POINTS AND SPACES

L. E. J. BROUWER

1. The gradual disengagement of mathematics from logic. Beginning with a historical review of the development of mathematical thought, we have to consider successively (cf. **10**, pp. 139–140):

(1) *The observational period.* For some familiar regularities of (outer or inner) experience of time and space, which, to any attainable degree of approximation, *seemed* invariable, absolute and sure invariability was postulated. These regularities were called *axioms* and were put into language. Thereupon extensive systems of properties were developed from the linguistic substratum of the axioms by means of *reasoning*, guided by experience but linguistically following and using the principles of classical logic. This logic was considered autonomous, and mathematics was considered more or less dependent on logic.

(2) *The revolution in science of space.* In the course of the 19th and the beginning of the 20th century, on the one hand geometry was gradually metamorphosed into a chapter of the science of numbers, and on the other hand Euclidean three-dimensional geometry lost its privileged character since a great number of other geometries originating from logical speculations, with properties distinct from the traditional but no less beautiful, found an arithmetical representation likewise.

(3) *The old formalist school.* Encouraged by the important part which had been played in the above metamorphosis of geometry by the *logico-linguistic method*, the old formalist school merged logic and mathematics into a single linguistic science, operating on meaningless words or symbols by means of logical rules, thus divesting logic and mathematics of their difference in character as well as of their autonomy.

(4) *The pre-intuitionist school*, by which autonomy and apriority were re-established for logic and established for the major part of "separable" mathematics. For the continuum however, this school on some occasions seems to have contented itself with an ever-unfinished and ever-denumerable set of real numbers which can never have a measure positively different from zero; on other occasions it seems to have stuffed the continuum with elements providing measure by means of some logical axiom. In both cases, in its further development of mathematics, it has unreservedly applied classical logic. So, logic and an introductory part of mathematics were autonomous here. The rest of mathematics was dependent on them.

(5) *The new formalist school*, by which autonomy and apriority were postulated for *mathematics of the second order*, i.e., for scientific consideration of the symbols

Received September 1, 1953. Lectures presented at the Seminar of the Canadian Mathematical Congress at Kingston, Ont., Aug. 10–31, 1953.

occurring in purified mathematical language, and of the rules of manipulating these symbols. This scientific consideration of language, later on called *metamathematics*, although using complete induction, apriorizes much less than pre-intuitionism. What it seems to have overlooked is that between perfection of mathematical language and perfection of mathematics proper, no clear connection can be seen.

(6) *The intervention of intuitionism* by two acts of which the first seems necessarily to lead to destructive and sterilizing consequences, whereas the second yields ample possibilities for recovery and new developments.

The first act of intuitionism completely separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic. It recognizes that mathematics is a languageless activity of the mind having its origin in the basic phenomenon of the perception of a *move of time*, which is the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the common substratum of all two-ities, the mental creation of the *empty two-ity*. This empty two-ity and the two unities of which it is composed, constitute the *basic mathematical systems*. And the basic operation of mathematical construction is the *mental creation of the two-ity of two mathematical systems previously acquired*, and the consideration of this two-ity as a new mathematical system.

It is introspectively realized how this basic operation, continually displaying *unaltered* retention by memory, successively generates each natural number, the infinitely proceeding sequence of the natural numbers, arbitrary finite sequences and infinitely proceeding sequences of mathematical systems previously acquired, finally a continually extending stock of mathematical systems corresponding to "separable" systems of classical mathematics.

The second act of intuitionism recognizes the possibility of generating new mathematical entities:

First, in the form of infinitely proceeding sequences whose terms *are chosen more or less freely from mathematical entities previously acquired*; in such a way that the freedom existing perhaps at the first choice may be irrevocably subjected, again and again, to progressive restrictions at subsequent choices, while all these restricting interventions, as well as the choices themselves, may, at any stage, be made to depend on possible future mathematical experiences of the creating subject;

Secondly, in the form of mathematical *species*, i.e., *properties supposable for mathematical entities previously acquired*, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be *equal* to it, equality having to be symmetric, reflexive, and transitive, and the empty two-ity being forbidden to be equalized to an empty unity. Mathematical entities for which the property in question holds, are called *elements* of the corresponding species.

In the edifice of mathematical thought based on the first and second act

https://doi.org/10.4153/CJM-1954-001-9 Published online by Cambridge University Press

of intuitionism, language plays no other part than that of an efficient, but never infallible or exact, technique for memorizing mathematical constructions, and for suggesting them to others; so that the wording of a mathematical theorem has no sense unless it indicates the construction either of an actual mathematical entity or of an incompatibility (e.g., the identity of the empty two-ity with an empty unity) out of some constructional condition imposed on a hypothetical mathematical system. So that mathematical language, in particular logic, can never by itself create new mathematical entities, nor deduce a mathematical state of things.

However, notwithstanding this rejection of classical logic as an instrument to discover mathematical truths, intuitionist mathematics has its general introspective theory of mathematical assertions, a theory which with some right may be called *intuitionist mathematical logic*, and to which belongs a theory of the *principle of the excluded third.*

In intuitionism this principle is also called the *principle of judgeability.* It is either (in its *simple* form) an assertion A' about a single primary assertion A or (in its *extended* form) a species (A'_σ) of assertions about the elements of a species (A_σ) of primary assertions saying that each A_σ can be *judged*, i.e., can either be proved to be true or be proved to be contradictory.

This principle of judgeability entails the following two corollaries which are weaker:

(i) *The principle of testability*, being (in its extended form) a species (A''_σ) of assertions about the elements of the species (A_σ) saying that each A_σ can be *tested*, i.e., can either be proved to be non-contradictory or be proved to be contradictory.

(ii) *The principle of reciprocity of complementarity*, being (in its extended form) a species (A'''_σ) of assertions about the elements of the species (A_σ), saying that each A_σ, if proved to be non-contradictory, can also be proved to be true.

In intuitionism, of course, all three of these principles, being assertions about assertions, are only then "realized," i.e., only then convey truths, when these truths have been experienced. On this basis it can be proved that the extended principles are not only not true, but even contradictory. On the other hand, in their simple form, all three of the principles are, although not true, at least non-contradictory.

The assertion of an incompatibility is called a *negative* assertion. In the field of negative assertions, the principle of reciprocity of complementarity is realized, and the principles of judgeability and testability are equivalent (**9**, pp. 1245–1246).

2. The refutation of the principle of the excluded third. The first act of intuitionism enables us to construct the linear rational grid. On the basis of this, by virtue of the second act of intuitionism, we introduce the linear continuum in the following way: By a *limiting number* we understand a (not necessarily predeterminate) convergent sequence of rational numbers. Then, regarding as self-explanatory the meaning of a *coincidence* of two limiting numbers, we call the species of limiting numbers coinciding with a given limiting number, a *limiting number core*. A predeterminate limiting number is also called a *sharp limiting number*, and a limiting number core containing a sharp limiting number is called a *sharp limiting number core*. The species of the limiting number cores is called the *linear continuum* or the *continuum*.

In order to furnish examples refuting the principle of the excluded third and its corollaries, we introduce the notion of a *drift* (cf. **9**, pp. 1246–1247). By a drift we understand the union γ of a convergent fundamental sequence of limiting number cores $c_1(\gamma)$, $c_2(\gamma)$, . . . called the *counting cores* of the drift, and the accumulation number core $c(\gamma)$ of this sequence, called the *kernel* of the drift, all counting cores lying apart from each other and from the kernel. (We say that a lies *apart* from b if there is some natural number n such that $|b - a| > 2^{-n}$.)

Let α be a mathematical assertion so far neither tested nor recognized as testable. Then, in connection with the assertion α and with a drift γ the creating subject can generate an infinitely proceeding sequence $R(\gamma, \alpha)$ of limiting number cores $c_1(\gamma, \alpha)$, $c_2(\gamma, \alpha)$, . . . according to the following direction: As long as during the choice of the $c_n(\gamma, \alpha)$ the creating subject has not experienced the truth of α [has neither experienced the truth nor the absurdity of α], each $c_n(\gamma, \alpha)$ is chosen equal to $c(\gamma)$. But as soon as between the choice of $c_{r-1}(\gamma, \alpha)$ and that of $c_r(\gamma, \alpha)$ the creating subject has experienced the truth of α [has either experienced the truth or the absurdity of α], $c_r(\gamma, \alpha)$, and likewise $c_{r+\nu}(\gamma, \alpha)$ for each natural number ν, is chosen equal to $c_r(\gamma)$. This sequence $R(\gamma, \alpha)$ converges to a limiting number core $C(\gamma, \alpha)$ [$D(\gamma, \alpha)$] which will be called a *conditional checking-core of γ through α* [*direct checking-core of γ through α*].

Let γ be a drift whose counting cores are rational and whose kernel is irrational. Then the assertion of the rationality of a $D(\gamma, \alpha)$ is not judgeable, but it *is* testable, because the assertion of irrationality of $D(\gamma, \alpha)$ would entail the simultaneous contradictority of the truth and the absurdity of α, which is an absurdity.

On the other hand, truth of α and rationality of $C(\gamma, \alpha)$ are equivalent. So the assertion of the rationality of $C(\gamma, \alpha)$ is neither judgeable nor testable. For, non-contradictority of rationality of $C(\gamma, \alpha)$ would entail non-contradictority of α, i.e. testability of α, which was presupposed not to exist. Furthermore, if some day α would prove to be non-contradictory without being true, rationality of $C(\gamma, \alpha)$ likewise would be non-contradictory without being true. So for rationality of $C(\gamma, \alpha)$, just as for α, non-contradictority would not be equivalent to truth.

Obviously the field of validity of the principle of the excluded third is identical with the intersection of the field of validity of the principle of testability and that of the principle of reciprocity of complementarity. Furthermore, the first field of validity is a *proper* subfield of each of the others, as is shown by the following examples:

Let A be the species of the direct checking-cores of drifts with rational counting cores, B the species of the irrational limiting number cores, C the union of A and B. Then all assertions of rationality of an element of C satisfy the principle of testability, while, as we have seen, there are assertions of rationality of an element of C not satisfying the principle of the excluded third.

Again, all assertions of equality of two limiting number cores satisfy the principle of reciprocity of complementarity, whereas there are assertions of equality of two limiting number cores not satisfying the principle of the excluded third.

In the domain of mathematical assertions the property of absurdity, like the property of truth, is a *universally additive property*, that is to say, if it holds for each element α of a species of assertions, it also holds for the assertion which is the union of the assertions α. *This property of universal additivity does not obtain for the property of non-contradictority.* However, non-contradictority *does* possess the weaker property of *finite additivity*, that is to say, if the assertions ρ and σ are non-contradictory, the assertion τ, which is the union of ρ and σ, is also non-contradictory.

Applying the latter theorem to the special non-contradictory assertions that are the enunciations of the principle of the excluded third for a single assertion, we see that a simultaneous enunciation of this principle for a finite number of assertions is likewise non-contradictory.

As to the long belief in the universal validity of the principle of the excluded third in mathematics, intuitionism considers it as a phenomenon of the history of civilization of the same kind as the old-time belief in the rationality of π or in the rotation of the firmament on an axis passing through the earth. And intuitionism tries to explain the long persistence of this dogma by two facts: first, the obvious non-contradictority of the principle for an arbitrary single assertion; secondly, the practical validity of the whole of classical logic for an extensive group of simple every-day phenomena. The latter fact apparently made such a strong impression that the *play* of thought which classical logic originally was, became a deep-rooted *habit* of thought which was considered not only as useful but even as aprioristic.

The above rejection of the universal truth of the principle of the excluded third in mathematics will make it plausible that intuitionist arguing requires a preliminary formulation of several definitions which sometimes split atomic notions of classical mathematics.

Two mathematical entities will be called *different* if their equality proves to be absurd. The notation for equality and difference will be $=$ and $\neq$ respectively.

Two infinite sequences of mathematical entities $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ will be said to be *equal*, or *identical*, if $a_\nu = b_\nu$ for each ν, and *distinct*, if a natural number can be indicated (or calculated) such that a_n and b_n are different.

A species is called *discrete* if any two of its elements can be proved either to be equal or to be different.

https://doi.org/10.4153/CJM-1954-001-9 Published online by Cambridge University Press

If the species M possesses an element which cannot possibly belong to the species N, we shall say that M *deviates* from N.

The species M will be called a *subspecies* of the species N, and we shall write $M \subset N$ if every element of M can be proved to belong to N. If, in addition, N deviates from M, then M is called a *proper* subspecies of N. If each element of N either belongs to M or cannot possibly belong to M, then M is called a *removable* subspecies of N.

Two species are said to be *equal*, or *identical*, if for each element of either of them an element of the other, equal to it, can be indicated. They are called *different* if their equality is absurd, and *congruent* if neither can deviate from the other.

Let M be the linear continuum, A and B the species of the rational and the irrational limiting number cores respectively, then M and the union of A and B are *congruent and different at the same time*!

A species which cannot possess an element is said to be *empty*. Two different species whose intersection is empty are called *disjoint*.

If M and N are disjoint subspecies of the species P, and the union of M and N is congruent to P, we shall say that P is *composed* of M and N, and that M and N are *conjugate subspecies* of P. Thus, e.g., the species of exponents of Fermat's equation which render it solvable and unsolvable respectively, are conjugate subspecies of the species of the natural numbers.

For a given P, for any subspecies M, a subspecies N can be indicated such that M and N are conjugate subspecies of P. This N, in general, is not even uniquely determined by P and M. Thus, e.g., if P is the linear continuum, and M the species of the irrational limiting number cores, then for N we may choose the species of those limiting number cores whose rationality is non-contradictory as well as the species of the rational limiting number cores.

If H and K are disjoint subspecies of the species P, and the union of H and K is identical with P, so that H and K are conjugate removable subspecies of P, we shall say that P *splits* into H and K. Thus, e.g., the species of the prime numbers and of the composite numbers are *conjugate removable subspecies* of the species of the natural numbers.

For an arbitrary proper subspecies H of P one cannot, in general, indicate a K such that H and K are conjugate removable subspecies of P. There are even species (e.g., the linear continuum) which possess no removable proper subspecies at all.

If V and W are conjugate subspecies of P, and if in addition V consists of those elements of P which cannot belong to W, and W of those elements of P which cannot belong to V, we shall say that P is *directly composed* of V and W, and that V and W are *directly conjugate subspecies* of P. Thus, e.g., the species consisting of those elements of P for which a certain negative property is true and absurd respectively, are directly conjugate subspecies of P.

If between two species M and N a (not necessarily predeterminate) 1-1 correspondence can be created, i.e., if M can be mapped onto N in such a way that equal and only equal elements of M have equal images in N, while each element of N is the image of some element of M, we shall say that M and N are *equipotential.*

A species which is equipotential to some natural number [to the infinite sequence of natural numbers] will be called *finite* [*denumerably infinite*].

A species which contains a denumerably infinite subspecies will be called *infinite.*

3. Spreads and fans. *Spreads* and *fans* are fundamental notions in intuitionism. Their introduction requires some further definitions.

By a *node of order n* we understand a sequence of n natural numbers ($n \geqslant 1$) called the *constituents* of the node.

A node p' of order $n + m$ ($m \geqslant 1$) will be called an mth *descendant* of the node p of order n, and p will be called the mth *ascendant* of p', if the sequence of constituents of p is an initial segment of the sequence of constituents of p'.

If $m = 1$, p' will also be called an *immediate descendant* of p and p the *immediate ascendant* of p′.

The species Q_p of the immediate descendants of the node p of order n considered in their natural order (i.e., ordered according to their last constituent) will be called a *row of nodes of order* $n + 1$ and the *ramifying row* of p, while p will be called the *dominant* of Q_p.

The species of the nodes of order 1 considered in their natural order will be called the *row of nodes of order* 1.

A finite sequence of nodes consisting of a node p_1 of order 1, an immediate descendant p_2 of p_1, an immediate descendant p_3 of p_2, . . ., up to an immediate descendant p_n of p_{n-1}, will be called a *rod of order n.*

An infinite (not necessarily predeterminate) sequence of nodes consisting of a node p_1 or order 1, an immediate descendant p_2 or p_1, an immediate descendant p_3 of p_2, and so on *ad infinitum*, will be called an *arrow.*

Naturally an arrow may grow in complete freedom, i.e., in the passage from p_ν to $p_{\nu+1}$, the choice of a new constituent for $p_{\nu+1}$ to be joined to those of p_ν may be completely free for each ν, for as long as the creating subject may desire. On the other hand this freedom in the generation of the arrow may at any stage be completely abolished, at the beginning or at any p_ν, by means of a law fixing all further nodes in advance. From this moment the arrow concerned will be called a *sharp arrow.* Furthermore, the freedom in the generation of the arrow, without being completely abolished, may, at any p_ν, undergo some restriction, and this restriction may be intensified at further p_ν's. Finally, all these interventions, by virtue of the second act of intuitionism, may, at any stage, be made to depend on possible future mathematical experiences of the creating subject.

Let ρ be a *natural denumeration* of the species of the nodes, i.e., a denumeration $a_1, a_2, \ldots$ of the nodes such that each node comes before its descendants, and before the nodes which it precedes in its row of nodes. Then, without knowledge of further details of this denumeration, as soon as in ρ for each a_ν a sequence

$$a_{\nu_1}, a_{\nu_2}, \ldots,$$

with ever increasing indices, can be indicated as its ramifying row, the sequence of constituents of any given a_ν can be reconstructed.

An example of a natural denumeration of the species of the nodes can be given as follows: Let G_n be the species of the nodes of order $\leqslant n$ and constituents $\leqslant n$, $G_{n,\nu}$ the species of the nodes of G_n of order ν, and A_n $(n \geqslant 2)$ the species of the nodes of G_n not belonging to G_{n-1}. Each $G_{n,\nu}$ is counted in such a way that p precedes q if the first constituent in which they differ is smaller for p than for q. If we then make each $G_{n,\nu}$ precede $G_{n,\nu+1}$ we get a natural denumeration Δ_n of G_n. Finally, by successively counting G_1 after Δ_1, A_2 after Δ_2, A_3 after Δ_3, and so on, we arrive at a natural denumeration of the species of the nodes.

We proceed to consider a (not necessarily predeterminate) species of nodes K containing:

(i) of the nodes of order 1, either all natural numbers or those and only those natural numbers which do not exceed a definite natural number m_0;

(ii) for each $n > 1$, of the nodes of order $n + 1$ which are immediate descendants of the node p of order n belonging to K, either all of them or those and only those whose $(n + 1)$st constituent joined to those of p does not exceed a definite natural number m_p.

Such a species of nodes K will be called a *spread direction*, and the species $w(K)$ of the arrows which consist of nodes of K will be called a *spread*.

The spread direction for which from the above alternatives always the first is chosen is called the *universal spread direction*, and the corresponding spread is called the *universal spread*.

A spread direction for which from the above alternatives always the second is chosen is called a *fan direction*, and the corresponding spread is called a *fan*.

As each spread direction is a subspecies of the universal spread direction USD (just as each spread is a subspecies of the universal spread US), any natural denumeration (in the above sense) of USD generates a natural denumeration of each spread direction. Furthermore, if $a_1, a_2, \ldots$ is a natural denumeration of a spread direction K, and for each a_ν a finite or denumerably infinite sequence

$$a_{\nu_1}, a_{\nu_2}, \ldots,$$

with ever increasing indices, can be indicated as its ramifying row, then for any given a_ν the sequence of its constituents in K can be reconstructed.

A node b of a spread direction K, together with its descendants in K, constitutes a removable subspecies $\pi_b(K)$ of K which will be called a *sector direction*, and the species $P_b(K)$ of the arrows composed of nodes of $\pi_b(K)$ will be called a *sector*. Both $\pi_b(K)$ and $P_b(K)$ will be said to be *dominated* by their "*top*" b. We shall speak of a *free sector* [*sector direction*], if b is of order 1, and of a *horned sector* [*sector direction*] of order n, if b is of order $n + 1$ $(n \geqslant 1)$. In the latter case the constituents of the immediate ascendant of b will be said to form the *horn* of the sector [sector direction].

https://doi.org/10.4153/CJM-1954-001-9 Published online by Cambridge University Press

A subspecies of the spread direction K will be called *thin* if none of its nodes is a descendant of any other of its nodes.

If a (not necessarily predeterminate) subspecies of the spread direction K has the property that no arrow of K can avoid it, it will be called a *crude block* of K. A crude block of K which is thin and removable will be called a *proper block* or simply a *block* of K.

The nodes of K which are not descendants of the block $B(K)$ of K constitute a removable subspecies $\tau_B(K)$ of K which will be called a *free stump*, and which we shall say is *carried* by the block $B(K)$.

A node b belonging to $\tau_B(K)$, together with its descendants in $\tau_B(K)$, constitutes a removable subspecies ${}_b\sigma_B(K)$ of $\tau_B(K)$ which will be called a *pyramid*, and which we shall say is *dominated* by its "*top*" b. We shall speak of a *free pyramid* if b is of order 1, and of a *horned pyramid* of order n if b is of order $n + 1$ $(n \geqslant 1)$. In the latter case the sequence of constituents of the immediate ascendant of b will be said to constitute the *horn* of ${}_b\sigma_B(K)$.

If from the free pyramid ${}_b\sigma_B(K)$ [from the horned pyramid ${}_b\sigma_B(K)$ of order n] we take away the top b, the remainder ${}_b\rho_B(K)$ (also in the case of its reducing to "nothing," if b belongs to B), will be called a *horned stump* of order 1 [of order $n + 1$]. The constituents of the removed top b will be said to constitute the *horn* of ${}_b\rho_B(K)$.

If from all nodes of a horned stump ${}_b\rho_B(K)$ the horn is taken away, the remainder will be a free stump ${}_b\tau_B(K)$. This holds also in the case of b belonging to B, if "nothing" is added to the species of the free stumps. If ${}_b\rho_B(K)$ was of order n, we shall call ${}_b\tau_B(K)$ a *free substump of $\tau_B(K)$ of rank n, dominated by b.*

To explain the notion of *absorption of a row of free substumps of rank n by a free substump of rank $n - 1$*, let $b_1, b_2, \ldots$ be a row of nodes of order n, dominated by the node a, and for each ν let β_ν be the last constituent of b_ν. For each ν, to each node of ${}_{b_\nu}\tau_B(K)$ we add β_ν as a first constituent, and to the horned stump of order 1 thus acquired we add the node β_ν, thus arriving at a "row" of free pyramids $\sigma_1, \sigma_2, \ldots$ whose union is ${}_a\tau_B(K)$, *a free substump of $\tau_B(K)$ of rank $n - 1$*. This process of absorption can also be effected if some or all of the ${}_{b_\nu}\tau_B(K)$ reduce to nothing.

In an analogous way, by absorption of a finite sequence or a fundamental sequence of free stumps of spread directions K_ν a free stump of a new spread direction K comes into being.

4. Well-ordered blocks and stumps. At this point, before continuing the study of spreads and fans, we have to insert some considerations about *well-ordered species*.

A discrete species D is said to be *completely ordered* if for any two *different* elements of D, say a and b, one of the two mutually exclusive relations $a < b$ (equivalent to $b > a$) and $a > b$ (equivalent to $b < a$) is realized, in such a way that $a < b$, $a = r$ and $b = s$ implies $r < s$, and $a < b$ and $b < c$ implies $a < c$.

https://doi.org/10.4153/CJM-1954-001-9 Published online by Cambridge University Press

Let R be a fundamental sequence [an ordered finite species] of disjoint completely ordered species N_ν. We construct a complete order of the union M of the N_ν in the following way: Let e' belong to N' and e'' to N''. Then we put $e' < e''$ in M if either $N' < N''$ in R or $N' = N'' = \bar{N}$ and $e' < e''$ in $\bar{N}$. Denoting the species M ordered in this way by $\bar{M}$, we write

$$\bar{M} = N_1 + N_2 + \ldots \; [\bar{M} = N_1 + N_2 + \ldots + N_m] \text{ or } \bar{M} = \sum_\nu N_\nu,$$

and we shall say that $\bar{M}$ is the *ordinal sum* of the N_ν. The generation of an ordinal sum will be called *ordinal addition*.

On the basis of this definition of ordinal addition we can generate a continually extending stock of well-ordered species according to the following rules:

(1) Each species containing one and only one element is a well-ordered species, and, as such, will be called a *basic species*.

(2) If, out of the available stock of well-ordered species previously acquired, a fundamental sequence of disjoint well-ordered species has been indicated, their addition will be called a *first generating operation*, and their ordinal sum will again be called a well-ordered species and, as such, will be added to the stock.

(3) If, out of the available stock of well-ordered species previously acquired, a non-vanishing ordered finite sequence of disjoint well-ordered species has been indicated, their addition will be called a *second generating operation*, and their ordinal sum will again be called a well-ordered species and, as such, will be added to the stock.

In the case that only the second, not the first, generating operation is effected, we speak of *bounded* well-ordered species.

Let F be a well-ordered species. All well-ordered species which, at some stage, have played a part during the construction of F will be called *constructional subspecies* of F. The constructional subspecies of F which have played a part in the final generating operation of F, will be denoted by F_ν (ν passing through the sequence of natural numbers or through an initial segment of it) and will be said to constitute the *row of constructional subspecies of order* 1 of F. The constructional subspecies of order 1 of F_{ν_1} will be denoted by $F_{\nu_1\nu}$ (ν varying as above) and will be said to constitute a *row of constructional subspecies of order* 2 of F. In general, the row of constructional subspecies of order 1 of $F_{\nu_1 \ldots \nu_k}$ will be denoted by $F_{\nu_1 \ldots \nu_k \nu}$ (ν varying as above) and will be said to constitute a *row of constructional subspecies of order* $k + 1$ of F. F itself will be considered as its own *constructional subspecies of order zero*.

In this way each basic species, that is, each element, of F, and each constructional subspecies of F, turns out to be a constructional subspecies of finite order (which order, however, for appropriately chosen constructional subspecies may increase indefinitely. This property is easily proved by the *inductive method*, i.e., by remarking that it holds if F is a basic species, and that when a generating operation is performed, it holds for the generated ordinal sum if it holds for the terms of the sum. By the same method we state that the species of sequences of indices of the constructional subspecies of a well-ordered species is a removable subspecies of USD, that every well-ordered species in whose construction the first generating operation has been effected at least once is denumerably infinite, and that every bounded well-ordered species is finite.

It is also by the inductive method that we shall prove the following theorem: *For each well-ordered species F there is a* 1-1 *correspondence between the species of its constructional subspecies of non-vanishing order and a free stump* τ *such that each sequence of indices of a constructional subspecies of F corresponds to an equal sequence of constituents of a node of* τ, *while a basic species of F corresponds to a node of the block carrying* τ, *and the union of an* F_ν *and its constructional subspecies corresponds to a free pyramid of* τ.

For, let

$$F_{\nu_1 \ldots \nu_{n-1} 1},\ F_{\nu_1 \ldots \nu_{n-1} 2}, \ldots$$

be a row of constructional subspecies of order n of F, and for each ν, let

$$F_{\nu_1 \ldots \nu_{n-1} \nu}$$

be provided with a 1-1 equality-mapping of the sequences of the indices following ν of its constructional subspecies onto a free stump $\tau_{B_\nu}(K_\nu)$ (containing as constituents of its nodes only indices of order $> n$ from F). Then the row

$$\tau_{B_1}(K_1),\ \tau_{B_2}(K_2), \ldots$$

can be considered as a row of free substumps of rank 1 of a free stump $\tau_B(K)$, by which it can be absorbed. Accomplishing this absorption, and assigning to each sequence of indices following ν_{n-1} of a constructional subspecies of $F_{\nu_1 \ldots \nu_{n-1}}$ an equal sequence of constituents of a node of $\tau_B(K)$, we arrive at a 1-1 equality-mapping of the sequences of the indices following ν_{n-1} of the constructional subspecies of $F_{\nu_1 \ldots \nu_{n-1}}$ onto $\tau_B(K)$. And if $F_{\nu_1 \ldots \nu_{n-1}}$ is a basic species of F the mapping as required by the theorem exists as a mapping of nothing onto "nothing."

Blocks and free stumps which can play the part of a $B(K)$ and a $\tau_B(K)$ as required by the above theorem will be called *well-ordered blocks* and *well-ordered free stumps* respectively. The free pyramids which are contained in a well-ordered free stump will be called *well-ordered free pyramids.* Horned pyramids which after removal of their horns become well-ordered free pyramids will be called *well-ordered horned pyramids.*

Obviously *each free stump corresponding to a bounded well-ordered species is finite.*

The above assignment of a sequence of indices to each constructional subspecies of non-vanishing order of a well-ordered species F was performed in a downward direction, but the same result can be obtained as well by an *upward* construction consisting in a gradual dressing-up of F parallel to its generation, according to the following prescriptions:

(i) At each ordinal addition of an ordered sequence d of basic species, to each of these species is assigned, as its only index, the natural number indicating its place in d.

https://doi.org/10.4153/CJM-1954-001-9 Published online by Cambridge University Press

(ii) At each ordinal addition of an ordered sequence d of well-ordered species previously acquired, for each of these species (and for each of their constructional subspecies) the natural number indicating its place in d is added to its adhering sequence of indices previously acquired, as a first index.

If, in an analogous way, for a given spread direction K in which a thin subspecies $C(K)$ has been indicated, we succeed in arriving at the free stump $\tau_B(K)$ by allowing in K the following sorts of acts:

(i) the qualification of a node of $C(K)$ as dominating a free substump "nothing,"

(ii) the formation of a free substump of rank $n-1$ by *absorbing a row of free substumps* of rank n,

then this gradual erection of the edifice of nodes of $\tau_B(K)$ (proving by the way that $C(K) = B$) is identical with the above upward construction of the edifice of sequences of indices of the constructional subspecies of a proper well-ordered species F, so that $\tau_B(K)$ is a *well-ordered* free stump and we may speak of a *well-ordered erection* of $\tau_B(K)$.

By extending a given free stump to its spread direction we see that a natural denumeration of the latter yields a natural denumeration of the former. So also the species of the sequences of indices of the constructional subspecies of a well-ordered species can be denumerated in a natural way.

We shall show by an example that not every block is a well-ordered block, and hence that not every free stump admits of a well-ordered erection.

Let K be a spread direction, and let β_ν be the species of the nodes of K of order ν. Let a v-union be a union of species β_ν with regard to which an infinite sequence of decisions $q_1, q_2, \ldots$ successively decides whether β_1 belongs to the union, whether β_2 belongs to the union, and so on, and let V be the species of the v-unions. Let α be a mathematical assertion so far neither tested nor recognized as testable, and let v_α be the element of V generated as follows: As long as in the course of the successive choices of the decisions q_ν the creating subject has neither experienced the truth nor the absurdity of α, each q_ν will be chosen to be negative; but as soon as between the choice of the decision q_{r-1} and that of the decision q_r the creating subject has experienced either the truth or the absurdity of α, q_r will be chosen to be affirmative and for each natural number ν, $q_{r+\nu}$ will again be chosen to be negative.

Obviously this v_α is a block of K of which we cannot say that it is a well-ordered block.

5. The fan theorem. If a (not necessarily predeterminate) subspecies $C(K)$ of the spread direction K has the property that every arrow of K *meets* $C(K)$, i.e., has a node in common with $C(K)$, this subspecies $C(K)$ will be called a *crude bar* of K. A crude bar of K which is thin will be called a *proper bar* or simply a *bar* of K.

The definition of a crude bar means that for every arrow α of K the order

$n(\alpha)$ of the postulated node of intersection with $C(K)$ must be computable, however complicated this calculation may be. For instance, the algorithm in question may indicate the calculation of a maximal order n_1 at which will appear a finite method of calculation of a further maximal order n_2 at which will appear a finite method of calculation of a further maximal order n_3 at which will appear a finite method of calculation of a further maximal order n_4 at which the postulated node of intersection must have been passed. And much higher degrees of complication are thinkable.

If $C(K)$ is a crude bar of K, then every node t of K has either been recognized as belonging to $C(K)$ or been provided with a constructive mathematical argument h_t proving that t is *barred* by $C(K)$, i.e., that every arrow passing through t has a node of intersection with $C(K)$.

For this mathematical argument h_t no other basis is available than the characterization of $C(K)$, and the species of constructional relations existing between the nodes of K. Now all these relations can be derived from the basic relations which for each node indicate its immediate predecessor in its row of nodes (or the non-existence of an immediate predecessor in its row of nodes), its immediate successor in its row of nodes (or the non-existence of an immediate successor in its row of nodes), its immediate ascendant (or the non-existence of an immediate ascendant), and the row of its immediate descendants. (Whether this system of basic relations is susceptible of further reductions, we shall leave undecided.) Consequently, if we split up the argument h_t into an argument k_t consisting exclusively of statements of atomic basic facts d and atomic immediately obvious inferences e, then, supposing $t = \nu_1 \ldots \nu_r$, the final inference of k_t must deduce the barred condition of t either from t being recognized as belonging to $C(K)$ or from the barred condition of $\nu_1 \ldots \nu_{r-1}$ (a so-called *ζ-inference*) or from the barred condition of $\nu_1 \ldots \nu_r\lambda$ for each λ (a so-called *ϝ-inference*). If, in particular, t is a node ν_1 of order 1, the final inference of k_t recognizing that t is barred, must either be the recognition of t as a node of $C(K)$ or the *ϝ-inference* deducing the barred condition of ν_1 from the barred condition of $\nu_1\lambda$ for each λ. So in the latter case the recognition of the barred condition of ν_1 has been *preceded* in k_t by the recognition of the barred condition of $\nu_1\lambda$ for each λ. From this follows that in k_t the recognition of the barred condition of $k_{\nu_1\nu_2}$ preceding that of k_{ν_1} must in its turn either be based on its belonging to $C(K)$ or have been preceded by the recognition of the barred condition of $k_{\nu_1\nu_2}\lambda$ for each λ, from which it has been deduced by a ϝ-inference; and so on.

Consequently, if t is a node of order 1, then in k_t appear

(1) a certain species of nodes N_t, including t and a certain thin subspecies $C_t(K)$ of $C(K)$,

(2) the species S_t of the statements of the barred condition of an element of N_t,

(3) a species I_t of ϝ-inferences connecting elements of S_t

such that each element of S_t is connected with the statement of the barred

https://doi.org/10.4153/CJM-1954-001-9 Published online by Cambridge University Press

condition of t by a finite sequence of elements of I_t, that each element of S_t, with the exception of the statements of the barred condition of an element of $C_t(K)$, has a row of predecessors in the argument with which it is connected by an element of I_t, and that each element of S_t, with the exception of the statement of the barred condition of t, has a successor in the argument with which it is connected by an element of I_t.

If we now take for t successively each node of order 1 of K, and consider the union k of the corresponding arguments k_t, then in k appear

(1) a certain species of nodes N, including all nodes of order 1 of K and a certain thin subspecies $C_0(K)$ of $C(K)$,

(2) the species S of the statements of the barred condition of an element of N,

(3) a species I of ρ-inferences connecting elements of S

such that each element of S is connected with the statement of the barred condition of a node of order 1 of K by a finite sequence of elements of I, that each element of S, with the exception of the statements of the barred condition of an element of $C_0(K)$, has a row of predecessors in the argument with which it is connected by an element of I, and that each element of S, with the exception of the statements of the barred condition of a node of order 1 of K, has a successor in the argument with which it is connected by an element of I.

In this way from the argument k we have extracted an argument k' which by performing acts of the two following sorts in K:

(i) taking an element of $C_0(K)$ as a basic pyramid consisting of barred nodes,

(ii) taking the union of a row of pyramids consisting of barred nodes previously acquired and the dominant of their row of tops, thus obtaining a new pyramid consisting of barred nodes,

has arrived at a row of free pyramids consisting of barred nodes whose row of tops is the row of nodes of order 1 of K.

This argument k' comes to the same as the argument k'' which by performing acts of the two following sorts in K:

(i) assigning to an element α of $C_0(K)$ a free substump "*nothing*" dominated by α,

(ii) having a row of free substumps consisting of barred nodes previously acquired, absorbed by a new free substump,

has arrived at a free stump of K consisting of barred nodes.

So, as was shown in §4, this argument k'' in its turn comes to the same as the *well-ordered erection of the species of nodes N as a well-ordered free stump of K, carried by the well-ordered block $C_0(K)$.*

With which we have deduced the

BAR THEOREM. *Every crude bar contains a well-ordered block.*[1]

[1]Cf. (**6**, pp. 63–65). The species μ_1 used there plays the role of the above species $C(K)$. The equivalence of the principles of the excluded third and of reciprocity of complementarity, mentioned there in a footnote by way of remark, subsequently has been recognized as non-existent. In fact, as was also shown in the present paper, the fields of validity of these two principles have turned out to be essentially different.

This theorem does *not* imply that every well-ordered block is a bar.

In the case that K is a fan direction, its well-ordered free stumps, on account of their correspondence to bounded well-ordered species, are all finite; so the above species of nodes N is finite, and there will be a finite maximum $O(N)$ for the order of its nodes. Furthermore *in this case the well-ordered block* $C_0(K)$ *is a bar.*

Now we easily prove the

FAN THEOREM. *Let K be a fan direction, and let us suppose that to each arrow α of K has been assigned a natural number $\mu(\alpha)$. Then a natural number s can be indicated such that, for any α, $\mu(\alpha)$ is determined at the sth node of α* (**6**, p. 66; **10**, p. 143).

For, since the natural number in question has to be known for each arrow of K at one of its nodes, the nodes yielding this knowledge constitute a species of nodes which each arrow of K is bound to meet, and which therefore is a crude bar $C(K)$ of K. Because this $C(K)$ contains a well-ordered block $C_0(K)$, and this well-ordered block $C_0(K)$ in the present case is finite and a bar of K, a maximum s can be indicated for the order of its nodes, so that each arrow α of K meets $C_0(K)$ not later than at its sth node. Hence, for each α, at its sth node, $\mu(\alpha)$ is determined.

6. The continuity theorem. The infinite sequence of natural numbers passes into a *located infinite sequence* $c_1, c_2, \ldots$ if for any two of its elements c_r and c_s a symmetric limiting number core function $\rho(c_r, c_s)$, called the *distance* of c_r and c_s, is indicated, which has the following properties:

(1) For $c_r = c_s$, $\rho(c_r, c_s) = 0$.

(2) For $c_r \neq c_s$, a natural number $f(c_r, c_s)$ can be indicated such that

$$\rho(c_r, c_s) > 2^{-f(c_r, c_s)}.$$

(3) $\rho(c_r, c_s) \leqslant \rho(c_r, c_t) + \rho(c_s, c_t)$.

(4) For each n a natural number $\mu(n)$ can be indicated such that, if we denote the union of $c_1, c_2, \ldots$ and $c_{\mu(n)}$ by ψ_n, then $\rho(c_\nu, \psi_n) \leqslant 4^{-n}$ for each ν.

We shall express the property (4) by saying that *the sequence $c_1, c_2, \ldots$ is approximated with any degree of accuracy by its successive initial segments.*

Let L be a located infinite sequence. An infinite sequence $a_1, a_2, \ldots$ of elements of L (among which equalities may occur) will be said to be *convergent* if for each n a natural number $\gamma(n)$ can be indicated such that

$$\rho(a_{\gamma(n)}, a_\nu) < 2^{-n}$$

for any $\nu > \gamma(n)$. A convergent infinite sequence of elements of L will also be called a *limiting element* of L. Regarding as self-explanatory the meaning of *coincidence of two limiting elements*, we shall call the species of the limiting elements of L coinciding with a given limiting element of L, a *point core of L.* The species RL of the point cores of L will be called a *located compact topological space.*

https://doi.org/10.4153/CJM-1954-001-9 Published online by Cambridge University Press

If in a spread direction [fan direction] and in the corresponding spread [fan] each constituent of a node is replaced by some mathematical entity in such a way that in each node $\nu_1\nu_2 \ldots \nu_n\nu_{n+1}$ the constituents $\nu_1, \nu_2, \ldots, \nu_n$ are replaced by the same mathematical entities as in the node $\nu_1\nu_2 \ldots \nu_n$, the result of this process will be called a *dressed spread direction* [*dressed fan direction*] with a corresponding *dressed spread* [*dressed fan*].

Let us consider a *dressed fan direction SRL* whose row of nodes $b_1, b_2, \ldots$ of order 1 consists of the elements of ψ_1, whilst each element of a row of nodes

$$b_{\nu_1\nu_2\ldots\nu_n1}, b_{\nu_1\nu_2\ldots\nu_n2}, \ldots$$

of order $n+1$ consists of the immediate ascendant of the row followed by a constituent for which is chosen successively each element of a subspecies of ψ_{n+1} which, though arbitrary to a certain extent, must include all elements of ψ_{n+1} at a distance $\leqslant 2.4^{-n}$ from the last constituent of $b_{\nu_1\ldots\nu_n}$, and must exclude all elements of ψ_{n+1} at a distance $\geqslant 3.4^{-n}$ from the last constituent of $b_{\nu_1\ldots\nu_n}$.

Each arrow of SRL defines a limiting element of L. For in each arrow of SRL each *accretion* of order n, i.e., each last constituent of a node of order n has a distance less than

$$(3.4^{-n} + 3.4^{-n-1} + 3.4^{-n-2} + \ldots) = 4^{-n+1}$$

from each of its descendant accretions of order $> n$.

Each limiting element of L coincides with an arrow of SRL. For, let $a_1, a_2, \ldots$ be a limiting element λ of L, and $\nu_1 < \nu_2 < \nu_3 \ldots$ an infinite sequence of increasing natural numbers such that

$$\rho(a_{\nu_n}, a_\nu) < 4^{-n-2}$$

for any $\nu > \nu_n$. If to each a_{ν_n} we assign an element σ_n of ψ_n at a distance $\leqslant 4^{-n} + 4^{-n-1}$ from a_{ν_n}, then from

$$\rho(\sigma_n, a_{\nu_n}) \leqslant 4^{-n} + 4^{-n-1}, \quad \rho(a_{\nu_n}, a_{\nu_{n+1}}) < 4^{-n-2}, \quad \rho(a_{\nu_{n+1}}, \sigma_{n+1}) \leqslant 4^{-n-1} + 4^{-n-2}$$

follows $\rho(\sigma_n, \sigma_{n+1}) < 2.4^{-n}$, so that *the infinite sequence* $\sigma_1, \sigma_2, \sigma_3, \ldots$ *generates an arrow of SRL* which, because $\rho(\sigma_n, a_{\nu_n}) \leqslant 4^{-n} + 4^{-n-1}$, coincides with λ.

If two limiting elements λ_1 and λ_2 of L are at a distance $< 4^{-n-2}$ from each other, then the distance of their respective a_{ν_n} is $< 3.4^{-n-2}$. Hence we can assign the same σ_n to both these a_{ν_n}, so that λ_1 and λ_2 correspond to two arrows of *SRL* which have an accretion of order n in common, and with which they coincide respectively.

On the other hand, two point cores of *RL* coinciding with two arrows of *SRL* respectively which have a common accretion of order n, are at a distance $\leqslant 2.4^{-n+1}$ from each other. So that we have proved:

LEMMA 1. *To each natural number p_3 a natural number p_4 can be assigned such that any two point cores of RL whose distance is $< 2^{-p_4}$ contain respectively two arrows of SRL which have their rod of order p_3 in common.*

And conversely:

https://doi.org/10.4153/CJM-1954-001-9 Published online by Cambridge University Press

LEMMA 2. *To each natural number p_1 a natural number p_2 can be assigned such that any two arrows of SRL which have their rod of order p_2 in common belong respectively to two point cores of RL which have a distance $< 2^{-p_1}$ from each other.*

Let $R'L'$ and $R''L''$ be two located compact topological spaces, and let I be a *full mapping* of $R'L'$ onto $R''L''$, i.e., an assignment of a point core of $R''L''$ to *each* point core of $R'L'$. Such a full mapping implies the assignment A of an arrow $\phi(E')$ of $S''R''L''$ to *each* arrow E' of $S'R'L'$ in such a way that to coinciding arrows E' coinciding arrows $\phi(E')$ are assigned.

Applying the fan theorem to this assignment A, we obtain:

LEMMA 3. *To each natural number p_2 a natural number p_3 can be assigned such that the rod of order p_2 of $\phi(E')$ is for each E' determined by its rod of order p_3, so that to any two arrows of $S'R'L'$ containing the same rod of order p_3 two respective arrows of $S''R''L''$ are assigned by A which contain the same rod of order p_2.*

By successive application of Lemmas 2, 3, and 1 we find that to each natural number p_1 there corresponds a natural number p_4 such that to each pair of point cores of $R'L'$ whose distance is $<2^{-p_4}$ the mapping I assigns a pair of point cores of $R''L''$ which have a distance $< 2^{-p_1}$ from each other.

This result establishes the

CONTINUITY THEOREM. *Every full mapping of a located compact topological space onto another located compact topological space is uniformly continuous.*

In particular, *a bounded function of a compact segment of the linear continuum is uniformly continuous* (**6**, p. 67; **10**, pp. 145–146).

REFERENCES

1. H. Weyl, *Über die neue Grundlagenkrise der Mathematik*, Math. Z., *10* (1921), 39–79.

2. A. Dresden, *Brouwer's contributions to the foundations of mathematics*, Bull. Amer. Math. Soc., *30* (1924), 31–40.

3. R. Wavre, *Y a-t-il une crise des mathématiques?* Revue de Métaphysique et de Morale, *31* (1924), 435–470.

4. ———, *Logique formelle et logique empiriste*, Revue de Métaphysique et de Morale, *33* (1926), 65–75.

5. P. Lévy, R. Wavre, and E. Borel, *Discussions*, Revue de Métaphysique et de Morale, *33* (1926), 253–258, 425–430, 545–551; *34* (1927), 271–276.

6. L. E. J. Brouwer, *Über Definitionsbereiche von Funktionen*, Math. Ann., *97* (1921), 60–75.

7. ———, *Wissenschaft, Mathematik und Sprache*, Monatsh. Math. Phys., *36* (1928), 153–164.

8. G. Mannoury, *La question vitale: A ou B?*, Nieuw Archief Wisk. (2), *21* (1943), 161–167.

9. L. E. J. Brouwer, *Consciousness, philosophy and mathematics*, Proc. Tenth International Congress of Philosophy (Amsterdam, 1948), 1235–1249.

10. ———, *Historical background, principles and methods of intuitionism*, South African J. Sci., Oct.-Nov. 1952, 139–146.

Blaricum, Holland