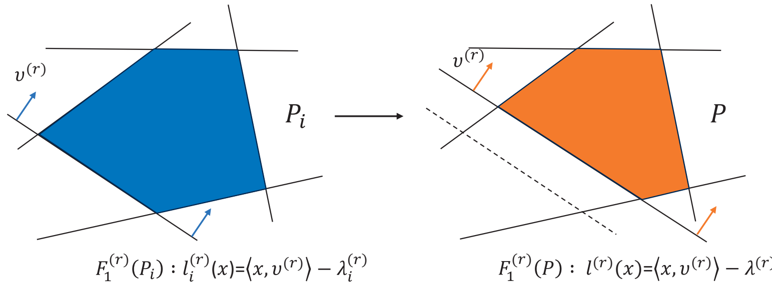

Figure 2.
A sequence of polytopes with constant normal vectors.

normal vector $\nu^{(r)}$, there exists a sequence of primitive normal vectors $\{\nu_i^{(r_i)}\}_i$ of $F^{(r_i)}(P_i)$ such that $\nu_i^{(r_i)} \to \nu^{(r)}$ $(i \to \infty)$.

Proof. As in the proof of Theorem 5.1.5 we can take a sequence of primitive normal vectors $\{\nu_i^{(1)}\}_i$ of $\{F^{(1)}(P_i)\}_i$, and it suffices to show that $\{\|\nu_i^{(1)}\|\}_i$ is bounded. Suppose that $\{\|\nu_i^{(1)}\|\}_i$ is unbounded. Consider a vertex of $F_{n-1}^{(1)}(P_i)$ and facets around it. We may assume that they are numbered as $r = 2, 3, \cdots, n$. Then for their primitive normal vectors we have

$$\left|\det\left(\frac{\nu_i^{(1)}}{\|\nu_i^{(1)}\|}, \frac{\nu_i^{(2)}}{\|\nu_i^{(2)}\|}, \cdots, \frac{\nu_i^{(n)}}{\|\nu_i^{(n)}\|}\right)\right| \leq \frac{1}{\|\nu_i^{(1)}\|} \to 0 \quad (i \to \infty).$$

It contradicts to our assumption $N_{n-1}(P) = \lim_{i\to\infty} N_{n-1}(P_i)$. ∎

5.2 From convergence of polytope to convergence of Guillemin metric

We first give the definition of equivariant (measured) Gromov–Hausdorff convergence as a special case of [7, Definition 1-3].

Definition 5.2.1. Let $X = (X, d)$ be a compact metric space and $\{X_i = (X_i, d_i)\}_i$ be a sequence of compact metric spaces. Suppose that there exists a group G which acts on X and each X_i in an effective and isometric way. Then $\{X_i\}_i$ converges to X in the *G-equivariant Gromov–Hausdorff topology* if there exist sequences of maps $\{f_i : X_i \to X\}_i$, group automorphisms $\{\rho_i : G \to G\}_i$ and positive numbers $\{\epsilon_i\}_i$ such that the following conditions hold.

1. $\epsilon_i \to 0$ as $i \to \infty$.
2. $|d_i(x, y) - d(f_i(x), f_i(y))| < \epsilon_i$ for all $x, y \in X_i$.
3. For any $p \in X$ there exists $x \in X_i$ such that $d(p, f_i(x)) < \epsilon_i$.
4. $d(f_i(gx), \rho_i(g)f_i(x)) < \epsilon_i$ for all $x \in X_i$ and $g \in G$.

When a map $f_i : X_i \to X$ satisfies 2, 3, and 4, respectively, we say that f_i is *almost isometric, almost surjective* and *almost equivariant*, respectively. This situation will be denoted by $X_i \xrightarrow{G\text{-eqGH}} X$ (or $X_i \to X$ for simplicity) and f_i's are called *approximation maps.*

Moreover if X (resp. $\{X_i\}_i$) is equipped with a G-invariant measure m (resp. m_i) in such a way that (X, m) (resp. (X_i, m_i)) is a metric measure space and the push forward

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measure $(f_i)_* m_i$ converges to m weakly, then we say $\{(X_i, m_i)\}_i$ converges to (X, m) in the G-equivariant *measured* Gromov–Hausdorff topology and we will denote $X_i \xrightarrow{G\text{-eqmGH}} X$.

When we consider about equivariant (measured) Gromov–Hausdorff convergence for Riemannian manifolds, we always assume that the manifolds are equipped with Riemannian distance and Riemannian measure.

As a corollary of Theorem 5.1.7 we have the following convergence theorem of symplectic toric manifolds. We emphasize that we do not put any assumptions on curvatures in our theorem below.

THEOREM 5.2.2. *For a sequence $\{P_i\}_i \subset \mathcal{D}_n$ suppose that $d^H(P_i, P) \to 0$ $(i \to \infty)$ for $P \in \mathcal{D}_n$ and $N_{n-1}(P) = \lim_{i\to\infty} N_{n-1}(P_i)$, where $N_{n-1}(\cdot)$ is the number of the facets. Then there exists a subsequence of $\{M_{P_i}\}_i$ which converges to M_P with respect to the corresponding Guillemin metrics in the T-equivariant Gromov–Hausdorff topology.*

Proof. We use the same notations as in Section 4.2 with suffix i. We may assume $N = N_{n-1}(P) = N_{n-1}(P_i) = N_i$. The proof of Theorem 5.1.7 implies that $\mathfrak{h}_i = \mathfrak{h}$ and $H_i = H$ for $i \gg 0$. Moreover as a corollary of Theorem 5.1.7 we have $\lambda_i^{(r)} \to \lambda^{(r)}$ $(i \to \infty)$ for the constants of the defining equations of P_i (after renumbering the facets). As a consequence $(\iota_i^* \circ \tilde{\mu}_i)^{-1}(0)$ converges to $(\iota^* \circ \tilde{\mu})^{-1}(0)$ in the equivariant Gromov–Hausdorff topology[5]. Then $\{M_{P_i} = (\iota_i^* \circ \tilde{\mu}_i)^{-1}(0)/H_i\}_i$ converges to $M_P = (\iota^* \circ \tilde{\mu})^{-1}(0)/H$ in the Gromov–Hausdorff topology by [7, Theorem 2-1]. Moreover the identifications $H_i = H$ induce identifications $T_i^n = T^N/H_i = T^N/H = T^n$, which makes the above convergence into the T-equivariant Gromov–Hausdorff topology. ∎

COROLLARY 5.2.3. *Under the same assumptions in Theorem 5.2.2, take a subsequence in $\{M_{P_i}\}_i$ which converges to M_P. Then M_{P_i} are T-equivariantly diffeomorphic to M_P for $i \gg 0$.*

Proof. By Theorem 5.1.7 we may assume that $\nu_i^{(r)} = \nu^{(r)}$ for $i \gg 0$. On the other hand each M_{P_i} is T-equivariantly diffeomorphic to the toric variety associated with the fan Σ_{P_i}. Note that Σ_{P_i} is determined by the normal vectors $\{\nu_i^{(r)}\}_r$ and it does not depend on $\{\lambda_i^{(r)}\}_r$ (See [3] for example). It implies the claim. ∎

REMARK 5.2.4. The following example shows that it cannot be expected that a convergence of Guillemin metrics to a Guillemin metric as in Theorem 5.2.2 occurs without the assumption $N_{n-1}(P) = \lim_{i\to\infty} N_{n-1}(P_i)$.

Consider a sequence of Delzant pentagon $\{P_i\}_i$ defined by 5 inequalities,

$$\xi_1 \geq 0,\ 1 - \xi_1 \geq 0,\ \xi_2 \geq 0,\ 1 - \xi_2 \geq 0,\ -\xi_1 - \xi_2 + (2 - 1/i) \geq 0,$$

which converges to a rectangle P as in Figure 3.

It is known that the symplectic toric manifolds M_{P_i} corresponding to each pentagon P_i are (diffeomorphic to) a 1 point blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$. On the other hand the symplectic toric manifold M_P corresponding to P is isometric to $\mathbb{C}P^1 \times \mathbb{C}P^1$ equipped with the Guillemin metric which is the product metric on $\mathbb{C}P^1$.

[5] In fact this convergence is nothing other than the Hausdorff convergence of a sequence of compact subsets in $\mathbb{R}^N$.

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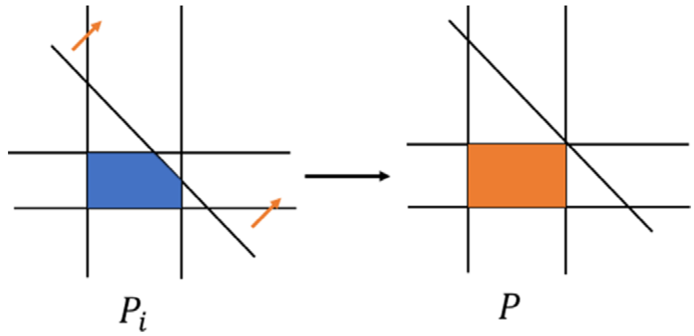

Figure 3.
A sequence of pentagons which converges to a rectangle.

The limiting process $i \to \infty$ gives a smooth convex function

$$g_\infty(\xi_1,\xi_2) = \frac{1}{2}(\xi_1 \log \xi_1 + (1-\xi_1)\log(1-\xi_1) + \xi_2 \log \xi_2 + (1-\xi_2)\log(1-\xi_2) \\ + (2-\xi_1-\xi_2)\log(2-\xi_1-\xi_2))$$

on P°. This g_∞ does not give the Guillemin metric on M_P. To deal with these subtle phenomena we have to consider finer structures on $\mathcal{D}_n$ or $\widetilde{\mathcal{D}}_n$ and incorporate *potential functions*. We will discuss such formulation in a subsequent paper.

5.3 From convergence of Guillemin metrics to convergence of polytopes

Now let us discuss the convergence of the opposite direction.

Hereafter for each $P \in \mathcal{D}_n$ we denote the symplectic toric manifold equipped with the Guillemin metric by $M_P = (M_P, \omega_P)$, and we use the Liouville volume form $\mathrm{vol}_{M_P} := \frac{(\omega_P)^{\wedge n}}{n!}$ on the symplectic toric manifold M_P. In this way we think M_P as a metric measure space.

THEOREM 5.3.1. *Let $\{P_i\}_i$ be a sequence in $\mathcal{D}_n$. Suppose that a sequence of symplectic toric manifolds $\{M_{P_i}\}_i$ converges to M_P for some $P \in \mathcal{D}_n$ with respect to their Guillemin metrics in the T-equivariant measured Gromov–Hausdorff topology. Let $\{f_i : M_{P_i} \to M_P\}_i$ be a sequence of approximation maps of the convergence. If $\{P_i\}_i$ are contained in a sufficiently large ball in $\mathbb{R}^n$, then we have*

$$\lim_{i\to\infty} f_i(M_{P_i}^T) = M_P^T,$$

where $M_{P_i}^T$ and M_P^T are the fixed point sets of T-actions. In particular we have

$$\liminf_{i\to\infty} \chi(M_{P_i}) \geq \chi(M_P),$$

where $\chi(\cdot)$ denotes the Euler characteristic.

Proof. For simplicity we denote $M_i := M_{P_i}$ and $M := M_P$.

Fix an arbitrary $\delta > 0$. We show that for any sequence $\{x_i \in M_i^T\}_i$ there exists $I \in \mathbb{N}$ such that $f_i(x_i) \in B(M^T, \delta)$ for any $i > I$. Suppose that there exists $\delta > 0$ such that $f_i(x_i) \notin B(M^T, \delta)$ for infinitely many i. For $\epsilon > 0$, we define δ_ϵ as the minimal $\delta' > 0$ such that if $y \notin B(M^T, \delta')$, then $\mathrm{Diam}(T \cdot y) \geq \epsilon$. Note that since M is compact such $\delta_\epsilon > 0$ exists and $\delta_\epsilon \to 0$ as $\epsilon \to 0$. Since f_i is almost T-equivariant we have

$$\epsilon_i > d(\rho_i(t) f_i(x_i), f_i(tx_i)) = d(\rho_i(t) f_i(x_i), f_i(x_i))$$

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for all $t \in T$, where $\{\epsilon_i\}_i$ is a sequence of positive numbers as in Definition 5.2.1 and d is the Riemannian distance of M. It implies that $\mathrm{Diam}(T \cdot f_i(x_i)) < 2\epsilon_i \to 0$ as $i \to \infty$. If we take i large enough so that $\delta_{\epsilon_i} < \delta$, then we have $f_i(x_i) \in B(M^T, \delta_{\epsilon_i})$. It contradicts to $f_i(x_i) \notin B(M^T, \delta)$.

Next we show that for any $\delta > 0$ there exists $i_0 \in \mathbb{N}$ such that

$$f_i^{-1}(M^T) \subset B(M_i^T, \delta)$$

holds for all $i > i_0$. If not then there exists $\delta > 0$ such that we can take $x_i \in f_i^{-1}(M^T)$ and $x_i \notin B(M_i^T, \delta)$ for infinitely many i. Since f_i is almost isometry and almost T-equivariant we have

$$\begin{aligned} d_i(tx_i, x_i) &< d(f_i(tx_i), f_i(x_i)) + \epsilon_i \\ &< d(f_i(tx_i), tf_i(x_i)) + \epsilon_i \\ &< 2\epsilon_i \end{aligned}$$

for all $t \in T$, where d_i is the Riemannian distance of M_i. It implies $\mathrm{Diam}(T \cdot x_i) < 4\epsilon_i$. On the other hand it is known that each $T \cdot x_i$ is a flat torus, and hence, $\mathrm{Diam}(T \cdot x_i) \to 0$ $(i \to \infty)$ implies $\mathrm{Vol}(T \cdot x_i) \to 0$ $(i \to \infty)$, where Vol is the Riemannian volume with respect to the induced Riemannian metric. Now consider a compact subset $P_i' := \mu_i(M_i \setminus B(M_i^T, \delta))$ of P_i. Since $\{M_i\}_i$ converges to M in the measured Gromov–Hausdorff topology $\{\mathrm{Vol}(M_i)\}_i$ converges to $\mathrm{Vol}(M)$. Duistermaat–Heckman's theorem implies that the Euclidean volumes of $\{P_i\}_i$ converge to that of P. In particular they are bounded below by a positive constant. Moreover since we assume that $\{P_i\}_i$ are contained in a ball, the sequence of convex polytopes $\{P_i\}_i$ converges to some convex body Q in the Hausdorff distance. As in the same way $\{P_i'\}_i$ converges to some compact subset Q' of Q. Let $Q^{(0)}$ be the limit point set of $\mu_i(M_i^T) = P_i^{(0)}$. Then we have $Q^{(0)} \cap Q' = \emptyset$. When we take $\delta' > 0$ small enough so that $\mathrm{dist}(Q^{(0)}, Q') > 2\delta'$ we have $\mathrm{dist}(P_i^{(0)}, P_i') > \delta'$. The formula of volumes of the orbits in [11] implies[6] that

$$\liminf_{i \to \infty} \mathrm{Vol}(T \cdot x_i) > 0.$$

It contradicts to $\lim_{i \to \infty} \mathrm{Vol}(T \cdot x_i) = 0$.

The inequality

$$\lim_{i \to \infty} \chi(M_{P_i}) \geq \chi(M_P),$$

follows from the fact that the Euler characteristic of symplectic toric manifold is equal to the number of fixed points. ∎

Hereafter we discuss the convergence of polytopes under the assumption in Theorem 5.3.1 without boundedness of $\{P_i\}_i$.

THEOREM 5.3.2. *Let $\{P_i\}_i$ be a sequence in $\mathcal{D}_n$. Suppose that a sequence of symplectic toric manifolds $\{M_{P_i}\}_i$ converges to M_P for some $P \in \mathcal{D}_n$ with respect to their Guillemin*

[6] Strictly speaking the formula in [11] can be applied when $\mu_i(x_i)$ is in the interior part of P_i. So the above argument shows that $\{x_i\}_i$ cannot be taken in such an interior part. As the next step we assume that $\{x_i\}_i$ sits in the inverse image of the interior part of codimension one face, and we deduce the contradiction. We proceed the same step for higher codimension face.

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metrics in the T-equivariant measured Gromov–Hausdorff topology. Let $\{f_i : M_{P_i} \to M_P\}_i$ *be a sequence of approximation maps of the convergence. We take and fix a section*[7] $S_i : P_i \to M_{P_i}$ *of the moment map* $\mu_i : M_{P_i} \to P_i$ *for each i. For each i we define* $\hat{f}_i : P_i \to P$ *by the composition* $\hat{f}_i := \mu \circ f_i \circ S_i$.

Under the above set-up there exists a subsequence of $\{\overline{\hat{f}_i(P_i)}\}_i$ *which converges to P in the* d^H*-topology.*

To show Theorem 5.3.2 we prepare two lemmas. Consider the same setting as in Theorem 5.3.2. Let $\mu_i : M_{P_i} \to P_i \subset \mathbb{R}^n$ and $\mu : M_P \to P \subset \mathbb{R}^n$ be the moment maps. For any $\varphi \in C_b(\mathbb{R}^n)$ we define $\tilde{\varphi} \in C(M_P)$ by $\tilde{\varphi} := \varphi \circ \mu$. Let $\{f_i\}_i$ be a family of approximation maps for $M_{P_i} \xrightarrow{T\text{-eqmGH}} M_P$. We define a sequence of measurable functions $\{\tilde{\varphi}_i : M_{P_i} \to \mathbb{R}\}_i$ by $\tilde{\varphi}_i := \tilde{\varphi} \circ f_i$. Let $\{(\mathrm{vol}_{M_{P_i}})_y\}_{y \in P_i}$ (resp. $\{(\mathrm{vol}_{M_P})_y\}_{y \in P}$) be a disintegration (See Appendix B) for $\mu_i : M_{P_i} \to P_i$ (resp. $\mu : M_P \to P$) and define a sequence of measurable functions $\{\varphi_i : P_i \to \mathbb{R}\}_i$ by

$$\varphi_i(y) := \int_{M_{P_i}} \tilde{\varphi}_i(x)(\mathrm{vol}_{M_{P_i}})_y(dx). \tag{5.3}$$

Lemma 5.3.3. *For any* $\varphi \in C_b(\mathbb{R}^n)$ *the sequence of measurable maps* $\{\varphi_i : P_i \to \mathbb{R}\}_i$ *satisfies*

$$\lim_{i \to \infty} \int_{P_i} \varphi_i \; d\mathcal{L}^n = \int_P \varphi \; d\mathcal{L}^n.$$

Proof. Note that by Duistermaat–Heckman's theorem we have $(\mu_i)_*(\mathrm{vol}_{M_{P_i}}) = \mathcal{L}^n|_{P_i}$. Since $(f_i)_*(\mathrm{vol}_{M_{P_i}})$ converges to vol_{M_P} the claim follows as follows.

$$\begin{aligned}
\int_{P_i} \varphi_i(y)\mathcal{L}^n(dy) &= \int_{P_i} \left(\int_{M_{P_i}} \tilde{\varphi}_i(x)(\mathrm{vol}_{M_{P_i}})_y(dx) \right) \mathcal{L}^n(dy) \\
&= \int_{M_{P_i}} \tilde{\varphi}_i(x)\mathrm{vol}_{M_{P_i}}(dx) \\
&= \int_{M_{P_i}} \tilde{\varphi}(f_i(x))\mathrm{vol}_{M_{P_i}}(dx) \\
&\xrightarrow[i \to \infty]{} \int_{M_P} \tilde{\varphi}(x)\mathrm{vol}_{M_P}(dx) \\
&= \int_P \left(\int_{M_P} \tilde{\varphi}(x)(\mathrm{vol}_{M_P})_y(dx) \right) \mathcal{L}^n(dy) \\
&= \int_P \left(\int_{\mu^{-1}(y)} \varphi(\mu(x))(\mathrm{vol}_{M_P})_y(dx) \right) \mathcal{L}^n(dy) \\
&= \int_P \left(\int_{\mu^{-1}(y)} \varphi(y)(\mathrm{vol}_{M_P})_y(dx) \right) \mathcal{L}^n(dy) \\
&= \int_P \varphi(y)\mathcal{L}^n(dy).
\end{aligned}$$

∎

[7] We do not assume the continuity of S_i. We only need the measurability of it.

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LEMMA 5.3.4. *As in the same setting in Theorem 5.3.2 we have*

$$\lim_{i\to\infty}\frac{1}{|P_i|}\int_{P_i}\varphi_i d\mathcal{L}^n=\lim_{i\to\infty}\frac{1}{|P_i|}\int_{P_i}\varphi\circ\hat{f}_i d\mathcal{L}^n.$$

for any $\varphi\in C_b(\mathbb{R}^n)$, *where* φ_i *are as in (5.3).*

Proof. Let $\{\rho_i:T^n\to T^n\}_i$ be a sequence of automorphisms as in Definition 5.2.1 for $M_{P_i}\xrightarrow{\text{eq}-\text{m}GH}M_P$. Fix $\eta>0$ and $\varphi\in C_b(\mathbb{R}^n)$. For any $y\in P_i$ we have

$$|\varphi_i(y)-\varphi(F_i(y))|\le\int_{\mu_i^{-1}(y)}|\varphi(\mu(f_i(x)))-\varphi(\mu(f_i(S_i(y))))|(\mathrm{vol}_{M_{P_i}})_y(dx). \tag{5.4}$$

Since for any $x\in\mu_i^{-1}(y)$ there exists $t_x\in T$ such that $x=t_x\cdot S_i(y)$ we have

$$\begin{aligned}\|\mu(f_i(x))-\mu(f_i(S_i(y)))\|&=\|\mu(f_i(t_x\cdot S_i(y)))-\mu(f_i(S_i(y)))\|\\&=\|\mu(f_i(t_x\cdot S_i(y)))-\mu(\rho_i(t_x)\cdot f_i(S_i(y)))\|.\end{aligned}$$

On the other hand since φ and μ are uniformly continuous and $\{M_{P_i}\}_i$ converges to M_P in the T-equivariant Gromov–Hausdorff topology there exists $i_0\in\mathbb{N}$ such that if $i>i_0$, then

$$|\varphi(\mu(f_i(x)))-\varphi(\mu(f_i(S_i(y))))|=|\varphi(\mu(f_i(x)))-\varphi(\mu(\rho_i(t_x)\cdot f_i(S_i(y))))|<\eta.$$

In particular we have

$$|\varphi_i(y)-\varphi(\hat{f}_i(y))|<\eta$$

in (5.4), and hence,

$$\frac{1}{|P_i|}\left|\int_{P_i}(\varphi_i(y)-\varphi(\hat{f}_i(y)))\mathcal{L}^n(dy)\right|<\eta.$$

Note that our assumption $M_{P_i}\xrightarrow{T\text{-eqmGH}}M_P$ and Duistermaat–Heckman's theorem imply $|P_i|=\mathrm{vol}_{M_{P_i}}(M_{P_i})\to|P|=\mathrm{vol}_{M_P}(M_P)$. Since $\eta>0$ is arbitrary the limit of $\dfrac{1}{|P_i|}\displaystyle\int_{P_i}\varphi_i(y)\mathcal{L}^n(dy)$ exists and we have the required equality

$$\lim_{i\to\infty}\frac{1}{|P_i|}\int_{P_i}\varphi_i(y)\mathcal{L}^n(dy)=\lim_{i\to\infty}\frac{1}{|P_i|}\int_{P_i}\varphi(\hat{f}_i(y))\mathcal{L}^n(dy).$$

□

Proof of Theorem 5.3.2. Let $\varphi\in C_b(\mathbb{R}^n)$. By Lemmas 5.3.3 and 5.3.4, we have a sequence of measurable maps $\{\hat{f}_i:P_i\to P\}_i$ and measurable functions $\{\varphi_i:P_i\to\mathbb{R}\}_i$ such that

$$\lim_{i\to\infty}\int_P\varphi(y)(\hat{f}_i)_*(\mathcal{L}^n)(dy)=\lim_{i\to\infty}\int_{P_i}\varphi_i(y)\mathcal{L}^n(dy)=\int_P\varphi(y)\mathcal{L}^n(dy).$$

Note that we have $|P_i|\to|P|$ $(i\to\infty)$ under our assumption, measured Gromov–Hausdorff convergence, and Duistermaat–Heckman's theorem. This equality implies that the sequence of probability measures $\{(\hat{f}_i)_*m_{P_i}\}_i$ converges weakly to m_P.

Now we show that

$$\inf_i\inf_{x\in P}(\hat{f}_i)_*(m_i)(B_\epsilon(x))>0 \tag{5.5}$$

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for all $\epsilon > 0$. If not, then there exists $\epsilon_0 > 0$, a sequence of natural numbers $\{i_j\}_j$ and a sequence of points $\{x_j\}_j$ in P such that

$$(\hat{f}_{i_j})_*(m_{i_j})(B_{\epsilon_0}(x_j)) \to 0 \quad (j \to \infty).$$

Since P is compact there is an accumulation point x_∞ of $\{x_j\}_j$ and we have

$$B_{\epsilon_0/2}(x_\infty) \subset B_{\epsilon_0}(x_j)$$

for infinitely many j. Then we have a contradiction

$$\begin{aligned} 0 < m_P(B_{\epsilon/2}(x_\infty)) &\le \liminf_j (\hat{f}_{i_j})_*(m_{i_j})(B_{\epsilon_0/2}(x_\infty)) \\ &\le \liminf_j (\hat{f}_{i_j})_*(m_{i_j})(B_{\epsilon_0}(x_j)) = 0. \end{aligned}$$

The weak convergence of $\{(\hat{f}_i)_* m_{P_i}\}_i$ to m_P implies the $\square$-convergence of a sequence of metric measure spaces $\{(P, (\hat{f}_i)_* m_{P_i})\}$ to (P, m_P) by [16, Proposition 4.12]. Moreover the $\square$-convergence and (5.5) imply the measured Gromov–Hausdorff convergence of $\{(P, (\hat{f}_i)_* m_{P_i})\}$ to (P, m_P) by [16, Remark 4.34], which in particular implies the Hausdorff convergence of $\{\mathrm{supp}((\hat{f}_i)_* m_{P_i}) = \overline{\hat{f}_i(P_i)}\}_i$ to $\mathrm{supp}(m_P) = P$. ∎

Remark 5.3.5. Regarding Theorem 5.3.2 it is natural to consider the convergence of $\{P_i\}_i$ itself to P in the Gromov–Hausdorff or d^H-topology. One can see that this is not true in the literal sense because of the ambiguity of the affine transformation groups G_n. We could address these problems in terms of the moduli space. Namely one may hope that if $\{M_{P_i}\}_i$ converges to M_P in the T-equivariant measured Gromov–Hausdorff topology, then there exists a sequence $\{\varphi_i\}_i$ in G_n such that $\{\varphi_i(P_i)\}_i$ converges to P in the Gromov–Hausdorff or d^H-topology. It would be useful to consider minimum variance elements explained in Remark 3.2.4.

§A. Preliminaries on probability measures and L^2-Wasserstein distance.

In this appendix we summarize several facts on probability measures and L^2-Wasserstein distance. For more details consult [18] for example.

Let $\mathscr{P}(\mathbb{R}^n)$ be the set of all complete Borel probability measures on $\mathbb{R}^n$. Consider the subset of $\mathscr{P}(\mathbb{R}^n)$ consisting of measures with finite quadratic moment,

$$\mathscr{P}_2(\mathbb{R}^n) := \left\{ m \in \mathscr{P}(\mathbb{R}^n) \;\middle|\; \exists o \in \mathbb{R}^n,\ \int_{\mathbb{R}^n} \|x - o\|^2 m(dx) < \infty \right\}.$$

A.1. Weak convergence and Prokhorov's theorem

Definition A.1.1. A sequence $\{m_i\}_i$ in $\mathscr{P}(\mathbb{R}^n)$ *converges weakly to* $m \in \mathscr{P}(\mathbb{R}^n)$

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} f(x) m_i(dx) = \int_{\mathbb{R}^n} f(x) m(dx)$$

for any bounded continuous function f on $\mathbb{R}^n$.

Theorem A.1.2. *For a sequence* $\{m_i\}_i$ *in* $\mathscr{P}(\mathbb{R}^n)$ *and* $m \in \mathscr{P}(\mathbb{R}^n)$ *the followings are equivalent.*

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1. $\{m_i\}_i$ *converges weakly to* m.
2. *For any open subset* U *in* $\mathbb{R}^n$ *we have* $\liminf_{i\to\infty} m_i(U) \geq m(U)$.
3. *For any closed subset* C *in* $\mathbb{R}^n$ *we have* $\limsup_{i\to\infty} m_i(C) \leq m(C)$.
4. *For any Borel subset* A *in* $\mathbb{R}^n$ *with* $m(\overline{A}\setminus A^\circ) = 0$ *we have* $\lim_{i\to\infty} m_i(A) = m(A)$.

THEOREM A.1.3 (Prokhorov's theorem). *A subset* $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^n)$ *is relatively compact with respect to the weak convergence topology if and only if for all* $\epsilon > 0$ *there exists a compact subset* $K \subset \mathbb{R}^n$ *such that*[8]

$$\sup_{m\in\mathcal{K}} m(\mathbb{R}^n \setminus K) < \epsilon.$$

For a weak convergent sequence of probability measure the following is well-known. See [2] for example.

THEOREM A.1.4. *If* $\{m_i\}_i \subset \mathcal{P}(\mathbb{R}^n)$ *has a weak convergent limit* $m \in \mathcal{P}(\mathbb{R}^n)$, *then for any* $x \in \mathrm{supp}\,(m)$ *there exists* $x_i \in \mathrm{supp}\,(m_i)$ *such that* $x_i \to x$.

A.2. L^2-Wasserstein distance of probability measures

For $m, m' \in \mathscr{P}_2(\mathbb{R}^n)$ let $\mathsf{Cpl}(m,m')$ be the set of all couplings between m and m'. Namely $\mathsf{Cpl}(m,m')$ is the set of measures $\xi \in \mathscr{P}(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any Borel subset A of $\mathbb{R}^n$ it satisfies

$$\begin{cases} \xi(A\times\mathbb{R}^n) = m(A) \\ \xi(\mathbb{R}^n \times A) = m'(A). \end{cases}$$

The L^2-Wasserstein distance between $m, m' \in \mathscr{P}_2(\mathbb{R}^m)$ is defined by

$$W_2(m,m') := \inf\left\{ \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|^2 \xi(dx,dy) \right)^{1/2} \,\middle|\, \xi \in \mathsf{Cpl}(m,m') \right\}.$$

It is known that W_2 is a metric on $\mathscr{P}_2(\mathbb{R}^n)$ and $(\mathscr{P}_2(\mathbb{R}^n), W_2)$ is a complete separable metric space with the following properties.

THEOREM A.2.1. *For a sequence* $\{m_i\}_i$ *in* $\mathscr{P}_2(\mathbb{R}^n)$ *and* $m \in \mathscr{P}_2(\mathbb{R}^n)$ *the followings are equivalent.*

1. $W_2(m_i, m) \to 0$ *(*$i \to \infty$*)*.
2. $\{m_i\}_i$ *converges weakly to* m *and*

$$\lim_{R\to\infty} \limsup_{i\to\infty} \int_{\mathbb{R}^n\setminus B(o,R)} \|x-o\|^2 m_i(dx) = 0.$$

3. *For any continuous function* φ *such that* $|\varphi(x)| \leq C(1+\|x_0 - x\|)^2$ *for some* $C > 0$, $x_0 \in \mathbb{R}^n$ *the following holds*

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} \varphi\, dm_i = \int_{\mathbb{R}^n} \varphi\, dm.$$

Recall that if for $m, m' \in \mathscr{P}_2(\mathbb{R}^n)$ there exists a Borel measurable map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T_* m = m'$ and $(\mathsf{id}\times T)_* m \in \mathsf{Opt}(m,m')$, then we say that the Monge problem for m, m' admits a solution and T is called a solution of the Monge problem.

[8] A subset $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^n)$ with this property is often called *tight*.

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Theorem A.2.2. *For $m, m' \in \mathscr{P}_2(\mathbb{R}^n)$ if $m \ll \mathcal{L}^n$, then there is a solution of the Monge problem for m and m'. The solution is unique in the following sense. For another solution $S : \mathbb{R}^n \to \mathbb{R}^n$ we have $m(\{T \neq S\}) = 0$.*

Corollary A.2.3. *For $m, m' \in \mathscr{P}_2(\mathbb{R}^n)$ with $m \ll \mathcal{L}^n$ and a sequence $\{m'_i\}_i$ in $\mathscr{P}_2(\mathbb{R}^n)$ which converges weakly to m', there exists a solution $T : \mathbb{R}^n \to \mathbb{R}^n$ of the Monge problem for m, m' and a sequence $\{T_i\}_i$ of solutions of the Monge problem for m, m'_i with*

$$m(\{x \in \mathbb{R}^n \mid |T_i(x) - T(x)| \geq \epsilon\}) \to 0 \ (i \to \infty).$$

§B. Disintegration theorem

We use the following type of disintegration theorem. See [8, Theorem 16.10.1] for example.

Theorem B.0.1. *Let X and Y be complete separable metric spaces. Let m be a σ-finite Borel probability measure and $f : X \to Y$ a Borel measurable map. Suppose that the push forward f_*m is a σ-finite measure on Y. Then there exists a family of probability measures $\{m_y\}_{y \in Y}$ on X such that for each Borel subset A the map*

$$Y \ni y \mapsto m_y(A) \in [0, 1]$$

is Borel measurable and for each Borel measurable function φ on X we have

$$\int_X \varphi \, dm = \int_Y \left(\int_X \varphi(x) m_y(dx) \right) f_*m(dy).$$

Moreover we have

$$m_y(f^{-1}(y)) = 1 \quad (y \in Y \ (f_*m\text{-}a.e)).$$

The above family of measures $\{m_y\}_{y \in Y}$ is called a *disintegration for* $f : X \to Y$.

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