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NILPOTENT-BY-ČERNIKOV CC-GROUPS

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Abstract

In this paper we study groups with Černikov conjugacy classes which are nilpotent-by-Černikov groups, giving full characterizations of them and applying the results obtained to some related areas.

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1. Introduction

This paper is mainly devoted to the characterization of groups in which $G/\zeta_c(G)$ is Černikov for some integer $c \geq 1$, that is, (nilpotent of class c)-by-Černikov groups, as well as some applications of the results obtained to other related areas. Thus the starting point here is the extension of some ideas first considered by P. Hall [3, 4].

P. Hall [3] has shown that a group G is finite-by-nilpotent if and only if $G/\zeta_j(G)$ is finite for some integer $j \geq 0$. In extending these ideas to Černikov groups we first note that the results do not hold for an arbitrary group G. A specific example of this is provided by the following construction. Let P be a Prüfer p-group, where p is an odd prime, and let α be the automorphism of P given by $x^\alpha = x^2$, $x \in P$. Let G be the split extension of P by $\langle\alpha\rangle$ so that G is a Černikov-by-abelian group. Then one can show that $G' = P = C_G(P)$ and $\zeta_c(G) = 1$, for every $c \geq 0$. Suppose now that

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there exists a nilpotent normal subgroup N of G such that G/N is Černikov. Then $N = P\langle\alpha^s\rangle$ for some $s \geq 1$. It is easy to show that $\zeta(N) = 1$, which is a contradiction. Therefore G cannot be nilpotent-by-Černikov.

To avoid this difficulty we consider *groups with Černikov conjugacy classes*, or *CC-groups*, a natural extension of the concept of FC-groups, which was first considered by Ya. D. Polovickiĭ [12]. A group G is said to be a CC-group if $G/C_G(x^G)$ is Černikov for each $x \in G$. Indeed the above example is a group G which is not a CC-group because $G/C_G([G, \alpha]) = G/P$ is not periodic.

Our group-theoretic notation is standard and is taken from Robinson [13]. Thus $\{\zeta_c(G) \mid c \geq 0\}$ denotes the upper central series of the group G and $\{\gamma_c(G) \mid c \geq 1\}$ its lower central series. The layout of this paper is as follows. Section 2 is devoted to the characterization of (nilpotent of class c)-by-Černikov and locally nilpotent-by-Černikov CC-groups (Corollaries 2.2 and 2.3 and Theorem 2.7). Also we shall give the Černikov version of P. Hall's results mentioned above (Theorem 2.5 and Corollary 2.6). Section 3 deals with a characterization of central-by-Černikov groups in terms of the structure of the factors $G/C_G(x^G)$, $x \in G$, of the CC-group G. What we shall do there is to bound the index of the radicable part of these Černikov groups and to show that this will characterize central-by-Černikov groups (Theorem 3.2). Finally, in Section 4, we shall make use of the results shown in Section 2 in order to obtain a result (Theorem 4.1), which is a contribution to the answer of an open question concerning groups with certain minimal conditions as those considered by the authors in [6].

2. Characterizations of Nilpotent-by-Černikov CC-groups

The factor group $G/\zeta(G)$ of an arbitrary CC-group G does not have to be periodic, which is one of the first difficulties that one always finds in dealing with CC-groups. However, if G is a residually Černikov CC-group, then the result is true, that is, $G/\zeta(G)$ is periodic (see Otal, Peña and Tomkinson [9, Lemma 6.2]). Since $G/\zeta(G)$ is residually Černikov, a consequence of the above fact is that $G/\zeta_c(G)$ is periodic for any CC-group G and any integer $c \geq 2$ (see González, Otal and Peña [2]). Thus, in a certain sense, the ordinal $c = 2$ is a boundary to periodicity and hence most of the results of this section have two parts depending on the values of c.

THEOREM 2.1. *Let G be a CC-group with a normal subgroup N such that G/N is radicable abelian.*

(1) *If N is abelian and $G/\zeta(G)$ is periodic, then G is abelian.*

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(2) *If* $c \geq 2$ *and* N *is nilpotent of class* c, *then* G *is nilpotent of class* c.

PROOF. (1) Let $x \in N$ and choose $n \geq 1$ such that $x^n \in \zeta(G)$. Since x^G is abelian, we have $[g, x]^n = [g, x]^n = 1$ for every $g \in G$, so that $[G, x]$ is Černikov with finite exponent and hence $[G, x]$ is finite. Then x^G is finite-by-cyclic and therefore $G/C_G(x^G)$ is finite. Since $x^G \leq N$ and N is abelian, we have that $N \leq C_G(x^G)$ and hence $G = C_G(x^G)$. Therefore N is central in G and, in particular, $G/\zeta(G)$ is abelian and $G' \leq \zeta(G)$.

Given g and h in G, we take $m \geq 1$ such that $g^m \in \zeta(G)$. Since G/N is radicable, there is some $y \in G$ such that $hN = y^m N$. Here N is central in G so that $[g, h] = [g, y^m]$ and, since G' is also central in G, we have $[g, y^m] = [g^m, y] = 1$. Thus $[g, h] = 1$ and G is abelian.

(2) Let c and N be as in the statement. We claim that $\gamma_c(N) \leq \zeta(G)$. For, let $x \in \gamma_c(N)$. Since $c \geq 2$, $x \in G'$ and so x has finite order, that is, x^G is Černikov. By hypothesis, $\gamma_c(N) \leq \zeta(N)$, so that x^G is Černikov and abelian. By Robinson [13, Corollary to 3.29.2], $G/C_G(x^G)$ is finite and, since $N \leq C_G(x^G)$, we find $G = C_G(x^G)$, that is, $x \in \zeta(G)$ as required. As a consequence we note that the quotient $N\zeta(G)/\zeta(G)$ is nilpotent of class $c - 1$.

We now show (2) by induction on c. Suppose first that $c = 2$. As we mentioned above, the quotient $G/\zeta(G)$ has central factor periodic. Here $N\zeta(G)/\zeta(G)$ is abelian so that, by (1), $G/\zeta(G)$ is abelian and G is nilpotent of class 2. Suppose now that $c > 2$. Considering the factor group $G/\zeta(G)$ and applying induction, we conclude that $G/\zeta(G)$ is nilpotent of class $c - 1$ and hence G is nilpotent of class c.

COROLLARY 2.2. *For a CC-group the following assertions are equivalent.*
(1) G *is abelian-by-Černikov and* $G/\zeta(G)$ *is periodic.*
(2) G *is abelian-by-finite.*
(3) G *is central-by-Černikov.*

PROOF. Clearly (3) implies (1) and it is easy to show that (2) implies (3) (see Otal and Peña [5, 2.1]).

(1) $\Rightarrow$ (2). Let A be an abelian normal subgroup of G such that G/A is Černikov. If R/A is the radicable part of G/A, then G/R is finite. By Theorem 2.1, R is abelian. Therefore G is abelian-by-finite.

In fact the above properties (2) and (3) are always equivalent. However we note that we cannot remove the condition on the periodicity of $G/\zeta(G)$ in Theorem 2.1 and Corollary 2.2 as the following example shows.

Example. Let $H = \langle x_i \mid i \geq 1\rangle$ and $K = \langle y_1 \mid i \geq 1\rangle$ be two copies of a Prüfer p-group. Then the direct product of H and K admits an automorphism z of infinite order given by $(x_i)^z = x_i$ and $(y_i)^z = x_i y_i$, for every $i \geq 1$. If G is the corresponding split extension of the direct product of H and K by $\langle z\rangle$, then it is easy to show that $G' = H = \zeta(G)$ so that G is a non-abelian CC-group. However $\langle z\rangle \cap \zeta(G) = 1$ and hence $G/\zeta(G)$ is not periodic.

Define $A = H\langle z\rangle$. Since $G' \leq A$, it is clear that A is a normal subgroup of G. Clearly A is abelian and $G/A \cong K$ is radicable abelian.

Corollary 2.3. *Let $c \geq 2$ be an integer. For a CC-group G the following assertions are equivalent.*

(1) *G is nilpotent of class c-by-Černikov.*

(2) *G is nilpotent of class c-by-finite.*

(3) *$G/\zeta_c(G)$ is Černikov.*

Proof. (3) $\Rightarrow$ (1) is trivial.

(2) $\Rightarrow$ (3). We may write $G = NX^G$, where N is a normal subgroup of G which is nilpotent of class c and X is a finite subset of G. By induction, it is easy to show that $\zeta_r(N) \cap C_G(X^G) \leq \zeta_r(G)$ for every integer $r \geq 0$. Thus $N \cap C_G(X^G) \leq \zeta_c(G)$ and, since G/N is finite, (3) follows.

(1) $\Rightarrow$ (2). Proceed as in the proof of Corollary 2.2, making use of Theorem 2.1.

We note that the conditions in Corollary 2.3 are not equivalent to the property of G being finite-by-nilpotent. For, let G be the split extension of a Prüfer p-group P, p odd, by a cyclic group of order 2. Then G is Černikov and it is easy to show that $\gamma_i(G) = P$ for every $i \geq 2$. Suppose that F is a finite normal subgroup of G. If $P \neq PF$, then $G = PF$, G/F is abelian and $P = G' \leq F$, a contradiction. Therefore F is contained in P. Let $g \in G$ such that $gF \in \zeta(G/F)$, that is, $[G, g] \leq F$. It follows that $g \in P$, otherwise $[G, g]$ is infinite. Then $g \in [G, g]$ so that $\zeta(G/F) = 1$. Therefore G/F cannot even be locally nilpotent.

As a consequence of a result of Polovickiĭ [11] and of the generalization of a result of P. Hall [4, Theorem 8.7] (see [13, 4.23, 4.21 and corollaries]) we may state a condition which is implied by those in the above corollaries.

Theorem 2.4. *Let $c \geq 0$. If $G/\zeta_c(G)$ is Černikov, then $\gamma_{c+1}(G)$ is Černikov.*

The converse of the above statement fails in general even in the case where the group G is an FC-group. For example, let G be a finite-by-abelian group G with $G/\zeta(G)$ infinite (see Tomkinson [14, Example 3.8]). Here G' is finite and then G is an FC-group and $G/\zeta(G)$ is residually finite. Then it follows that $G/\zeta(G)$ cannot be Černikov.

Despite that, we have the following characterization in the residually Černikov case which completes the information given in Corollaries 2.2 and 2.3.

THEOREM 2.5. *Let $c \geq 0$ and let G be a residually Černikov CC-group. Then $G/\zeta_c(G)$ is Černikov if and only if $\gamma_{c+1}(G)$ is Černikov.*

PROOF. Since G is residually Černikov and $\gamma_{c+1}(G)$ is Černikov, there exists a normal subgroup N of G such that G/N is Černikov and $N \cap \gamma_{c+1}(G) = 1$. Then the latter implies that

$$[N, \underbrace{G, \dots, G}_{\leftarrow c \rightarrow}] = 1.$$

Since $\zeta_c(G)$ is the largest subgroup of G satisfying the above condition, it follows that $N \leq \zeta_c(G)$. Then $G/\zeta_c(G)$ is Černikov.

In studying the converse of Theorem 2.4, a positive result is the next one, which is a consequence of Theorem 2.5.

COROLLARY 2.6. *Let $c \geq 1$. If G is a CC-group and $\gamma_c(G)$ is Černikov, then $G/\zeta_c(G)$ is Černikov.*

PROOF. If $c = 1$, then the conclusion is trivial. Suppose that $c > 1$ and the result holds for $c - 1$. The factor group $\overline{G} = G/\zeta(G)$ is residually Černikov and $\gamma_c(\overline{G})$ is Černikov so that, by Theorem 2.5, $\overline{G}/\zeta_{c-1}(\overline{G})$ is Černikov. Then the result follows since $\zeta_{c-1}(\overline{G}) = \zeta_c(G)/\zeta(G)$.

We note that, if G is a CC-group, $k = k(G)$ is the least non-negative integer such that $\gamma_{k+1}(G)$ is Černikov and $j = j(G)$ is the least non-negative integer such that $G/\zeta_j(G)$ is Černikov, then $k(G) \leq j(G) \leq k(G) + 1$. If G is again the group described in Tomkinson [14], Example 3.8, then $k(G) = 1$ and $j(G) = 2$. We note that an infinite elementary abelian group G is an example in which $j(G) = 1 = k(G)$. Therefore the above inequalities are the best possible even for FC-groups.

Concerning infinite ordinals, we first note that a result such as Theorem 2.4 does not work for these ordinals when we consider the class of finite groups. For example, let G be a locally dihedral 2-group, that is, the non-abelian

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split extension of a Prüfer 2-group C by a cyclic group of order 2. Here G is Černikov, $\gamma_{\omega+1}(G) = G' = C$ but $G/\zeta_\omega(G)$ has order 2. In our case, we have that the properties "locally nilpotent" and "hypercentral" are identical for CC-groups (see Otal and Peña [5, 2.1]). Then the extension of Corollaries 2.2 and 2.3 for these ordinals is given by the next result, which is a slight generalization of a result shown by the authors in [8].

THEOREM 2.7. *Let G be a CC-group with $G/\zeta(G)$ periodic. Then the following conditions are equivalent.*

(1) *G is locally nilpotent-by-finite.*

(2) *G is locally nilpotent-by-Černikov.*

(3) *G is Černikov-by-locally nilpotent.*

(4) *There is a finite set X of G such that G/S is locally nilpotent, where S is the torsion subgroup of X^G.*

PROOF. Let H be the locally nilpotent radical of G. By Otal and Peña [7, Theorem B], G/H is residually finite and then G/H is Černikov if and only if G/H is finite. This establishes the equivalence between (1) and (2). Assume (1) and write $G = HX^G$, where X is a finite subset of G. If S is the torsion subgroup of X^G, then, by Alcázar and Otal [1, Lemma 1], S is Černikov and X^G/S is abelian. Moreover $[X^G, H] \leq X^G \cap G' \leq S$ so that $[X^G/S, HS/S] = 1$ and therefore $X^G/S \leq \zeta(G/S)$. Thus $(G/S)/\zeta(G/S)$ is locally nilpotent and so is G/S. Hence (1) $\Rightarrow$ (4) and it is clear that (4) $\Rightarrow$ (3).

Thus it remains to show (3) $\Rightarrow$ (2). Let N be a Černikov normal subgroup of G such that G/N is locally nilpotent. Put $C = C_G(N)$. Since $G/\zeta(G)$ is periodic, G/C is a periodic group of automorphisms of the Černikov group N and so G/C is also Černikov (see Robinson [13, Theorem 3.29]). Now $C/C \cap N$ is locally nilpotent and $C \cap N \leq \zeta(C)$ so that $C/\zeta(C)$ is locally nilpotent. Therefore C is locally nilpotent and G is locally nilpotent-by-Černikov.

3. A characterization of central-by-Černikov groups

A central-by-Černikov group is a very special type of CC-group (see Theorem 2.4) and these groups have been characterized in Corollary 2.2. Certainly any CC-group has a local system of central-by-Černikov groups, because a CC-group is locally an FC-group and a finitely generated FC-group is central-by-finite (see Tomkinson [14]). However the above description does not give too much information on the structure of a CC-group.

This section gives a new characterization of central-by-Černikov groups from another point of view. The starting point of this study is the structure of the normal closures of the Černikov subgroups of a CC-group.

Even if H is a Černikov subgroup of a periodic CC-group G, the normal closure H^G of H need not be Černikov (Otal, Peña and Tomkinson [9, Example 6.8]), although H^G can be embedded in a subgroup of a direct product of countably many Černikov groups (Polovickiĭ [10, Theorem 6]). Furthermore we have the following result.

LEMMA 3.1. *If H is a Černikov subgroup of the periodic CC-group G, then H^G is a central-by-Černikov group.*

PROOF. Let A be the radicable part of H and let T be a transversal from H to A. Then T^G is Černikov so that $T^G = BF$, where B is the radicable part of T^G and F is a finite group. It is known that the properties radicable and semi-radicable are identical for CC-groups (see Otal and Peña [5]) so that we have that an arbitrary join of radicable subgroups of G is radicable. By the results of [2, Section 4], A^G and $A^G B$ are radicable abelian. Therefore $H^G = (A^G B)F$ is abelian-by-finite and then, by Corollary 2.2, H^G is central-by-Černikov.

Suppose now that G is a central-by-Černikov group. If X is a finite subset of G, then $G/C_G(X^G)$ is a quotient of $C/\zeta(G)$ and the index of the radicable part of $G/C_G(X^G)$ in $G/C_G(X^G)$ is bounded by that of $G/\zeta(G)$ in $G/\zeta(G)$. This trivial remark gives sense to the following definition. If G is a CC-group and $n \geq 1$ is a fixed integer, then G *is said to be a CC-group of bounded index* n, or simply *a CC-group of bounded index*, if the index of the radicable part of $G/C_G(X^G)$ in $G/C_G(X^G)$ is at most n for each finite subset X of G. Then a central-by-Černikov group is of bounded index and this type of CC-group exhausts the class of CC-groups of bounded index.

THEOREM 3.2. *For a group G the following conditions are equivalent.*
(1) *G is a CC-group of bounded index and $G/\zeta(G)$ is periodic.*
(2) *G is a central-by-Černikov group.*

PROOF. It suffices to show that $(1) \Rightarrow (2)$. Let X be a finite subset of G such that $G/C_G(X^G)$ has maximal index k. Let $D/C_G(X^G)$ be the radicable part of $G/C_G(X^G)$ so that $|G/D| = k$. If $g \in G$ and $Y = X \cup \{g\}$ then $G/C_G(X^G)$ is isomorphic to a quotient of $G/C_G(Y^G)$ and the maximality of k shows that $D/C_G(Y^G)$ is the radicable part of $G/C_G(Y^G)$. In particular $D/C_D(g^G)$ is radicable Černikov and so abelian. Therefore $D/\zeta(G)$ is abelian.

Let $x \in D$ with $x^m \in \zeta(G)$. Since $[D, x]$ is central in G we have $[y, x]^m = [y, x^m] = 1$ for each $y \in D$. Therefore $[D, x]$ has finite exponent and so is finite. Then x^D is finite-by-cyclic and $D/C_D(x^D)$ is finite. Since $D/C_D(x^D)$ is radicable, $x \in \zeta(D)$ and so D is abelian. Hence G is abelian-by-finite and, by Corollary 2.2, G is central-by-Černikov.

González, Otal and Peña [2] have constructed a radicable CC-group G in which G' is a Prüfer p-group and $G/\zeta(G)$ is a torsion-free abelian group. This example has bounded index 1 and shows that we cannot remove the periodicity in Theorem 3.2. Also the direct product H of an infinite number of copies of the above G is again a radicable CC-group and therefore has bounded index 1 but H' is not Černikov. We note that the example given in Section 2 is Černikov-by-abelian but is not of bounded index. Thus, for an arbitrary CC-group G, there is no relationship between the properties G' *Černikov* and G *is of bounded index*; that is, there is no analogue of BFC-groups (compare with B. H. Neumann's well-known characterization of BFC-groups: Robinson [13, Theorem 4.35]). Nevertheless we have the following positive result.

COROLLARY 3.3. *If G is a CC-group of bounded index then G' is central-by-Černikov.*

PROOF. We know that $G/\zeta_2(G)$ is periodic. By the Three Subgroup Lemma, $[\zeta_2(G), G'] = 1$ and so $G/C_G(G')$ is periodic. Clearly $G/C_G(G')$ is of bounded index and so, by Theorems 3.2 and 2.4, $G'/\zeta(G')$ is Černikov.

4. Groups with abelian-by-Černikov proper subgroups

In [6], the authors have studied groups in which every proper subgroup is abelian-by-Černikov and characterized them in the periodic case. The main result in this direction is [6, Theorem 2], which asserts that a periodic locally graded group G is abelian-by-Černikov if and only if every proper subgroup of G is abelian-by-Černikov. Here G is said to be locally graded if every non-trivial finitely generated subgroup of G contains a proper subgroup of finite index. This condition is necessary to exclude Tarski groups and assures that the groups in consideration are locally finite ([6, 2.1]). We left in [6] as an open question the study of the structure of these groups in the non-periodic case. At this time we do not know whether or not there exists a group G which is not abelian-by-Černikov but in which every proper subgroup is abelian-by-Černikov, that is, a minimal non (abelian-by-Černikov) group. Making use of the results of Section 2 of the present paper, we are able to deduce some properties that such a minimal has to possess. In fact we show the next result, which is an extension of [6, Theorem 2].

THEOREM 4.1. *Let G be a minimal non (abelian-by-Černikov) group having no infinite simple quotients. Then G is an $\mathscr{F}$-perfect (nilpotent of class-two)-by-Černkov group and, in particular, G is soluble. More specifically, G has a normal CC-subgroup B such that G/B is Černikov radicable and $B/\zeta(B)$ is not periodic.*

To prove this theorem we need an auxiliary result.

LEMMA 4.2. *An abelian-by-Černikov group G contains a characteristic nilpotent of class two CC-subgroup N such that G/N is Černikov.*

PROOF. Let A be an abelian normal subgroup of G such that G/A is Černikov. If C is the CC-centre of G (see Robinson [13, Part 1, p. 127]), then C is a characteristic CC-subgroup of G and it is clear that $A \leq C$, so that G/C is Černikov. By Corollary 2.3, $C/\zeta_2(C)$ is Černikov. Then $N = \zeta_2(C)$ satisfies the result.

PROOF OF THEOREM 4.1. Suppose that H is a proper normal subgroup of G and G/H is finite. Then H contains an abelian normal subgroup A such that H/A is Černikov. If L is the core of A in G, then H/L is Černikov and so is G/L, which is a contradiction. Therefore G is $\mathscr{F}$-perfect.

Since G has no infinite simple quotients, it follows that G is not simple. If N is a proper and non-trivial normal subgroup of G, then, by the result of the above paragraph, G/N has to be infinite. Again G/N cannot be simple and, since it is $\mathscr{F}$-perfect, we may have an infinite proper and non-trivial image of G/N. Iterating the argument, we conclude that G is the union of a tower of proper normal subgroups.

Let N be a member of that tower. By Lemma 4.2, N contains a nilpotent of class two normal subgroup K of G such that N/K is Černikov. Let R/K be the radicable part of N/K so that R is normal in G, R/K is abelian and N/R is finite. Then $G/C_G(N/R)$ is finite and, since G is $\mathscr{F}$-perfect,

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we have $[G, N] \leq R$. In particular $N' \leq R$ and then N is soluble and, in particular, $G' \neq G$. Then G/G' is a non-trivial radicable abelian group so that G has a proper normal subgroup V containing G' such that G/V is a quasicyclic group. By Lemma 4.2, V contains a nilpotent of class two normal subgroup B of G, which is a CC-group and such that G/B is Černikov. By Corollary 2.2, $B/\zeta(B)$ cannot be periodic, otherwise $B/\zeta(B)$ and $G/\zeta(B)$ would be Černikov. Hence the proof is now complete.

As in [6], the condition of having no infinite simple quotients is imposed to avoid Tarski groups. In the periodic case, the locally graded condition and results from the classification of the finite simple groups ensure that this condition holds ([6, 2.1]).

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