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CM Periods, CM Regulators, and Hypergeometric Functions, I

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Abstract. We prove the Gross–Deligne conjecture on CM periods for motives associated with H^2 of certain surfaces fibered over the projective line. Then we prove for the same motives a formula which expresses the K_1-regulators in terms of hypergeometric functions ${}_3F_2$, and obtain a new example of non-trivial regulators.

1 Introduction

Periods and regulators of a motive over a number field are very important invariants, whose arithmetic significance can be seen from their conjectural relations with values of the L-function at integers. Such conjectures include those of Birch–Swinnerton-Dyer, Deligne, Bloch, Beilinson and Bloch–Kato. If the motive has complex multiplication (CM) by a number field, especially by an abelian field, those invariants take a special form.

If A is an abelian variety with CM by a subfield of the N-th cyclotomic field, its periods are written in terms of values of the gamma function at $\frac{1}{N}\mathbb{Z}$. When A is an elliptic curve, the formula is due to Lerch [15] and was rediscovered by Chowla–Selberg [8]. Gross [13] gave a geometric proof of a generalization of the formula and proposed a conjecture for any motivic Hodge–de Rham structure with CM by an abelian field, whose precise form was given by Deligne. Using Shimura's monomial relation [23], Anderson [1] proved the formula for CM abelian varieties by reducing to the case of Fermat curves.

In this paper, we study a surface X fibered over $\mathbb{P}^1$ (t-line) with the general fiber defined by $y^p = x^a(1-x)^b(t^l-x)^{p-b}$, where l and p are distinct prime numbers. It admits an action of μ_{lp} and its second cohomology modulo the image of classes supported at singular fibers gives a Hodge–de Rham structure $H = (H_{\mathrm{dR}}, H_B)$ with multiplication by $K := \mathbb{Q}(\mu_{pl})$ (see §2.2). We shall prove that H_B is one-dimensional over K (Theorem 4.12). For each embedding $\chi\colon K \hookrightarrow \mathbb{C}$, let H^χ be the eigencomponent. We shall determine its period and the Hodge type independently, and prove the Gross–Deligne conjecture.

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Theorem 1.1 (Period formula, see Theorem 5.4) *For each $\chi\colon K \hookrightarrow \mathbb{C}$, let $\chi(\zeta_p) = \zeta_p^n$, $\chi(\zeta_l) = \zeta_l^m$, and put $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$, $\mu = \{\frac{m}{l}\}$. Then we have*

$$\mathrm{Per}(H^\chi) \sim_{K'^\times} B(\beta, \mu) B(1-\beta, \beta - \alpha + \mu),$$

where $K' := \mathbb{Q}(\mu_{2lp})$, and the Gross–Deligne conjecture holds.

On the other hand, regulators of the Fermat curve of degree N are written in terms of values at 1 of hypergeometric functions ${}_3F_2$ with parameters in $\frac{1}{N}\mathbb{Z}$ [18]. The conjectural relation with L-values is verified for some cases in [19,20]. Recall that the beta function is related to the value of Gauss' hypergeometric function ${}_2F_1$ at 1. It is also suggestive that the classical polylogarithm can be written as

$$\mathrm{Li}_k(x) = x \cdot {}_{k+1}F_k \left(\begin{matrix} 1, 1, \dots, 1 \\ 2, \dots, 2 \end{matrix} ; x \right),$$

and hence special values of Dirichlet L-functions are written in terms of ${}_{k+1}F_k$-values.

For the surface X, we consider the Beilinson regulator [7] from the motivic cohomology to the Deligne cohomology

$$r_{\mathscr{D}}\colon H^3_{\mathscr{M}}(X, \mathbb{Q}(2)) \longrightarrow H^3_{\mathscr{D}}(X_{\mathbb{C}}, \mathbb{Q}(2)).$$

In terms of algebraic K-theory, we have $H^3_{\mathscr{M}}(X, \mathbb{Q}(2)) = (K_1(X) \otimes_{\mathbb{Z}} \mathbb{Q})^{(2)}$ (the second eigenspace for the Adams operations). Let Z_1 be the union of fibers over μ_l and consider the image of $H^3_{\mathscr{M},Z_1}(X, \mathbb{Q}(2)) \to H^3_{\mathscr{M}}(X, \mathbb{Q}(2))$. The Deligne cohomology can be regarded as functionals on $F^1 H^2_{\mathrm{dR}}(X)$ up to periods, and we restrict them to $F^1 H_{\mathrm{dR}}$.

Theorem 1.2 (Regulator formula, see Theorem 6.5) *Let χ be an embedding such that $H^\chi_{\mathrm{dR}} \subset F^1 H_{\mathrm{dR}}$. Then, for any $z \in H^3_{\mathscr{M},Z_1}(X, \mathbb{Q}(2))$ and $\omega \in H^\chi_{\mathrm{dR}}$, we have*

$$r_{\mathscr{D}}(z)(\omega) \sim_{K^\times} B(1-\alpha, \beta) \cdot {}_3F_2 \left(\begin{matrix} 1-\alpha, \beta, \beta - \alpha + \mu \\ 1 - \alpha + \beta, \beta - \alpha + \mu + 1 \end{matrix} ; 1 \right),$$

where α, β, μ are as before.

Moreover, we shall show the non-vanishing of the regulator image under a mild assumption (Theorem 6.6).

Regarding these examples, it is tempting to ask if the regulators and hence the L-values of a motive with CM by an abelian field can be written in terms of values of ${}_{k+1}F_k$, with k depending on the weight. In a forthcoming paper [4], we shall study more general fibrations of varieties over $\mathbb{P}^1$ with multiplication by a number field whose relative H^1 has a special type of monodromy.

Concerning the period conjecture, there is a result of Maillot–Roessler [16] using Arakelov theory on the absolute value of the period. Recently, Fresán [12] proved the formula for the alternating product of the determinants for any smooth projective variety with a finite order automorphism by reducing to a result of Saito–Terasoma [22]. Since we prove $\dim_K H_B = 1$ and $H^1(X) = H^3(X) = 0$, the Gross–Deligne conjecture for our H follows from Fresán's result. However, we need our precise computations for the study of regulators. Our method is quite different from previous works mentioned above. A crucial step is to compute explicitly Deligne's canonical extension $\mathscr{H}_e$ of the Gauss–Manin connection on the relative first de Rham cohomology. Our fibration is smooth outside $D := \{0, \infty\} \cup \mu_l$, and there is a connection

$$\nabla : \mathscr{H}_e \longrightarrow \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e.$$

We will describe it explicitly and determine the Hodge structure of H. The 1-periods of the fiber are Gauss hypergeometric functions ${}_2F_1$. By the integral representation of Euler type, the 2-periods of X are first written in terms of ${}_3F_2$-values, which then turn out to be ${}_2F_1$-values. The conjecture follows by comparing these computations.

It is more delicate in general to compute the regulators of given motivic elements, even for a fibration of curves. Here we use a new technique [3], originally unpublished, but now included in the appendix of the present paper. Via the canonical extension, we shall represent elements of F^1H_{dR} by certain rational 2-forms. Then the regulators are expressed as integrals of those rational forms over Lefschetz thimbles, which are again written in terms of ${}_3F_2$-values.

This paper proceeds as follows. In Section 2, we fix the setting and compute the 1-periods of the fiber and 2-periods of X. In Section 3, we determine the Gauss–Manin connection and the canonical extension. In Section 4, we determine the Hodge structure and show that H_B is one-dimensional over K. In Section 5, we give a basis of F^1H_{dR} and verify the Gross–Deligne conjecture. In Section 6, we prove the regulator formula and discuss the non-vanishing. The appendix, due to the first author, provides the technique to compute the regulators.

1.1 Notations

Throughout this paper, $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. For each positive integer N, μ_N denotes the group of N-th roots of unity and we put $\zeta_N = e^{2\pi i/N}$. For a real number x, we write $x = \lfloor x \rfloor + \{x\}$ with $\lfloor x \rfloor \in \mathbb{Z}$, $0 \le \{x\} < 1$, and put $\lceil x \rceil = -\lfloor -x \rfloor$. For $\alpha \in \mathbb{C}$ and an integer $n \ge 0$, $(\alpha)_n = \prod_{i=0}^{n-1}(\alpha + i)$ is the Pochhammer symbol and the generalized hypergeometric function is defined by

$$ {}_pF_q\left(\begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n} \frac{x^n}{n!}.$$

We often drop the subscripts from ${}_pF_q$. It converges at $x = 1$ when $\mathrm{Re}(\sum_j \beta_j - \sum_i \alpha_i) > 0$. We use the standard notation for the product of Γ-values

$$\Gamma\left(\begin{matrix}\alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q\end{matrix}\right) = \frac{\prod_{i=1}^{p}\Gamma(\alpha_i)}{\prod_{j=1}^{q}\Gamma(\beta_j)}.$$

For a variety X over $\overline{\mathbb{Q}}$, $H^n_{\mathrm{dR}}(X) = H^n_{\mathrm{dR}}(X/\overline{\mathbb{Q}})$ denotes the algebraic de Rham cohomology and $H^n(X, \mathbb{Q})$ denotes the Betti cohomology of the analytic manifold $X(\mathbb{C})$, or the associated mixed Hodge structure.

2 Preliminaries

2.1 The Setting

Let p, l be distinct prime numbers and a, b, c be integers with $0 < a, b, c < p$ (we shall soon assume that $b + c = p$). We define a fibration of curves $f\colon X \to \mathbb{P}^1$ as follows. Let $g\colon Y \to \mathbb{P}^1$ be a proper flat morphism over $\overline{\mathbb{Q}}$ whose fiber Y_t at $t \in \mathbb{P}^1$ is the normalization of the curve defined by $y^p = x^a(1-x)^b(t-x)^c$. Then g is smooth outside $\{0, 1, \infty\}$ and, by the Riemann–Hurwitz formula, the genus of the generic fiber is $p - 1$. The fiber Y_1 is a union of $\mathbb{P}^1$ intersecting transversally with each other. We have an automorphism σ of order p of Y over $\mathbb{P}^1$ defined by $\sigma(x, y) = (x, \zeta_p^{-1} y)$.

Let $g^{(l)}\colon Y^{(l)} \to \mathbb{P}^1$ be the base change of g by the morphism $\mathbb{P}^1 \to \mathbb{P}^1; t \mapsto t^l$. The action of σ extends naturally to $Y^{(l)}$. On the other hand, the automorphism $\tau(t) = \zeta_l t$ of $\mathbb{P}^1$ induces an automorphism τ of $Y^{(l)}$ over Y. There is a desingularization X of $Y^{(l)}$ such that σ and τ extend to automorphisms of X respectively over $\mathbb{P}^1$ and Y (for example, if one takes a sequence of blow-ups only at the singular points, then σ and τ extend automatically). As a result, we obtain a fibration $f\colon X \to \mathbb{P}^1$ of curves in the commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y^{(l)} & \longrightarrow & Y \\ & \searrow^{f} & \downarrow{\scriptstyle g^{(l)}} & \square & \downarrow{\scriptstyle g} \\ & & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

and for $t \notin \{0, \infty\} \cup \mu_l$, the fiber X_t is isomorphic to Y_{t^l}.

2.2 CM Hodge–de Rham structures

A *Hodge–de Rham structure* is a quadruple $H = (H_{\mathrm{dR}}, H_B, \iota, F^\bullet)$ consisting of

- a finite-dimensional $\overline{\mathbb{Q}}$-vector space H_{dR},
- a finite-dimensional $\mathbb{Q}$-vector space H_B,
- an isomorphism

$$\iota\colon H_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to H_B \otimes_{\mathbb{Q}} \mathbb{C},$$

  and
- a descending filtration $F^\bullet H_{\mathrm{dR}}$ that induces a Hodge structure on H_B via ι.

For a proper smooth variety X over $\overline{\mathbb{Q}}$, its n-th de Rham and Betti cohomology groups, the comparison isomorphism, and the Hodge filtration define a Hodge–de Rham structure $H^n(X)$.

Let K be a finite extension of $\mathbb{Q}$. We say that H admits a *K-multiplication* if we are given K-actions on H_{dR} and H_B that are compatible with ι and $F^\bullet$. Moreover, we say that H *has CM by* K if $\dim_K H_B = 1$. For each embedding $\chi\colon K \hookrightarrow \mathbb{C}$, let H^χ_{dR}, $H^\chi_B := (H_B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\chi$ denote the subspace on which K acts as the multiplication via χ. If $\dim_K H_B = 1$, then these subspaces are 1-dimensional over $\overline{\mathbb{Q}}$. Choosing any bases $\omega_{\mathrm{dR}} \in H^\chi_{\mathrm{dR}}$ and $\omega_B \in H^\chi_B$, we define the *period* $\mathrm{Per}(H^\chi) \in \mathbb{C}^\times$ by $\iota(\omega_{\mathrm{dR}}) = \mathrm{Per}(H^\chi)\omega_B$. By the ambiguity of the choices, $\mathrm{Per}(H^\chi)$ is only well defined up to $\overline{\mathbb{Q}}^\times$. If $(H_{\mathrm{dR}}, F^\bullet)$ is already defined over K, the period is well defined up to $K^\times$.

Let X be as in Section 2.1 and let $Z = X \times_{\mathbb{P}^1} (\{0, \infty\} \cup \mu_l)$ be the union of the singular fibers. Note that Z is stable under the actions of σ and τ. Put $R = \mathbb{Q}[\sigma, \tau]$, $K = \mathbb{Q}(\mu_{lp})$ and regard K as an R-algebra by $\sigma \mapsto \zeta_p$, $\tau \mapsto \zeta_l$. The Hodge–de Rham structure we consider in this paper is $H := \operatorname{Coker}(H^2_Z(X) \to H^2(X)) \otimes_R K$. It admits a K-multiplication, and we shall show that $\operatorname{rank}_K H_B = 1$ (Theorem 4.12). An embedding $\chi: K \hookrightarrow \mathbb{C}$ is identified with an element $h \in (\mathbb{Z}/lp\mathbb{Z})^\times$ such that $\chi(\zeta_{lp}) = \zeta_{lp}^h$. If

$$\operatorname{Coker}(H^2_Z(X) \to H^2(X)) = \bigoplus_{\substack{m \in \mathbb{Z}/l\mathbb{Z}, \\ n \in \mathbb{Z}/p\mathbb{Z}}} H^{(m,n)}$$

denotes the decomposition into the eigenspaces on which τ (resp. σ) acts by ζ_l^m (resp. ζ_p^n), we have $H = \bigoplus_{m \neq 0, n \neq 0} H^{(m,n)}$.

2.3 Periods of the Fiber

For $n = 1, \ldots, p-1$ and integers i, j, k, put a rational 1-form on Y_t by

$$\omega_n^{ijk} = \frac{x^i (1-x)^j (t-x)^k}{y^n} dx.$$

Then we have

$$\sigma^* \omega_n^{ijk} = \zeta_p^n \omega_n^{ijk}. \tag{2.1}$$

Let $0 < t < 1$ and δ_0 be a path on Y_t from $(0,0)$ to $(t,0)$ defined by

$$x = ts, \ \ y = \sqrt[p]{x^a (1-x)^b (t-x)^c}.$$

Let δ_1 be a path on Y_t from $(t,0)$ to $(1,0)$ defined by

$$x = t + (1-t)s, \ \ y = \varepsilon^c \sqrt[p]{x^a (1-x)^b (x-t)^c},$$

where we put

$$\varepsilon = \begin{cases} i & \text{if } p = 2, \\ -1 & \text{if } p \text{ is odd.} \end{cases}$$

If we put $\kappa_m = (1-\sigma)_* \delta_m$, $(m = 0, 1)$, these define 1-cycles on Y_t, and we have

$$\int_{\kappa_m} \omega_n^{ijk} = \int_{\delta_m} (1-\sigma)^* \omega_n^{ijk} = (1 - \zeta_p^n) \int_{\delta_m} \omega_n^{ijk}. \tag{2.2}$$

Lemma 2.1 *Fix integers $i, j, k \geq 0$. For $n = 1, \ldots, p-1$, put*

$$\alpha = \frac{na}{p} - i, \quad \beta = \frac{nb}{p} - j, \quad \gamma = \frac{nc}{p} - k.$$

Then we have

$$\int_{\delta_0} \omega_n^{ijk} = B(1-\alpha, 1-\gamma) \cdot t^{1-\alpha-\gamma} F\left({1-\alpha, \beta \atop 2-\alpha-\gamma} ; t \right),$$

$$\int_{\delta_1} \omega_n^{ijk} = \varepsilon^{p\gamma} B(1-\beta, 1-\gamma) \cdot (1-t)^{1-\beta-\gamma} F\left({\alpha, 1-\beta \atop 2-\beta-\gamma} ; 1-t \right).$$

Proof The first equality follows directly from Euler's integral representation of the Gauss hypergeometric function ${}_2F_1$:

$$B(b, c-b)\cdot F\left({a, b \atop c}; t\right) = \int_0^1 (1-tx)^{-a} x^{b-1}(1-x)^{c-b-1}\,dx$$

(let $a = \beta$, $b = 1-\alpha$, $c = 2-\alpha-\gamma$). The second one follows from the same formula and the transformation formula

$$F\left({a, c-b \atop c}; 1-\frac{1}{t}\right) = t^a \cdot F\left({a, b \atop c}; 1-t\right). \qquad \blacksquare$$

2.4 Cohomology of the Fiber

We have decompositions

$$H^1(Y_t, \mathbb{C}) = \bigoplus_{n=1}^{p-1} H^1(Y_t, \mathbb{C})^{(n)}, \quad H_1(Y_t, \mathbb{Q}(\mu_p)) = \bigoplus_{n=1}^{p-1} H_1(Y_t, \mathbb{Q}(\mu_p))^{(n)},$$

where ${}^{(n)}$ denotes the subspace on which σ^* (resp. σ_*) acts as the multiplication by ζ_p^n. Note that $H^1(Y_t, \mathbb{C})^{(0)} = 0$ since Y_t/μ_p is a rational curve. The natural paring induces a non-degenerate pairing $H^1(Y_t, \mathbb{C})^{(n)} \otimes H_1(Y_t, \mathbb{Q}(\zeta_p))^{(n)} \to \mathbb{C}$. We shall give bases of these spaces under a certain assumption.

Lemma 2.2 *Let $n = 1, \dots, p-1$ and $i, j, k \geq 0$ be integers.*

(i) *If $p \nmid a+b+c$, then ω_n^{ijk} is a differential form of the second kind.*

(ii) *Moreover, ω_n^{ijk} is holomorphic if and only if*

$$i \geq \frac{na+1}{p} - 1, \quad j \geq \frac{nb+1}{p} - 1, \quad k \geq \frac{nc+1}{p} - 1,$$
$$i + j + k \leq \frac{n(a+b+c)-1}{p} - 1.$$

Proof See [2, (18)] (but see [2, (13)] for the correct sign in the fourth inequality). $\blacksquare$

Henceforth, we assume $b + c = p$. Then the condition $p \nmid a+b+c$ is automatically satisfied. By Lemma 2.2, ω_n^{ijk} is holomorphic if and only if

$$i = \left\lceil \frac{na+1}{p} \right\rceil - 1, \quad j = \left\lceil \frac{nb+1}{p} \right\rceil - 1, \quad k = \left\lceil \frac{nc+1}{p} \right\rceil - 1,$$

and we write this ω_n^{ijk} simply as ω_n. The α, β, γ in Lemma 2.1 become

$$\alpha = \left\{\frac{na}{p}\right\}, \quad \beta = \left\{\frac{nb}{p}\right\}, \quad \gamma = \left\{\frac{nc}{p}\right\} = 1-\beta.$$

In particular, $0 < \alpha, \beta, \gamma < 1$. Although these depend on n, we shall suppress n from the notation. By Lemma 2.1, we have

$$\begin{aligned} \int_{\delta_0} \omega_n &= B(1-\alpha, \beta)\cdot t^{\beta-\alpha} F\left({1-\alpha, \beta \atop 1-\alpha+\beta}; t\right), \\ \int_{\delta_1} \omega_n &= -\varepsilon^{p\beta} B(1-\beta, \beta)\cdot F\left({\alpha, 1-\beta \atop 1}; 1-t\right). \end{aligned} \tag{2.3}$$

For each n, let i, j, k be as above and put $\eta_n = \omega_n^{i,j+1,k}$. Then β is replaced by $\beta - 1$ in Lemma 2.1 and we obtain

$$\int_{\delta_0} \eta_n = B(1-\alpha,\beta)\cdot t^{\beta-\alpha} F\left(\begin{matrix}1-\alpha,\beta-1\\ 1-\alpha+\beta\end{matrix};t\right),$$
(2.4)
$$\int_{\delta_1} \eta_n = -\varepsilon^{p\beta} B(1-\beta,\beta)\cdot(1-\beta)(1-t)F\left(\begin{matrix}\alpha,2-\beta\\ 2\end{matrix};1-t\right).$$

Here we used $B(2-\beta,\beta) = (1-\beta)B(1-\beta,\beta)$.

Proposition 2.3 *Let $n = 1,\dots,p-1$ and $0 < t < 1$. Then $\{\omega_n,\eta_n\}$ is a basis of $H^1(Y_t,\mathbb{C})^{(n)}$.*

Proof By (2.1), (2.2), (2.3), and (2.4), ω_n, η_n are non-trivial elements of $H^1(Y_t,\mathbb{C})^{(n)}$. Since ω_n is holomorphic and η_n is not, they are linearly independent. Since

$$\dim H^1(Y_t,\mathbb{C}) = 2(p-1),$$

the proposition follows. ■

Proposition 2.4 *Let $n = 1,\dots,p-1$ and $0 < t < 1$.*

(i) *The projections of κ_0,κ_1 form a basis of $H_1(Y_t,\mathbb{Q}(\mu_p))^{(n)}$.*

(ii) *As a $\mathbb{Q}[\sigma]$-module, $H_1(Y_t,\mathbb{Q})$ is generated by κ_0 and κ_1.*

Proof The period matrix is

$$M_n(t) = \begin{pmatrix}\int_{\kappa_0}\omega_n & \int_{\kappa_0}\eta_n\\ \int_{\kappa_1}\omega_n & \int_{\kappa_1}\eta_n\end{pmatrix}.$$

It suffices to show that $\det M_n(t) \neq 0$. Since $\prod_{n=1}^{p-1}\det M_n(t)$ is constant, it coincides with its limit as $t \to 1$. Hence the proposition follows from the lemma below. ■

Lemma 2.5 *We have*

$$\lim_{t\to 1}\det M_n(t) = \varepsilon^{p\beta}(1-\zeta_p^n)^2\cdot\frac{B(\beta,1-\beta)}{1-\alpha}.$$

Proof By (2.2), (2.3), (2.4), we have

$$\det M_n(t) = -\,\varepsilon^{p\beta}(1-\zeta_p^n)^2 B(1-\alpha,\beta)B(1-\beta,\beta)t^{\beta-\alpha}$$
$$\times\det\begin{pmatrix}F\left(\begin{smallmatrix}1-\alpha,\beta\\ 1-\alpha+\beta\end{smallmatrix};t\right) & F\left(\begin{smallmatrix}1-\alpha,\beta-1\\ 1-\alpha+\beta\end{smallmatrix};t\right)\\ F\left(\begin{smallmatrix}\alpha,1-\beta\\ 1\end{smallmatrix};1-t\right) & (1-\beta)(1-t)F\left(\begin{smallmatrix}\alpha,2-\beta\\ 2\end{smallmatrix};1-t\right)\end{pmatrix}.$$

First, we have

$$\lim_{t\to 1}(1-t)F\left(\begin{matrix}1-\alpha,\beta\\ 1-\alpha+\beta\end{matrix};t\right) = 0.$$

This follows from the transformation formula (*cf.* [11, p. 74 (2)])

$$F\left(\begin{matrix}1-\alpha,\beta\\ 1-\alpha+\beta\end{matrix};t\right) = \frac{1}{B(1-\alpha,\beta)}\sum_{n=0}^{\infty}\frac{(1-\alpha)_n(\beta)_n}{(n!)^2}(k_n-\log(1-t))(1-t)^n,$$
$$k_n := 2\psi(n+1)-\psi(1-\alpha+n)-\psi(\beta+n)$$

where $\psi(t) = \Gamma'(t)/\Gamma(t)$ is the digamma function. On the other hand, by Euler's formula, we have

$$F\left(\begin{matrix}1-\alpha, \beta-1\\ 1-\alpha+\beta\end{matrix}; 1\right) = \Gamma\left(\begin{matrix}1-\alpha+\beta\\ 2-\alpha, \beta\end{matrix};\right) = \frac{1}{(1-\alpha)B(1-\alpha,\beta)}.$$

Hence the lemma follows. ∎

2.5 Periods of X

Now we consider the fibration $f: X \to \mathbb{P}^1$. Recall that $X_t \simeq Y_{t^l}$. By abuse of notation, for each $s = 0, 1$, let δ_s (resp. κ_s) be the path (resp. loop) on X_t which corresponds to the one on Y_{t^l} defined in §2.3. For each s, let Δ_s be the 2-simplex obtained by sweeping δ_s along $0 \le t \le 1$. Since δ_s is vanishing as $t \to s$, the Lefschetz thimble $(1-\sigma)_*\Delta_s$ has boundary on the fiber X_{1-s}. We shall use $(1-\sigma)_*\Delta_1$ (resp. $(1-\sigma)_*\Delta_0$) to compute the periods (resp. regulators). Again by abuse of notation, let ω_n denote the pullback to X of the rational 1-form ω_n on Y defined in §2.4. For $n = 1, \dots, p-1$ and an integer m, define rational 2-forms on X by

$$\omega_{m,n} = t^m \frac{dt}{t} \wedge \omega_n, \quad \eta_{m,n} = t^m \frac{dt}{t} \wedge \eta_n.$$

We have evidently, $(\tau^i\sigma^j)^*\omega_{m,n} = \zeta_l^{mi}\zeta_p^{nj}\omega_{m,n}$ and $(\tau^i\sigma^j)^*\eta_{m,n} = \zeta_l^{mi}\zeta_p^{nj}\eta_{m,n}$.

Proposition 2.6 *Let $n = 1, \dots, p-1$ and $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$ as before. For an integer m, put $\mu = m/l$.*

(i) *If $\mu > \alpha - \beta$, then we have*

$$\int_{\Delta_1} \omega_{m,n} = -\frac{\varepsilon^{p\beta}}{l} \cdot B(\beta,\mu)B(1-\beta, \beta-\alpha+\mu),$$
$$\int_{\Delta_1} \eta_{m,n} = -\frac{\varepsilon^{p\beta}(1-\beta)}{l(1-\alpha+\mu)} \cdot B(\beta,\mu)B(1-\beta, \beta-\alpha+\mu).$$

(ii) *We have*

$$\int_{\Delta_0} \omega_{m,n} = \frac{B(1-\alpha,\beta)}{l(\beta-\alpha+\mu)} \cdot F\left(\begin{matrix}1-\alpha, \beta, \beta-\alpha+\mu\\ 1-\alpha+\beta, \beta-\alpha+\mu+1\end{matrix}; 1\right),$$
$$\int_{\Delta_0} \eta_{m,n} = \frac{B(1-\alpha,\beta)}{l(\beta-\alpha+\mu)} \cdot F\left(\begin{matrix}1-\alpha, \beta-1, \beta-\alpha+\mu\\ 1-\alpha+\beta, \beta-\alpha+\mu+1\end{matrix}; 1\right).$$

Proof Recall the integral representation of ${}_3F_2$ (*cf.* [24, (4.1.2)]):

$$\Gamma\left(\begin{matrix}c, e-c\\ e\end{matrix}\right) F\left(\begin{matrix}a, b, c\\ d, e\end{matrix}; t\right) = \int_0^1 F\left(\begin{matrix}a, b\\ d\end{matrix}; tx\right) x^{c-1}(1-x)^{e-c-1}\, dx.$$

By (2.3), we have

$$\begin{aligned}\int_{\Delta_1}\omega_{m,n} &= -\varepsilon^{p\beta}B(\beta,1-\beta)\int_0^1 F\left({\alpha,1-\beta \atop 1};1-t^l\right)t^{m-1}\,dt\\ &= -\varepsilon^{p\beta}\frac{B(\beta,1-\beta)}{l}\int_0^1 F\left({\alpha,1-\beta \atop 1};1-t\right)t^{\mu-1}\,dt\\ &= -\varepsilon^{p\beta}\frac{B(\beta,1-\beta)}{l}\int_0^1 F\left({\alpha,1-\beta \atop 1};t\right)(1-t)^{\mu-1}\,dt\\ &= -\varepsilon^{p\beta}\frac{B(\beta,1-\beta)}{l\mu}F\left({\alpha,1-\beta,1 \atop 1,\mu+1};1\right)\\ &= -\varepsilon^{p\beta}\frac{B(\beta,1-\beta)}{l\mu}F\left({\alpha,1-\beta \atop \mu+1};1\right),\end{aligned}$$

which converges by the assumption. Using Euler's formula

$$F\left({a,b \atop c};1\right)=\Gamma\left({c,c-a-b \atop c-a,c-b}\right)\quad(\mathrm{Re}(c-a-b)>0)$$

and the functional equations

$$\Gamma(x+1)=x\Gamma(x),\quad B(x,y)=\Gamma\left({x,y \atop x+y}\right),$$

we obtain the first equality of (i). The others follow similarly, using (2.4) for $\eta_{m,n}$. ∎

3 Canonical Extension

In this section, we compute the Gauss–Manin connection of the fibration and determine its canonical extension to $\mathbb{P}^1$.

3.1 Gauss–Manin Connection

Let us start with the fibration $g\colon Y\to\mathbb{P}^1$; for a while, t denotes the coordinate of the base scheme of g. Put $T=\mathbb{P}^1\smallsetminus\{0,1,\infty\}$, $Y_T=Y\times_{\mathbb{P}^1}T$. Then the restriction $g\colon Y_T\to T$ is smooth. Put

$$\mathscr{H}=R^1g_*\Omega^\bullet_{Y_T/T},\quad \Omega^1_T=\Omega^1_{T/\overline{\mathbb{Q}}},$$

and let $\nabla\colon\mathscr{H}\to\Omega^1_T\otimes\mathscr{H}$ be the Gauss–Manin connection. For each $n=1,\dots,p-1$, let $\mathscr{H}^{(n)}\subset\mathscr{H}$ be the subbundle on which σ^* acts as the multiplication by ζ_p^n. Then $\mathscr{H}^{(n)}$ is locally generated by ω_n,η_n as defined in §2.4, and the Hodge filtration $F^1\mathscr{H}^{(n)}$ is generated by ω_n.

Proposition 3.1 *For $n=1,\dots,p-1$, the Gauss–Manin connection*

$$\nabla\colon\mathscr{H}^{(n)}\to\Omega^1_T\otimes\mathscr{H}^{(n)}$$

is given by

$$(\nabla\omega_n,\nabla\eta_n)=\frac{dt}{t}\otimes(\omega_n,\eta_n)\begin{pmatrix}1-\beta & 0\\ 0 & 1-\alpha\end{pmatrix}\begin{pmatrix}-1 & -1\\ (1-t)^{-1} & 1\end{pmatrix},$$

where we put $\alpha=\{\frac{na}{p}\}$, $\beta=\{\frac{nb}{p}\}$ as before.

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Proof We use the following standard derivation relations among Gauss hypergeometric functions [24, (1.4.1.1), (1.4.1.6)]:

$$\frac{d}{dt}F\left({a,b \atop c};t\right)=\frac{ab}{c}F\left({a+1,b+1 \atop c+1};t\right), \tag{3.1}$$

$$\frac{d}{dt}\left(t^{c-1}F\left({a,b \atop c};t\right)\right)=(c-1)t^{c-2}F\left({a,b \atop c-1};t\right). \tag{3.2}$$

We also use the following contiguous relations (see [24, (1.4.1), (1.4.3), (1.4.5), (1.4.9), (1.4.13)]):

$$(c-2a+(a-b)t)F+a(1-t)F[a+1]=(c-a)F[a-1], \tag{3.3}$$

$$(c-a-b)F+a(1-t)F[a+1]=(c-b)F[b-1], \tag{3.4}$$

$$(c-a-1)F+aF[a+1]=(c-1)F[c-1], \tag{3.5}$$

$$(a-1+(1+b-c)t)F+(c-a)F[a-1]=(c-1)(1-t)F[c-1], \tag{3.6}$$

$$c(1-t)F+(c-a)tF[c+1]=cF[b-1]. \tag{3.7}$$

Here, $F=F\left({a,b \atop c};t\right)$ and the notation $F[a+1]$, for example, means $F\left({a+1,b \atop c};t\right)$.

We are reduced to show

$$t\frac{d}{dt}M_n(t)=M_n(t)\begin{pmatrix}1-\beta & 0\\ 0 & 1-\alpha\end{pmatrix}\begin{pmatrix}-1 & -1\\ (1-t)^{-1} & 1\end{pmatrix}. \tag{3.8}$$

We prove this for each row vector. For the first row vector, put

$$(f(t),g(t))=\left(t^{\beta-\alpha}F\left({1-\alpha,\beta \atop 1-\alpha+\beta};t\right),t^{\beta-\alpha}F\left({1-\alpha,\beta-1 \atop 1-\alpha+\beta};t\right)\right).$$

First, consider the case $\alpha\neq\beta$. By (3.2), we have

$$t\frac{d}{dt}(f(t),g(t))=\left((\beta-\alpha)t^{\beta-\alpha}F\left({1-\alpha,\beta \atop -\alpha+\beta};t\right),(\beta-\alpha)t^{\beta-\alpha}F\left({1-\alpha,\beta-1 \atop -\alpha+\beta};t\right)\right).$$

Applying (3.6) to $F\left({\beta,1-\alpha \atop 1-\alpha+\beta};t\right)$, we obtain

$$t\frac{d}{dt}f(t)=-(1-\beta)f(t)+(1-\alpha)(1-t)^{-1}g(t).$$

Applying (3.5) to $F\left({\beta-1,1-\alpha \atop 1-\alpha+\beta};t\right)$, we obtain $t\frac{d}{dt}g(t)=-(1-\beta)f(t)+(1-\alpha)g(t)$. Hence we are done. Now consider the case $\alpha=\beta$. Then

$$(f(t),g(t))=\left(F\left({1-\alpha,\alpha \atop 1};t\right),F\left({1-\alpha,\alpha-1 \atop 1};t\right)\right).$$

By (3.1), we have

$$\frac{d}{dt}(f(t),g(t))=\left((1-\alpha)\alpha F\left({2-\alpha,1+\alpha \atop 2};t\right),-(1-\alpha)^2F\left({2-\alpha,\alpha \atop 2};t\right)\right).$$

Applying (3.7) to $F\left({2-\alpha,1+\alpha \atop 1};t\right)$, we have

$$t\frac{d}{dt}f(t)=\alpha(1-t)F\left({2-\alpha,1+\alpha \atop 1};t\right)-\alpha F\left({2-\alpha,\alpha \atop 1};t\right). \tag{3.9}$$

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Applying (3.4) to $F\left({1-\alpha,1+\alpha \atop 1};t\right)$, we have

$$(1-\alpha)(1-t)F\left({2-\alpha,1+\alpha \atop 1};t\right)=F\left({1-\alpha,1+\alpha \atop 1};t\right)-\alpha f(t). \tag{3.10}$$

Applying (3.3) to $F\left({\alpha,1-\alpha \atop 1};t\right)$, we have

$$\alpha(1-t)F\left({1-\alpha,1+\alpha \atop 1};t\right)=(2\alpha-1)(1-t)f(t)+(1-\alpha)g(t). \tag{3.11}$$

Applying (3.4) to $F\left({1-\alpha,\alpha \atop 1};t\right)$, we have

$$(1-t)F\left({2-\alpha,\alpha \atop 1};t\right)=g(t). \tag{3.12}$$

Combining (3.9)–(3.12), we obtain $t\frac{d}{dt}f(t)=(1-\alpha)\left(-f(t)+(1-t)^{-1}g(t)\right)$. Applying (3.7) to $F\left({\alpha,2-\alpha \atop 1};t\right)$, we have

$$\begin{aligned} t\frac{d}{dt}g(t)&=(1-\alpha)\left(-F\left({1-\alpha,\alpha \atop 1};t\right)+(1-t)F\left({2-\alpha,\alpha \atop 1};t\right)\right)\\ &\overset{(3.12)}{=}(1-\alpha)(-f(t)+g(t)). \end{aligned}$$

In both cases $\alpha\neq\beta$ and $\alpha=\beta$, we have proved (3.8) for the first row vector. For the second row vector, put

$$(u(t),v(t))=\left(F\left({\alpha,1-\beta \atop 1};1-t\right),(1-\beta)(1-t)F\left({\alpha,2-\beta \atop 2};1-t\right)\right).$$

Then by (3.1) and (3.2) we have

$$\frac{d}{dt}(u(t),v(t))=-(1-\beta)\left(\alpha F\left({\alpha+1,2-\beta \atop 2};1-t\right),F\left({\alpha,2-\beta \atop 1};1-t\right)\right).$$

Applying (3.7) to $F\left({\alpha,2-\beta \atop 1};1-t\right)$, we obtain

$$t\frac{d}{dt}v(t)=-(1-\beta)u(t)+(1-\alpha)v(t). \tag{3.13}$$

Applying (3.4) to $F\left({\alpha,2-\beta \atop 2};1-t\right)$, we have

$$t\frac{d}{dt}u(t)=(\beta-\alpha)(1-t)^{-1}v(t)-(1-\beta)\beta\cdot F\left({\alpha,1-\beta \atop 2};1-t\right). \tag{3.14}$$

Applying (3.6) to $F\left({2-\beta,\alpha \atop 2};1-t\right)$, we have

$$\begin{aligned} (1-\beta)\beta\cdot F\left({\alpha,1-\beta \atop 2};1-t\right)&=\left(-(1-\beta)(1-t)^{-1}+1-\alpha\right)v(t)-t\frac{d}{dt}v(t)\\ &\overset{(3.13)}{=}(1-\beta)\left(u(t)-(1-t)^{-1}v(t)\right). \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$t\frac{d}{dt}u(t)=-(1-\beta)u(t)+(1-\alpha)(1-t)^{-1}v(t).$$

Hence we have proved (3.8) for the second row vector. ∎

3.2 Canonical Extension

Now we return to the fibration $f: X \to \mathbb{P}^1$, and from now on t denotes the coordinate of the base scheme of f. Put $D = \{0, \infty\} \cup \mu_l$, $T = \mathbb{P}^1 \smallsetminus D$, $U = X \times_{\mathbb{P}^1} T$, $\mathscr{H} = R^1 f_* \Omega^\bullet_{U/T}$, and let $\nabla: \mathscr{H} \to \Omega^1_T \otimes \mathscr{H}$ be the Gauss–Manin connection. The following is immediate from Proposition 3.1.

Proposition 3.2 *For $n = 1, \dots, p-1$, the Gauss–Manin connection $\nabla: \mathscr{H}^{(n)} \to \Omega^1_T \otimes \mathscr{H}^{(n)}$ is given by*

$$(\nabla\omega_n, \nabla\eta_n) = l\frac{dt}{t} \otimes (\omega_n, \eta_n)\begin{pmatrix}1-\beta & 0\\ 0 & 1-\alpha\end{pmatrix}\begin{pmatrix}-1 & -1\\ \frac{1}{1-t^l} & 1\end{pmatrix}$$
$$= l\frac{ds}{s} \otimes (\omega_n, \eta_n)\begin{pmatrix}1-\beta & 0\\ 0 & 1-\alpha\end{pmatrix}\begin{pmatrix}1 & 1\\ \frac{s^l}{1-s^l} & -1\end{pmatrix}, \quad s = 1/t.$$

Let $j: T \to \mathbb{P}^1$ denote the embedding. Let $\Omega^1_{\mathbb{P}^1}(\log D)$ be the sheaf of differentials on $\mathbb{P}^1$ with logarithmic poles along D. Then Deligne's *canonical extension* ([9, 5.1]) $\nabla: \mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e$ is defined to be the unique sub-bundle of $j_*\mathscr{H}$ satisfying the following properties:

- $\nabla(\mathscr{H}_e) \subset \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e$,
- for each $t \in D$, all the eigenvalues of $\mathrm{Res}_t(\nabla)$ lie in the interval $[0,1)$, where $\mathrm{Res}_t(\nabla)$ denotes the residue at t of the connection matrix.

In fact, we have $\mathscr{H}_e = R^1 f_* \Omega^\bullet_{X/\mathbb{P}^1}(\log Z)$ (recall $Z = X \times_{\mathbb{P}^1} (\{0,\infty\} \cup \mu_l)$) by Steenbrink [25, (2.18), (2.20)]. This is determined as follows.

Proposition 3.3 *For $n = 1, \dots, p-1$, local bases of $\mathscr{H}_e^{(n)}$ at $t \in D$ are given as follows.*

$$\mathscr{H}_e^{(n)}|_0 = \begin{cases}\langle \omega_n - \eta_n, t^{\lceil(\alpha-\beta)l\rceil}((1-\beta)\omega_n - (1-\alpha)\eta_n)\rangle & \text{if } \alpha \neq \beta,\\ \langle \omega_n, \eta_n\rangle & \text{if } \alpha = \beta,\end{cases}$$

$$\mathscr{H}_e^{(n)}|_\infty = \begin{cases}\langle t^{\lfloor(1-\beta)l\rfloor}((1-\alpha-\beta)\omega_n + (1-\alpha)t^{-l}\eta_n), t^{\lfloor\alpha l\rfloor - l}\eta_n)\rangle & \text{if } \alpha+\beta \neq 1,\\ \langle t^{\lfloor\alpha l\rfloor}\omega_n, t^{\lfloor\alpha l\rfloor - l}\eta_n\rangle & \text{if } \alpha+\beta = 1,\end{cases}$$

$$\mathscr{H}_e^{(n)}|_\zeta = \langle\omega_n, \eta_n\rangle \quad (\zeta \in \mu_l).$$

The residue matrices with respect to these bases are

$$\mathrm{Res}_0(\nabla) = \begin{cases}\begin{pmatrix}0 & 0\\ 0 & \{(\beta-\alpha)l\}\end{pmatrix} & \text{if } \alpha \neq \beta,\\ l(1-\alpha)\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix} & \text{if } \alpha = \beta,\end{cases}$$

$$\mathrm{Res}_\infty(\nabla) = \begin{cases}\begin{pmatrix}\{(1-\beta)l\} & 0\\ 0 & \{\alpha l\}\end{pmatrix} & \text{if } \alpha+\beta \neq 1,\\ \begin{pmatrix}\{\alpha l\} & 0\\ (\alpha-1)l & \{\alpha l\}\end{pmatrix} & \text{if } \alpha+\beta = 1,\end{cases}$$

$$\mathrm{Res}_\zeta(\nabla) = -(1-\alpha)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Proof Let A be the matrix of the connection from Proposition 3.2. For each $t \in D$, we shall find a matrix P with coefficients in local sections of $j_*\mathscr{O}_U$ such that $(\omega_n, \eta_n)P$

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is a local basis of $\mathscr{H}_e$ at t. The connection matrix with respect to this basis is given by the gauge transformation $A_P := P^{-1}AP + P^{-1}P'$, where $P' = \frac{d}{dt}P$. For $t = 0$, we let

$$P = \begin{pmatrix} 1 & 1-\beta \\ -1 & -(1-\alpha) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t^{\lceil(\alpha-\beta)l\rceil} \end{pmatrix}$$

if $\alpha \neq \beta$, and $P = I$ (the unit matrix) if $\alpha = \beta$. For $t = \zeta \in \mu_l$, we let $P = I$. Finally for $t = \infty$, we let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & t^{-l} \end{pmatrix} \begin{pmatrix} 1-\alpha-\beta & 0 \\ 1-\alpha & 1 \end{pmatrix} \begin{pmatrix} t^{\lfloor(1-\beta)l\rfloor} & 0 \\ 0 & t^{\lfloor\alpha l\rfloor} \end{pmatrix}$$

if $\alpha + \beta \neq 1$, and

$$P = \begin{pmatrix} t^{\lfloor\alpha l\rfloor} & 0 \\ 0 & t^{\lfloor\alpha l\rfloor - l} \end{pmatrix}$$

if $\alpha + \beta = 1$. Then one verifies that A_P satisfies the desired properties and its residue is given as stated. ■

To see the Hodge filtration, we rewrite the above bases as follows.

Corollary 3.4 *Let $n = 1, \dots, p-1$.*

$$\mathscr{H}_e^{(n)}|_{t=0} = \begin{cases} \langle \omega_n, t^{-\lfloor(\beta-\alpha)l\rfloor}((1-\beta)\omega_n - (1-\alpha)\eta_n)\rangle & \text{if } \alpha \leq \beta, \\ \langle t^{\lceil(\alpha-\beta)l\rceil}\omega_n, \omega_n - \eta_n\rangle & \text{if } \alpha > \beta. \end{cases}$$

$$\mathscr{H}_e^{(n)}|_{t=\infty} = \begin{cases} \langle t^{\lfloor(1-\beta)l\rfloor}\omega_n, t^{\lfloor\alpha l\rfloor - l}\eta_n)\rangle & \text{if } \lfloor\alpha l\rfloor \geq \lfloor(1-\beta)l\rfloor, \\ \langle t^{\lfloor\alpha l\rfloor}\omega_n, t^{\lfloor(1-\beta)l\rfloor}((1-\alpha-\beta)\omega_n + (1-\alpha)t^{-l}\eta_n)\rangle & \\ & \text{if } \lfloor\alpha l\rfloor < \lfloor(1-\beta)l\rfloor. \end{cases}$$

$$\mathscr{H}_e^{(n)}|_{t=\zeta} = \langle \omega_n, \eta_n\rangle \quad (\zeta \in \mu_l).$$

Write $\mathscr{O} = \mathscr{O}_{\mathbb{P}^1}$ and define $F^1\mathscr{H}_e = \mathscr{H}_e \cap j_*(F^1\mathscr{H})$. Then we immediately have the following corollary.

Corollary 3.5 *Let $n = 1, \dots, p-1$.*

(i) *We have $F^1\mathscr{H}_e^{(n)} = \mathscr{O}(i)t^j\omega_n$ with*

$$(i,j) = \begin{cases} (\lfloor(1-\beta)l\rfloor, 0) & \text{if } \lfloor\alpha l\rfloor \geq \lfloor(1-\beta)l\rfloor,\ \alpha \leq \beta, \\ (\lfloor(1-\beta)l\rfloor - \lceil(\alpha-\beta)l\rceil, \lceil(\alpha-\beta)l\rceil) & \text{if } \lfloor\alpha l\rfloor \geq \lfloor(1-\beta)l\rfloor,\ \alpha > \beta, \\ (\lfloor\alpha l\rfloor, 0) & \text{if } \lfloor\alpha l\rfloor < \lfloor(1-\beta)l\rfloor,\ \alpha \leq \beta, \\ (\lfloor\alpha l\rfloor - \lceil(\alpha-\beta)l\rceil, \lceil(\alpha-\beta)l\rceil) & \text{if } \lfloor\alpha l\rfloor < \lfloor(1-\beta)l\rfloor,\ \alpha > \beta. \end{cases}$$

(ii) *According to the four cases as above, we have*

$$\mathrm{Gr}_F^0\,\mathscr{H}_e^{(n)} = \begin{cases} \mathscr{O}(-\lceil(1-\alpha)l\rceil + \lfloor(\beta-\alpha)l\rfloor)t^{-\lfloor(\beta-\alpha)l\rfloor}((1-\beta)\omega_n - (1-\alpha)\eta_n), \\ \mathscr{O}(-\lceil(1-\alpha)l\rceil)(\omega_n - \eta_n), \\ \mathscr{O}(\lfloor(\beta-\alpha)l\rfloor - \lceil\beta l\rceil)t^{-\lfloor(\beta-\alpha)l\rfloor} \\ \qquad\qquad \times\big((1-\alpha-\beta)t^l\omega_n - (1-\beta)\omega_n + (1-\alpha)\eta_n\big), \\ \mathscr{O}(-\lceil\beta l\rceil)\big((1-\alpha-\beta)t^l\omega_n - (1-\alpha)(\omega_n - \eta_n)\big). \end{cases}$$

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Here, by abuse of notation, the images of ω_n, η_n in $\mathrm{Gr}_F^1 \mathscr{H}_e^{(n)}$ are denoted by the same letters.

Corollary 3.6 *For each $\zeta \in \mu_l$, X_ζ is a normal crossing divisor in X with rational irreducible components.*

Proof By Proposition 3.3, the local monodromy of $H^1(X_t, \mathbb{Q})$ at $t = \zeta$ is unipotent, hence X_ζ is normal crossing [21, Theorem 1]. By the Clemens–Schmid exact sequence [17, §4 (a)], $H^1(X_\zeta, \mathbb{Q})$ is the kernel of the log local monodromy $N: H^1(X_t, \mathbb{Q}) \to H^1(X_t, \mathbb{Q})$. The cohomology group $H^1(X_t, \mathbb{Q})$ carries a limiting mixed Hodge structure and N is a morphism of mixed Hodge structures of type $(-1, -1)$. Since rank $N = \frac{1}{2} \dim H^1(X_t, \mathbb{Q})$ by Proposition 3.3, we have $\mathrm{Gr}_1^W H^1(X_t, \mathbb{Q}) = 0$ and $W_0 H^1(X_t, \mathbb{Q}) = \mathrm{Ker}(N)$. Hence $H^1(X_\zeta)$ is of pure weight 0, and all the irreducible components of X_ζ are rational. ■

4 Hodge Numbers

In this section, we determine the Hodge numbers of the eigencomponents of our H and prove that it has CM by K, *i.e.*, $\dim_K H_B = 1$.

4.1 Localization Sequence

Let the notations be as in Section 3.2 and put $Z = X \smallsetminus U$. We have the localization sequence $H_Z^2(X) \to H^2(X) \to H^2(U) \to H_Z^3(X) \to H^3(X)$ both for the de Rham and Betti cohomologies. Let $\langle Z \rangle$ denote the image of the first map. Recall that we defined (§2.2) the Hodge–de Rham structure $H = H^2(X)/\langle Z \rangle \otimes_R K$.

Proposition 4.1 $H^1(X) = H^3(X) = 0$.

Proof By Poincaré duality, it suffices to show $H^1(X, \mathbb{Q}) = 0$. Since $H^1(X, \mathbb{Q}) \hookrightarrow W_1 H^1(U, \mathbb{Q})$, where $W_\bullet$ denotes the weight filtration, it suffices to show the vanishing of the latter. Using the Leray spectral sequence, we have an exact sequence

$$0 \longrightarrow H^1(T, \mathbb{Q}) \longrightarrow H^1(U, \mathbb{Q}) \longrightarrow H^0(T, R^1 f_* \mathbb{Q}) \longrightarrow 0.$$

By the computation of $\mathrm{Res}_\infty(\nabla)$ in Proposition 3.3, for $n = 1, \dots, p-1$, the local monodromy around $t = \infty$ of $H^1(X_t, \mathbb{C})^{(n)}$ does not have 1 as an eigenvalue. Hence we have $H^0(T, R^1 f_* \mathbb{Q}) = 0$ (recall that $H^1(X_t, \mathbb{C})^{(0)} = 0$). Since $H^1(T, \mathbb{Q})$ is of weight 2, we have $W_1 H^1(U, \mathbb{Q}) = 0$. ■

As a result, we have an exact sequence on the de Rham side [14, Chapter II, Theorem (3.3), Proposition (3.4)]

$$0 \longrightarrow H^2_{\mathrm{dR}}(X)/\langle Z \rangle \longrightarrow H^2_{\mathrm{dR}}(U) \xrightarrow{\partial} H_1^{\mathrm{dR}}(Z) \longrightarrow 0.$$

The middle term is described by the canonical extension as follows. The Leray spectral sequence yields an exact sequence

$$0 \longrightarrow H^1(T, \mathscr{H}) \longrightarrow H^2_{\mathrm{dR}}(U) \longrightarrow H^0(T, R^2 f_* \Omega^\bullet_{U/T}) \longrightarrow 0.$$

Since σ^* acts on $R^2 f_* \Omega^\bullet_{U/T}$ trivially, we have $H^1(T, \mathscr{H}^{(n)}) \simeq H^2_{\mathrm{dR}}(U)^{(n)}$ for $n = 1, \dots, p-1$. Put a complex of sheaves on $\mathbb{P}^1$ as $\mathscr{E} = [\mathscr{H}_e \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e]$. Then the map of complexes

$$\begin{array}{ccc} \mathscr{H}_e & \longrightarrow & \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e \\ \downarrow & & \downarrow \\ j_*\mathscr{H} & \longrightarrow & j_*(\Omega^1_T \otimes \mathscr{H}) \end{array}$$

induces an isomorphism $H^1(\mathbb{P}^1, \mathscr{E}) \simeq H^1(T, \mathscr{H})$, and the first group carries a mixed Hodge structure [26, Theorem (4.1)] and its Hodge filtration is given as follows [26, (4.10)]:

(4.1)
$$\begin{aligned} F^0 H^1(\mathbb{P}^1, \mathscr{E}) &= H^1(\mathbb{P}^1, \mathscr{E}), \\ F^1 H^1(\mathbb{P}^1, \mathscr{E}) &= H^1(\mathbb{P}^1, F^1\mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e), \\ F^2 H^1(\mathbb{P}^1, \mathscr{E}) &= H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1\mathscr{H}_e). \end{aligned}$$

It follows that

(4.2)
$$\begin{aligned} \mathrm{Gr}^0_F H^1(\mathbb{P}^1, \mathscr{E}) &= H^1(\mathbb{P}^1, \mathrm{Gr}^0_F \mathscr{H}_e), \\ \mathrm{Gr}^1_F H^1(\mathbb{P}^1, \mathscr{E}) &= \mathrm{Coker}\Big(H^0(\mathbb{P}^1, F^1\mathscr{H}_e) \xrightarrow{\overline{\nabla}} H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathrm{Gr}^0_F \mathscr{H}_e)\Big), \end{aligned}$$

where $\overline{\nabla}$ is the map induced from the composition of ∇ and the projection $\mathscr{H}_e \to \mathrm{Gr}^0_F \mathscr{H}_e$.

4.2 Residues

For each $t \in D$, let $\partial_t \colon H^2_{\mathrm{dR}}(U) \to H_1^{\mathrm{dR}}(X_t)$ be the t-component of the coboundary map ∂. Let $N_t \subset \mathscr{H}_{e,t}$ be the image of the composite

$$\Gamma(U_t, \mathscr{H}_e) \xrightarrow{\nabla} \Gamma(U_t, \Omega^1_{\mathbb{P}^1}(\log t) \otimes \mathscr{H}_e) \xrightarrow{\mathrm{Res}_t} \mathscr{H}_{e,t},$$

where U_t is a small open neighborhood of t. Then it is not difficult to show that the diagram

$$\begin{array}{ccc} H^1(\mathbb{P}^1, \mathscr{E}) & \xrightarrow{\subset} & H^2_{\mathrm{dR}}(U) \\ \downarrow{\scriptstyle \mathrm{Res}_t} & & \downarrow{\scriptstyle \partial_t} \\ \mathscr{H}_{e,t}/N_t & \xrightarrow{\simeq} & H_1^{\mathrm{dR}}(X_t) \end{array}$$

commutes, where the lower map is an isomorphism. The following is immediate from Proposition 3.3.

Proposition 4.2 *For $n = 1, \dots, p-1$, we have*

$$\begin{aligned} N_0^{(n)} &= \big\langle t^{\lceil (\alpha-\beta) l \rceil}((1-\beta)\omega_n - (1-\alpha)\eta_n) \big\rangle, \\ N_\infty^{(n)} &= \mathscr{H}_{e,\infty}, \\ N_\zeta^{(n)} &= \langle \eta_n \rangle \quad \textit{for } \zeta \in \mu_l. \end{aligned}$$

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Therefore, we have

$$\dim H_1^{\mathrm{dR}}(X_t)^{(n)} = \begin{cases} 1 & \text{if } t = 0 \text{ or } t \in \mu_l, \\ 0 & \text{if } t = \infty. \end{cases}$$

Later, we shall use the following.

Lemma 4.3 *Let $n = 1, \dots, p-1$.*

(i) *If $\alpha \le \beta$, then $t^m \omega_n|_{t=0} \in N_0^{(n)}$ if $m > 0$, and $\notin N_0^{(n)}$ if $m = 0$.*

(ii) *If $\alpha > \beta$, then $t^m \omega_n|_{t=0} \in N_0^{(n)}$ if $m \ge \lceil (\alpha - \beta) l \rceil$.*

Proof By Corollary 3.4 and Proposition 4.2, this is trivial except when $\alpha > \beta$ and $m = \lceil (\alpha - \beta) l \rceil$. In this case, we have

$$t^m \omega_n|_{t=0} = t^m \omega_n|_0 + \frac{1-\alpha}{\alpha - \beta} t^m (\omega_n - \eta_n)|_{t=0} = \frac{t^m ((1-\beta)\omega_n - (1-\alpha)\eta_n)|_{t=0}}{\alpha - \beta} \in N_0^{(n)}.$$

∎

4.3 Hodge Numbers

For each $n = 1, \dots, p-1$, we obtained an exact sequence

$$(4.3)\quad 0 \longrightarrow (H_{\mathrm{dR}}^2(X)/\langle Z \rangle)^{(n)} \longrightarrow H^1(\mathbb{P}^1, \mathscr{E}^{(n)}) \xrightarrow{\mathrm{Res}} \mathscr{H}_{e,0}^{(n)}/N_0^{(n)} \oplus \bigoplus_{\zeta \in \mu_l} \mathscr{H}_{e,\zeta}^{(n)}/N_\zeta^{(n)} \longrightarrow 0.$$

First, we give a basis of F^2. By (4.1), we have an embedding

$$\iota\colon F^2(H_{\mathrm{dR}}^2(X)/\langle Z \rangle)^{(n)} \hookrightarrow \Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathscr{H}_e^{(n)}).$$

By this, we identify $F^2(H_{\mathrm{dR}}^2(X)/\langle Z \rangle)^{(n)}$ with the elements of the right-hand side having trivial residues. Recall the rational 2-forms $\omega_{m,n} = t^m \frac{dt}{t} \otimes \omega_n$.

Proposition 4.4 *For each $n = 1, \dots, p-1$, a basis of $F^2(H_{\mathrm{dR}}^2(X)/\langle Z \rangle)^{(n)}$ is given by $\{\omega_{m,n} \mid m \in I_n^2\}$, where*

$$I_n^2 := \big\{ m \mid \max\{1, \lceil (\alpha - \beta) l \rceil\} \le m \le \min\{\lfloor \alpha l \rfloor, \lfloor (1-\beta) l \rfloor\} \big\}.$$

In particular, $\dim F^2(H_{\mathrm{dR}}^2(X)/\langle Z \rangle)^{(n)} = \min\{\lfloor \alpha l \rfloor, \lfloor (1-\beta) l \rfloor\} - \max\{0, \lfloor (\alpha - \beta) l \rfloor\}$.

Proof Let $F^1 \mathscr{H}_e^{(n)} = \mathscr{O}(i) t^j \omega_n$ be as in Corollary 3.5 (i). One easily sees that a basis of $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log D) \otimes F^1 \mathscr{H}_e^{(n)})$ is given by

$$\omega_{m,n}\ (j \le m \le i + j), \qquad t^j \frac{dt}{t - \zeta} \otimes \omega_n\ (\zeta \in \mu_l).$$

For the first type, the residues at $\zeta \in \mu_l$ are trivial. By Lemma 4.3, $\mathrm{Res}_0(\omega_{m,n}) = t^m \omega_n$ is trivial for $m \ge j$ unless $\alpha \le \beta$ and $m = 0$. For the second type, it has trivial residues except at ζ and

$$\operatorname{Res}_\zeta\Big(t^j \frac{dt}{t-\zeta} \otimes \omega_n \Big) = t^j \omega_n,$$

which is non-trivial by Proposition 4.2. These show that a basis of

$$F^2(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)}$$

is given by $\omega_{m,n}$ with $j \le m \le i+j$ and $m \ne j = 0$ if $\alpha \le \beta$. Hence the proposition follows from Corollary 3.5 (i). ∎

Since $(\mathscr{H}_{e,0}/N_0)^{(n)}$ and $(\mathscr{H}_{e,\zeta}/N_\zeta)^{(n)}$ are all 1-dimensional, the above proof implies the following.

Corollary 4.5 *For $n = 1, \dots, p-1$, we have*

$$\operatorname{Res}(F^2H^1(\mathbb{P}^1, \mathscr{E}^{(n)})) = \begin{cases} (\mathscr{H}_{e,0}/N_0)^{(n)} \oplus \bigoplus_{\zeta\in\mu_l} (\mathscr{H}_{e,\zeta}/N_\zeta)^{(n)} & \text{if } \alpha \le \beta, \\ \bigoplus_{\zeta\in\mu_l} (\mathscr{H}_{e,\zeta}/N_\zeta)^{(n)} & \text{if } \alpha > \beta. \end{cases}$$

Corollary 4.6 *Suppose that $p < l$. Then we have $F^2(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} \ne 0$ for any $n = 1, \dots, p-1$.*

Proof Since $\alpha, 1-\beta \ge 1/p$, we have $l\alpha, l(1-\beta) > 1$. Since $\beta \ge 1/p$ and $\alpha \le 1 - 1/p$, we have $(\alpha-\beta)l < \alpha l - 1, (1-\beta)l - 1$. Hence we have $I_n^2 \ne \varnothing$. ∎

Now we determine the other Hodge numbers.

Lemma 4.7 *Let $n = 1, \dots, p-1$.*

(i) *If $\alpha \le \beta$, then we have $\operatorname{Gr}^1_F(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} = \operatorname{Gr}^1_F H^1(\mathbb{P}^1, \mathscr{E}^{(n)})$.*

(ii) *If $\alpha > \beta$, then we have an exact sequence*

$$0 \longrightarrow \operatorname{Gr}^1_F(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} \longrightarrow \operatorname{Gr}^1_F H^1(\mathbb{P}^1, \mathscr{E}^{(n)}) \xrightarrow{\operatorname{Res}_0} (\mathscr{H}_{e,0}/N_0)^{(n)} \longrightarrow 0.$$

Proof By (4.3) and Corollary 4.5, we are left to show the non-triviality of Res_0 in the case (ii). If $\lfloor \alpha l \rfloor \ge \lfloor (1-\beta)l \rfloor$, consider

$$\frac{dt}{t(1-t^l)} \otimes (\omega_n - \eta_n).$$

By Corollary 3.5 (ii), this is an element of $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes \operatorname{Gr}^0_F \mathscr{H}_e^{(n)})$. Its residue at 0 is $\omega_n - \eta_n \not\equiv 0 \pmod{N_0}$ by Proposition 4.2. If $\lfloor \alpha l \rfloor < \lfloor (1-\beta)l \rfloor$, consider similarly

$$\frac{dt}{t(1-t^l)} \otimes ((1-\alpha-\beta)t^l\omega_n - (1-\alpha)(\omega - \eta_n)),$$

whose residue at 0 is $-(1-\alpha)(\omega_n - \eta_n) \not\equiv 0 \pmod{N_0}$. ∎

Proposition 4.8 *For each $n = 1, \dots, p-1$, we have*

$$\dim \operatorname{Gr}^1_F(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} = \big|\lfloor \alpha l \rfloor - \lfloor (1-\beta)l \rfloor\big| + \lfloor |\alpha-\beta| l \rfloor.$$

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Proof First we show that the map

$$\overline{\nabla}\colon H^0(\mathbb{P}^1, F^1\mathscr{H}_e^{(n)}) \to H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)\otimes \mathrm{Gr}_F^0 \mathscr{H}_e^{(n)})$$

is injective. Let $F^1\mathscr{H}_e^{(n)} = \mathscr{O}(i)t^j\omega_n$ as in Corollary 3.5 (i). Then $H^0(\mathbb{P}^1, F^1\mathscr{H}_e^{(n)})$ has a basis $\{\omega_{m,n} \mid j \le m \le i+j\}$, and

$$\nabla\omega_{m,n} = \frac{dt}{t}t^m\Big\{(m-l(1-\beta))\omega_n + \frac{l(1-\alpha)}{1-t^l}\eta_n\Big\} \equiv l(1-\alpha)\frac{dt}{t(1-t^l)}t^m\eta_n \not\equiv 0$$

modulo $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)\otimes F^1\mathscr{H}_e^{(n)})$. Since $0 \le i < l$ in every case, $\omega_{m,n}$ belong to different eigenspaces with respect to the τ-action. Hence the non-vanishing implies the injectivity.

By Corollary 3.5 (ii), we have $\mathrm{Gr}_F^0 \mathscr{H}_e^{(n)} \simeq \mathscr{O}(k)$, where

$$k := \begin{cases} -\lceil(1-\alpha)l\rceil + \lfloor(\beta-\alpha)l\rfloor & \text{if } \lfloor\alpha l\rfloor \ge \lfloor(1-\beta)l\rfloor, \alpha \le \beta, \\ -\lceil(1-\alpha)l\rceil & \text{if } \lfloor\alpha l\rfloor \ge \lfloor(1-\beta)l\rfloor, \alpha > \beta, \\ \lfloor(\beta-\alpha)l\rfloor - \lceil\beta l\rceil & \text{if } \lfloor\alpha l\rfloor < \lfloor(1-\beta)l\rfloor, \alpha \le \beta, \\ -\lceil\beta l\rceil & \text{if } \lfloor\alpha l\rfloor < \lfloor(1-\beta)l\rfloor, \alpha > \beta. \end{cases}$$

Note that $k < 0$ in any case. One sees that $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)\otimes\mathscr{O}(k))$ has a basis

$$\frac{t^m}{1-t^l}\frac{dt}{t}\otimes\omega_n \quad (0 \le m \le l+k).$$

By (4.2) and the above injectivity, we have

$$\begin{aligned}\dim \mathrm{Gr}_F^1 H^1(\mathbb{P}^1, \mathscr{E}^{(n)}) &= \dim H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D)\otimes\mathscr{O}(k)) - \dim H^0(\mathbb{P}^1, \mathscr{O}(i)) \\ &= (l+k+1) - (i+1) = l+k-i.\end{aligned}$$

By Corollary 3.5 (i) and Lemma 4.7, we obtain the desired formula. ■

Corollary 4.9 *Assume that $p < l$ and $p > 2$ when $a = b$. Then we have*

$$\mathrm{Gr}_F^1(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} \ne 0$$

for any $n = 1, \dots, p-1$.

Proof If $a \ne b$, then $\lfloor|\alpha-\beta|l\rfloor \ge \lfloor\frac{l}{p}\rfloor \ge 1$. If $a = b$, then $\alpha \ne 1-\alpha$ since $p > 2$, and hence $\big|\lfloor\alpha l\rfloor - \lfloor(1-\alpha)l\rfloor\big| \ge 1$. ■

Proposition 4.10 *For each $n = 1, \dots, p-1$, we have*

$$\dim \mathrm{Gr}_F^0(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} = \min\{\lfloor(1-\alpha)l\rfloor, \lfloor\beta l\rfloor\} - \max\{0, \lfloor(\beta-\alpha)l\rfloor\}.$$

Proof By (4.2), Corollary 4.5, and Lemma 4.7, we have

$$\mathrm{Gr}_F^0(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} = H^1(\mathbb{P}^1, \mathrm{Gr}_F^0 \mathscr{H}_e^{(n)}) = H^1(\mathbb{P}^1, \mathscr{O}(k)),$$

where k is as in the proof of Proposition 4.8. Since $k < 0$, we have

$$\dim H^1(\mathbb{P}^1, \mathscr{O}(k)) = \dim H^0(\mathbb{P}^1, \mathscr{O}(-k-2)) = -k-1.$$

Hence the proposition follows. ■

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Remark 4.11 In fact, Proposition 4.10 is equivalent to the dimension formula in Proposition 4.4. Note that the complex conjugation switches n (resp. α, β) and $p-n$ (resp. $1-\alpha, 1-\beta$).

Theorem 4.12 *The Hodge–de Rham structure* $H = (H^2(X)/\langle Z\rangle) \otimes_R K$ *has CM by* K, i.e., $\dim_K H_B = 1$.

Proof Combining Propositions 4.4, 4.8, and 4.10, one verifies that

$$\dim(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} = l-1$$

for each $n = 1,\dots,p-1$. It follows that $\dim_{\mathbb{Q}} H_B \leq (l-1)(p-1) = [K:\mathbb{Q}]$. It remains to show that $H \neq 0$, for which it suffices to show that τ is not the identity on $H^2_{\mathrm{dR}}(X)/\langle Z\rangle$. If $p < l$, this follows from Proposition 4.4 and Corollary 4.6. The general case follows from Proposition 5.2 below. ∎

5 Periods

We compute the periods of our H and verify the Gross–Deligne conjecture, for which it will suffice to consider $F^1 H_{\mathrm{dR}}$.

5.1 Basis of $F^1 H_{\mathrm{dR}}$

Recall that, by (4.3), we can identify $F^1(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)}$ with the elements of

$$F^1 H^1(\mathbb{P}^1, \mathscr{E}^{(n)})$$

having trivial residues. Furthermore, they are identified with rational 2-forms by the following lemma. Put $T_1 = \mathbb{P}^1 \setminus \{0,\infty\}$.

Lemma 5.1 *For each* $n = 1,\dots,p-1$, *there is a natural injection*

$$\iota\colon F^1(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)} \hookrightarrow \Gamma(T_1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1\mathscr{H}_e^{(n)}).$$

Proof By (4.1) and (4.3), it suffices to show the existence of an injection

$$H^1(\mathbb{P}^1, F^1\mathscr{E}^{(n)}) \hookrightarrow \Gamma(T_1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1\mathscr{H}_e^{(n)}),$$

where we put $F^1\mathscr{E} = [F^1\mathscr{H}_e \to \Omega^1_{\mathbb{P}^1}(\log D)\otimes\mathscr{H}_e]$. Consider the commutative diagram in Figure 1, where the right vertical sequence is exact. By Proposition 3.3, $\overline{\nabla}$ is an isomorphism on T_1. Therefore, we have an isomorphism

$$\Gamma(T_1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1\mathscr{H}_e^{(n)}) \xrightarrow{\simeq} H^1(T_1, F^1\mathscr{E}^{(n)}).$$

It remains to show the injectivity of $H^1(\mathbb{P}^1, F^1\mathscr{E}^{(n)}) \to H^1(T_1, F^1\mathscr{E}^{(n)})$. This follows from the fact that $H^1(\mathbb{P}^1, F^1\mathscr{E}) \to H^1(\mathbb{P}^1, \mathscr{E})$ is injective and $H^1(\mathbb{P}^1, \mathscr{E}) \to H^1(T_1, \mathscr{E})$ is an isomorphism. ∎

$$
\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & \Omega^1_{\mathbb{P}^1}(\log D) \otimes F^1\mathscr{H}_e^{(n)} \\
 & & \downarrow \\
F^1\mathscr{H}_e^{(n)} & \xrightarrow{\nabla} & \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e^{(n)} \\
\downarrow {\scriptstyle =} & & \downarrow \\
F^1\mathscr{H}_e^{(n)} & \xrightarrow{\overline{\nabla}} & \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathrm{Gr}^0_F\,\mathscr{H}_e^{(n)} \\
 & & \downarrow \\
 & & 0
\end{array}
$$

Figure 1

Under the identification via ι, we have the following.

Proposition 5.2 *For each $n = 1, \dots, p-1$, a basis of $F^1(H^2_{\mathrm{dR}}(X)/\langle Z \rangle)^{(n)}$ is given by $\{\omega_{m,n} \mid m \in I^1_n\}$, where*

$$
I^1_n := \begin{cases} \{-\lfloor(\beta-\alpha)l\rfloor, \dots, -1\} \cup \{1, \dots, \max\{\lfloor \alpha l \rfloor, \lfloor (1-\beta)l \rfloor\}\} & \text{if } \alpha < \beta, \\ \{1, \dots, \max\{\lfloor \alpha l \rfloor, \lfloor (1-\beta)l \rfloor\}\} & \text{if } \alpha \geq \beta. \end{cases}
$$

Recall that $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$.

Proof It is routine to verify that $|I^1_n| = \dim F^1(H^2_{\mathrm{dR}}(X)/\langle Z \rangle)^{(n)}$ using Propositions 4.4 and 4.8. Therefore, it suffices to show that

$$
\omega_{m,n} \in F^1(H^2_{\mathrm{dR}}(X)/\langle Z \rangle)^{(n)}
$$

if $m \in I^1_n$. We construct Čech cocycles representing elements of $H^1(\mathbb{P}^1, F^1\mathscr{E}^{(n)})$ with trivial residues which correspond to $\omega_{m,n}$. Take a covering $\mathbb{P}^1 = U_0 \cup U_\infty$, where $U_0 := \mathbb{P}^1 \smallsetminus \{\infty\}$, $U_\infty := \mathbb{P}^1 \smallsetminus \{0\}$; note that $T_1 = U_0 \cap U_\infty$. A Čech cocycle in this case is a triple

$$
(\psi, \varphi_0, \varphi_\infty) \in \Gamma(T_1, F^1\mathscr{H}_e^{(n)}) \oplus \bigoplus_{t=0,\infty} \Gamma(U_t, \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e^{(n)})
$$

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satisfying $\nabla\psi = \varphi_0|_{T_1} - \varphi_\infty|_{T_1}$. We construct such cocycles in four ways. By Proposition 3.2, we have

$$l^{-1}\nabla(t^m\omega_n)$$

$$= (\mu - 1 + \beta)\omega_{m,n} + \frac{1-\alpha}{1-t^l}\eta_{m,n} \tag{5.1}$$

$$= \Big(\mu - \alpha - \frac{1-\alpha-\beta}{1-t^l}\Big)\omega_{m,n} + \frac{t^l}{1-t^l}\Big((1-\alpha-\beta)\omega_{m,n} + \frac{1-\alpha}{t^l}\eta_{m,n}\Big) \tag{5.2}$$

$$= \Big(\mu + (1-\beta)\frac{t^l}{1-t^l}\Big)\omega_{m,n} - \frac{1}{1-t^l}\Big((1-\beta)\omega_{m,n} - (1-\alpha)\eta_{m,n}\Big) \tag{5.3}$$

$$= \Big(\mu - \alpha + \beta + (1-\alpha)\frac{1-t^l}{1-t^l}\Big)\omega_{m,n} - \frac{1-\alpha}{1-t^l}(\omega_{m,n} - \eta_{m,n}). \tag{5.4}$$

Put $j = \max\{0, \lceil(\alpha-\beta)l\rceil\}$, $k = \min\{\lfloor\alpha l\rfloor, \lfloor(1-\beta)l\rfloor\}$.

(i) Suppose that $\lfloor\alpha l\rfloor \geq \lfloor(1-\beta)l\rfloor$. Let $\psi = l^{-1}t^m\omega_n$,

$$\varphi_0 = (\mu - 1 + \beta)\omega_{m,n},\ \varphi_\infty = -\frac{1-\alpha}{1-t^l}\eta_{m,n}.$$

By (5.1) and Corollary 3.4, these define a cocycle if $j \leq m \leq \lfloor\alpha l\rfloor$. By Proposition 4.2, it has no residues unless $m = 0$, and hence defines an element of $F^1(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)}$ if

$$j \leq m \leq \lfloor\alpha l\rfloor, \quad m \neq 0.$$

(ii) Suppose that $\lfloor\alpha l\rfloor < \lfloor(1-\beta)l\rfloor$. Then by (5.2) and Corollary 3.4, $\psi = l^{-1}t^m\omega_n$,

$$\varphi_0 = \Big(\mu - \alpha - \frac{1-\alpha-\beta}{1-t^l}\Big)\omega_{m,n},$$

$$\varphi_\infty = -\frac{t^l}{1-t^l}\Big((1-\alpha-\beta)\omega_{m,n} + (1-\alpha)t^{-l}\eta_{m,n}\Big)$$

define a cocycle if $j \leq m \leq \lfloor(1-\beta)l\rfloor$. To kill the residues, we use Lemma 5.3 below. Then by letting

$$\varphi_0 = (\mu-\alpha)\omega_{m,n}, \quad \varphi_\infty = (1-\alpha-\beta)\omega_{m,n} - \frac{1-\alpha}{1-t^l}\eta_{m,n},$$

we obtain an element of $F^1(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)}$ for $j \leq m \leq \lfloor(1-\beta)l\rfloor$, $m \neq 0$.

(iii) Suppose that $\alpha \leq \beta$. Then by (5.3) and Corollary 3.4, $\psi = -l^{-1}t^m\omega_n$,

$$\varphi_0 = \frac{1}{1-t^l}((1-\beta)\omega_{m,n} - (1-\alpha)\eta_{m,n}), \quad \varphi_\infty = \Big(\mu + (1-\beta)\frac{t^l}{1-t^l}\Big)\omega_{m,n}$$

define a cocycle if $-\lfloor(\beta-\alpha)l\rfloor \leq m \leq k$. If $m < 0$, we can kill the residues using Lemma 5.3, and $\varphi_0 = (1-\beta)\omega_{m,n} - \frac{1-\alpha}{1-t^l}\eta_{m,n}$, and $\varphi_\infty = \mu\omega_{m,n}$ define an element of $F^1(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)}$ for $-\lfloor(\beta-\alpha)l\rfloor \leq m < 0$.

(iv) Finally suppose that $\alpha > \beta$. Then, by (5.4) and Corollary 3.4, $-l^{-1}t^m\omega_n$,

$$\varphi_0 = \frac{1-\alpha}{1-t^l}(\omega_{m,n} - \eta_{m,n}), \quad \varphi_\infty = \Big(\mu - \alpha + \beta + (1-\alpha)\frac{t^l}{1-t^l}\Big)\omega_{m,n}$$

define a cocycle if $0 \leq m \leq k$. If $m \neq 0$, we can use Lemma 5.3 to kill the residues and

$$\varphi_0 = (1-\alpha)\omega_{m,n} - \frac{1-\alpha}{1-t^l}\eta_{m,n}, \quad \varphi_\infty = (\mu - 1 + \beta)\omega_{m,n}$$

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define an element of $F^1(H^2_{\mathrm{dR}}(X)/\langle Z\rangle)^{(n)}$ for $0 < m \le k$. Combining (iii) and (i) (or (ii)), we obtain the first case of the proposition. For the second case, combine (iv) and (i) (or (ii)), just noting that $k \ge j - 1 = \lfloor(\alpha - \beta)l\rfloor$. ■

Lemma 5.3 *If $j \le m < l$, $m \ne 0$, then*

$$\frac{1}{1-t^l} \otimes \omega_{m,n} \in \Gamma(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(\log D) \otimes \mathscr{H}_e^{(n)}),$$

and it has trivial residues at $t = 0, \infty$.

Proof This is immediate from Corollary 3.4 and Lemma 4.3. ■

5.2 Period Formula

We prove the period formula which verifies the conjecture of Gross–Deligne [13, §4] (but see Remark 5.6 below). We identify an embedding $\chi: K \hookrightarrow \mathbb{C}$ with the element $h \in (\mathbb{Z}/lp\mathbb{Z})^\times$ such that $\chi(\zeta_{lp}) = \zeta_{lp}^h$, and write $H^{(h)}$ instead of H^χ. For each $h \in (\mathbb{Z}/lp\mathbb{Z})^\times$, let $(p(h), 2-p(h))$ be the Hodge type of $H^{(h)}$. Put $K' = \mathbb{Q}(\mu_{2lp})$ ($K = K'$ if lp is odd).

Theorem 5.4 *Define a function $\varepsilon: \mathbb{Z}/lp\mathbb{Z} \to \mathbb{Z}$ by*

$$\varepsilon(i) = \begin{cases} 1 & \text{if } i \equiv lb, p, l(p-b), l(b-a)+p \pmod{lp}, \\ -1 & \text{if } i \equiv lb+p, l(p-a)+p \pmod{lp}, \\ 0 & \text{otherwise}. \end{cases}$$

Then, for any $h \in (\mathbb{Z}/lp\mathbb{Z})^\times$, we have

$$p(h) = \sum_{i\in\mathbb{Z}/lp\mathbb{Z}} \varepsilon(i)\Big\{-\frac{hi}{lp}\Big\} \quad \text{and} \quad \mathrm{Per}(H^{(h)}) \sim_{K'^\times} \prod_{i\in\mathbb{Z}/lp\mathbb{Z}} \Gamma\Big(\Big\{\frac{hi}{lp}\Big\}\Big)^{\varepsilon(i)}.$$

Proof For real numbers x, y with $0 < x, y < 1$, $x + y \ne 1$, put

$$\delta(x,y) := \{-x\} + \{-y\} - \{-(x+y)\} = \begin{cases} 1 & \text{if } x+y<1, \\ 0 & \text{if } x+y>1. \end{cases}$$

Then we have $\varphi(h) := \sum_i \varepsilon(i)\{-\frac{hi}{lp}\} = \delta(\beta,\mu) + \delta(1-\beta, \{\beta-\alpha+\mu\})$, where we put $\alpha = \{ha/p\}$, $\beta = \{hb/p\}$, $\mu = \{h/l\}$. First, we have $\varphi(h) = 2$ if and only if

$$\beta + \mu < 1, \quad 1 - \beta + \{\beta - \alpha + \mu\} < 1.$$

Letting $m = l\mu$, the first condition becomes $m < (1-\beta)l$, *i.e.*, $m \le \lfloor(1-\beta)l\rfloor$. Similarly, the second condition is equivalent to

$$(\alpha \le \beta, m < \alpha l) \quad \text{or} \quad \big(\alpha > \beta, (\alpha-\beta)l < m < \alpha l\big).$$

Comparing with Proposition 4.4, we have $p(h) = 2$ if and only if $\varphi(h) = 2$. Secondly, since $p(h) + p(-h) = \varphi(h) + \varphi(-h) = 2$, we have $p(h) = 0$ if and only of $\varphi(h) = 0$. Since $p(h), \varphi(h) \in \{0,1,2\}$, we have $p(h) = \varphi(h)$ for any h.

For the second statement, we compute the periods over the 2-cycle

$$(1-\tau)_*(1-\sigma)_*\Delta_1.$$

Since $(1-\zeta_l)(1-\zeta_p)$ is invertible in K, it reduces to the periods over Δ_1 (Proposition 2.6 (i)). First consider the two cases:

(i) $\alpha \le \beta$ and $p(h) \ge 1$,
(ii) $\alpha > \beta$ and $p(h) = 2$.

By Propositions 4.4 and 5.2, $H^{(h)}$ is generated by $\omega_{m,n}$ satisfying $\lceil(\alpha-\beta)l\rceil \le m$ in both cases, which is equivalent to $\alpha - \beta < \mu := m/l$. This is the assumption of Proposition 2.6 (i) and we obtain the desired formula.

The other cases are reduced to the ones above. If we replace χ with χ^{-1}, then h (resp. $\alpha, \beta, p(h)$) is replaced with $-h$ (resp. $1-\alpha, 1-\beta, 2-p(h)$). By Lemma 5.5, the cup-product $H^2(X)\otimes H^2(X) \to \mathbb{Q}(-2)$ induces an auto-duality on H, under which H^χ is dual to $H^{\chi^{-1}}$. Hence we have $\operatorname{Per}(H^{(h)})\cdot \operatorname{Per}(H^{(-h)}) \sim_{K^\times} (2\pi i)^2$. On the other hand, recall the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \sim_{K'^\times} 2\pi i,$$

for any $x \in \frac{1}{lp}\mathbb{Z}\smallsetminus\mathbb{Z}$. Therefore, the case where $\alpha \le \beta$ and $p(h) = 0$ (resp. $\alpha > \beta$ and $p(h) \ge 1$) is equivalent to case (ii) (resp. (i)). ■

Lemma 5.5 *Put $H^2(X)_Z = \operatorname{Ker}(H^2(X) \to H^2(Z))$. Then the composition*

$$H^2(X)_Z \hookrightarrow H^2(X) \twoheadrightarrow H^2(X)/\langle Z\rangle$$

induces an isomorphism of Hodge–de Rham structures $H^2(X)_Z \otimes_R K \simeq H$.

Proof This follows from the fact that the kernel of the composite

$$H^2_Z(X,\mathbb{C}) \to H^2(X,\mathbb{C}) \to H^2(Z,\mathbb{C})$$

is one-dimensional by Zariski's lemma [6, III, (8.2)]. ■

Remark 5.6 Our definition of ε is slightly different from [13]; $\varepsilon(i)$ here is $\varepsilon(-i)$, where Gross looks at the values $\Gamma(1-\{hi/lp\})^{\varepsilon(i)}$. The former conforms to the definition of the Stickelberger element as

$$\sum_{h\in(\mathbb{Z}/N\mathbb{Z})^\times} \left\{-\frac{h}{N}\right\}\sigma_h^{-1},$$

where $\sigma_h \in \operatorname{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ sends an N-th root of unity to its h-th power.

6 Regulators

After explaining the regulator map we are considering, we prove Theorem 1.2 from the introduction and its consequences on the non-vanishing.

6.1 Formulation

The Deligne cohomology of $X_\mathbb{C} := X \times_{\operatorname{Spec}\overline{\mathbb{Q}}} \operatorname{Spec}\mathbb{C}$ with coefficients in $\mathbb{Q}(2)$ is defined to be the hypercohomology of the complex $\mathbb{Q}(2) \to \mathscr{O}_{X_\mathbb{C}} \to \Omega^1_{X_\mathbb{C}/\mathbb{C}}$, where $\mathbb{Q}(2) := (2\pi i)^2\mathbb{Q}$ is placed in degree 0. Consider the Beilinson regulator map [7] from the motivic cohomology $r_{\mathscr{D}}: H^3_{\mathscr{M}}(X, \mathbb{Q}(2)) \to H^3_{\mathscr{D}}(X_{\mathbb{C}}, \mathbb{Q}(2))$. We have a natural isomorphism $H^3_{\mathscr{D}}(X_{\mathbb{C}}, \mathbb{Q}(2)) \simeq H^2(X, \mathbb{C})/\big(F^2 + H^2(X, \mathbb{Q}(2))\big)$, and the Carlson isomorphism

$$H^2(X, \mathbb{C})/(F^2 + H^2(X, \mathbb{Q}(2))) \simeq \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q}(2))).$$

Here MHS denotes the abelian category of $\mathbb{Q}$-mixed Hodge structures. By Poincaré duality $H^2(X, \mathbb{Q}(2)) \simeq H_2(X, \mathbb{Q})$, we obtain an identification

$$H^3_{\mathscr{D}}(X_{\mathbb{C}}, \mathbb{Q}(2)) \simeq \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q})).$$

Let $Z \subset X$ be as before and consider the regulator map

$$r_{\mathscr{D},Z}: H^3_{\mathscr{M},Z}(X, \mathbb{Q}(2)) \to H^3_{\mathscr{D},Z}(X, \mathbb{Q}(2)) \simeq H_1(Z, \mathbb{Q})$$

from the motivic cohomology supported on Z. Since $H_1(X, \mathbb{Q}) = 0$ by Proposition 4.1, we have an exact sequence of mixed Hodge structures

$$H_2(Z, \mathbb{Q}) \longrightarrow H_2(X, \mathbb{Q}) \longrightarrow H_2(X, Z; \mathbb{Q}) \xrightarrow{\partial} H_1(Z, \mathbb{Q}) \longrightarrow 0.$$

If we denote the image of the first map by $\langle Z \rangle$, we have the connecting homomorphism $\rho: H_1(Z, \mathbb{Q}) \cap H^{0,0} \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle)$, where $H^{0,0}$ denotes the Hodge $(0,0)$-component of $H_1(Z, \mathbb{C})$. By the lemma and Remark 6.2, ρ describes the restriction of $r_{\mathscr{D}}$ to the image of $H^3_{\mathscr{M},Z}(X, \mathbb{Q}(2))$.

Lemma 6.1 *The diagram below is commutative up to sign.*

$$\begin{array}{ccccc}
H^3_{\mathscr{M},Z}(X, \mathbb{Q}(2)) & \xrightarrow{r_{\mathscr{D},Z}} & H_1(Z, \mathbb{Q}) \cap H^{0,0} & \xrightarrow{\rho} & \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle) \\
\downarrow & & & & \uparrow \\
H^3_{\mathscr{M}}(X, \mathbb{Q}(2)) & \xrightarrow{r_{\mathscr{D}}} & H^3_{\mathscr{D}}(X_{\mathbb{C}}, \mathbb{Q}(2)) & \xrightarrow{\simeq} & \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q}))
\end{array}$$

where the vertical maps are the natural ones.

Proof See [5, Theorem 11.2]. ∎

Remark 6.2 The right vertical arrow is surjective since $\mathrm{Ext}^2_{\mathrm{MHS}} = 0$. Its kernel is topologically generated by decomposable elements, *i.e.*, the image of

$$(\mathrm{CH}_1(Z) \otimes \overline{\mathbb{Q}}^{\times}) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^3_{\mathscr{M},Z}(X, \mathbb{Q}(2)).$$

Also, it is not difficult to show that $r_{\mathscr{D},Z}$ is surjective.

6.2 Regulator Formula

Now we regard the extension classes as functionals (up to period functionals). Let $H^2(X)_Z = \mathrm{Ker}(H^2(X) \to H^2(Z))$ as before. Since $H^2(X, \mathbb{Q})_Z \simeq (H_2(X, \mathbb{Q})/\langle Z \rangle)^*$, we have

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_2(X, \mathbb{Q})/\langle Z \rangle) \simeq (F^1 H^2(X, \mathbb{C})_Z)^* / \,\mathrm{Image}\, H_2(X, \mathbb{Q}),$$

where $*$ denotes the $\mathbb{C}$-linear dual. By Lemma 5.5, ρ induces a map

$$\rho: (H_1(Z, \mathbb{Q}) \cap H^{0,0}) \otimes_R K \to (F^1 H_{\mathbb{C}})^* / H_B^{\vee},$$

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where $H_{\mathbb{C}} := H_B \otimes_{\mathbb{Q}} \mathbb{C}$ and $H^\vee$ denotes the dual Hodge–de Rham structure of H.

Put $Z_1 = \bigsqcup_{\zeta \in \mu_l} X_\zeta$. We shall describe the restriction of ρ to $H_1(Z_1, \mathbb{Q}) \otimes_R K$. Recall that $H_1(Z_1, \mathbb{Q}) \subset H^{0,0}$ (Corollary 3.6). We have, in fact, the following.

Lemma 6.3 *We have an isomorphism* $H_1(Z_1, \mathbb{Q}) \otimes_R K \xrightarrow{\simeq} H_1(Z, \mathbb{Q}) \otimes_R K$.

Proof By Proposition 4.2, τ acts trivially on $H_1(X_0, \mathbb{Q})$ and $H_1(X_\infty, \mathbb{Q}) = 0$. ∎

Let $(1-\sigma)_*\Delta_0 \in H_2(X, Z_1 ; \mathbb{Q})$ be the Lefschetz thimble defined in Section 2.5, and let $H_2(X, Z_1 ; \mathbb{Q})_{\mathrm{Lef}} \subset H_2(X, Z_1 ; \mathbb{Q})$ denote the R-submodule generated by this element.

Lemma 6.4 *The restriction of the boundary map*

$$\partial\colon H_2(X, Z_1; \mathbb{Q})_{\mathrm{Lef}} \otimes_R K \longrightarrow H_1(Z_1, \mathbb{Q}) \otimes_R K$$

is surjective and $H_1(Z_1, \mathbb{Q}) \otimes_R K$ *is one-dimensional over* K.

Proof By Proposition 4.2, $\dim_{\mathbb{Q}} H_1(X_\zeta, \mathbb{Q}) = p-1$ for $\zeta \in \mu_l$. Since τ permutes the components of Z_1, $H_1(Z_1, \mathbb{Q}) \otimes_R K$ is one-dimensional over K. Whereas κ_0 and κ_1 generate $H_1(X_t, \mathbb{Q})$ (Proposition 2.4 (ii)), κ_1 vanishes as $t \to 1$ by definition. Therefore κ_0 does not vanish, *i.e.*, $\partial((1-\sigma)_*\Delta_0)$ is non-trivial in $H_1(X_1, \mathbb{Q})$, hence is in $H_1(Z_1, \mathbb{Q}) \otimes_R K$. ∎

Now we state our main theorem. For $x \in K$, let x_* (resp. x^*) denote its action on homology (resp. cohomology). Since $1-\zeta_p$ is invertible in K, we write

$$((1-\zeta_p)^{-1})_*(1-\sigma)_*\Delta_0 \in H_1(X, Z_1 ; \mathbb{Q}) \otimes_R K$$

simply as Δ_0. For each m and n, define an embedding $\chi_{m,n}\colon K \hookrightarrow \mathbb{C}$ by

$$\chi_{m,n}(\zeta_l) = \zeta_l^m, \quad \chi_{m,n}(\zeta_p) = \zeta_p^n.$$

Theorem 6.5 *Let* $\gamma \in H_1(Z_1, \mathbb{Q}) \otimes_R K$ *and take* $x \in K$ *such that* $\gamma = x_*\partial\Delta_0$. *Let* $\{\omega_{m,n} \mid n = 1, \dots, p-1, m \in I_n^1\}$ *be the basis of* $F^1 H_{\mathrm{dR}}$ *given in Proposition 5.2. Then we have*

$$\rho(\gamma)(\omega_{m,n}) = \chi_{m,n}(x) \frac{B(1-\alpha, \beta)}{l(\beta-\alpha+\mu)} \cdot F\left({1-\alpha, \beta, \beta-\alpha+\mu \atop 1-\alpha+\beta, \beta-\alpha+\mu+1} ; 1 \right),$$

where $\alpha = \{\frac{na}{p}\}$, $\beta = \{\frac{nb}{p}\}$, $\mu = \frac{m}{l}$.

Proof We apply Theorem A.3 of the appendix to our situation, where $D = Z_1$ and $X^\circ = X \smallsetminus (X_0 \cup X_\infty)$ (see the proof of Lemma 5.1). Note that $H_{\mathbb{C}} \simeq H^2_{\mathrm{dR}}(X_{\mathbb{C}})_0 \otimes_R K$ by Lemma 5.5 since τ acts trivially on $H^2_{\mathrm{dR}}(e(\mathbb{P}^1_{\mathbb{C}}))$ (see §A.2 for the notations).

Put $\Gamma = (1-\tau)_*(1-\sigma)_*\Delta_0$. Since $\Gamma \in H_2(X, Z_1 ; \mathbb{Q})$ does not necessarily come from $H_2(X^\circ, Z_1 ; \mathbb{Q})$, we take a detour. Let Γ' be the Lefschetz thimble given by sweeping $(1-\sigma)_*\delta_0$ along the path $\kappa_1 + \kappa_2 + \kappa_3$ in $T \smallsetminus \{0, \infty\}$, where κ_1 is the line segment from ζ to $\varepsilon\zeta$ ($\varepsilon > 0$), κ_2 is the arc from $\varepsilon\zeta$ to ε, and κ_3 is the line segment from ε to 1. Then $\Gamma' \in H_2(X^\circ, Z_1 ; \mathbb{Q})$ and $\gamma := \partial(\Gamma) = \partial(\Gamma')$. Theorem A.3 yields $\rho(\gamma)(\omega_{m,n}) = \int_{\Gamma'} \omega_{m,n}$. The right integral is computed similarly as Proposition 2.6

(ii), and letting $\varepsilon \to 0$, we obtain the theorem for $x = (1-\zeta_l)(1-\zeta_p)$. The general case follows by the cyclicity of $H_1(Z_1, \mathbb{Q}) \otimes_R K$. ∎

6.3 Non-vanishing

We prove the non-vanishing of ρ under a mild assumption. The situation is different depending on whether $a + b = p$ or not.

If $a + b \neq p$, the regulator does not vanish even in the Deligne cohomology with $\mathbb{R}$-coefficients, or equivalently, the extension group of $\mathbb{R}$-mixed Hodge structures

$$\operatorname{Ext}^1_{\mathbb{R}\mathrm{MHS}}(\mathbb{R}, H_{\mathbb{R}}) \simeq (F^1 H_{\mathbb{C}})^* / H_{\mathbb{R}}^{\vee},$$

where $H_{\mathbb{R}} = H_B \otimes_{\mathbb{Q}} \mathbb{R}$, $H_{\mathbb{C}} = H_B \otimes_{\mathbb{Q}} \mathbb{C}$. Note that $\dim_{\mathbb{R}}(F^1 H_{\mathbb{C}})^*/H_{\mathbb{R}}^{\vee} = \dim_{\overline{\mathbb{Q}}} \operatorname{Gr}^1_F H_{\mathrm{dR}}$. Let $\rho_{\mathbb{R}} \colon H_1(Z_1, \mathbb{Q}) \otimes_R K \to (F^1 H_{\mathbb{C}})^*/H_{\mathbb{R}}^{\vee}$ be the composition of ρ and the natural surjection.

Theorem 6.6 *Suppose that $p < l$ and $a + b \neq p$ (so $p > 2$). Then $\rho_{\mathbb{R}}$ is non-trivial. In particular,* $\dim_{\mathbb{Q}} \rho_{\mathbb{R}}(H_1(Z_1, \mathbb{Q}) \otimes_R K) = (l-1)(p-1)$.

Proof By restricting the functionals to $F^1 H_{\mathbb{R}} := F^1 H_{\mathbb{C}} \cap H_{\mathbb{R}}$ and taking the imaginary part, we obtain a $K \cap \mathbb{R}$-linear map $\rho'_{\mathbb{R}} \colon H_1(Z_1, \mathbb{Q}) \otimes_R K \to \operatorname{Hom}(F^1 H_{\mathbb{R}}, i\mathbb{R})$. For each $n = 1, \dots, p-1$, we have $\alpha \neq 1 - \beta$ by the assumption. Hence $|\alpha - (1-\beta)| \geq 1/p > 1/l$ and there exists an m satisfying

$$\min\{\lfloor \alpha l \rfloor, \lfloor (1-\beta) l \rfloor\} < m \leq \max\{\lfloor \alpha l \rfloor, \lfloor (1-\beta) l \rfloor\}. \tag{6.1}$$

Then we have $\omega_{m,n} \in \operatorname{Gr}^1_F H_{\mathrm{dR}}$ by Propositions 4.4 and 5.2. Since $m > \lfloor (\alpha - \beta) l \rfloor$, we have $\mu := m/l > \alpha - \beta$, hence we can apply Proposition 2.6 (i) to compute the period

$$\Omega_{m,n} := \int_{\Delta_1} \omega_{m,n} = -\frac{(-1)^{p\beta}}{l} B(\beta, \mu) B(1-\beta, \beta - \alpha + \mu).$$

Put a normalization as $\widetilde{\omega}_{m,n} = \Omega_{m,n}^{-1} \omega_{m,n}$. Then we have

$$\int_{x_* \Delta_1} \widetilde{\omega}_{m,n} = \int_{\Delta_1} x^* \widetilde{\omega}_{m,n} = \chi_{m,n}(x),$$

for any $x \in K$. If we let $n' = p - n$, $\alpha' = \{n'a/p\} = 1 - \alpha$, $\beta' = \{n'b/p\} = 1 - \beta$, $m' = l - m$, and $\mu' = \{m'/l\} = 1 - \mu$, then these satisfy the assumption (6.1). Hence, $\widetilde{\omega}_{m',n'}$ is defined and we have $\int_{x_* \Delta_1} \widetilde{\omega}_{m',n'} = \overline{\chi_{m,n}(x)}$, for any $x \in K$. Since $H_B^{\vee}$ is generated as a K-module by $((1-\zeta_l)^{-1}(1-\zeta_p)^{-1})_*(1-\tau)_*(1-\sigma)_* \Delta_1$, that we simply denote Δ_1 as before, we have $\overline{\widetilde{\omega}_{m,n}} = \widetilde{\omega}_{m',n'}$ and hence

$$\widetilde{\omega}_{m,n} + \widetilde{\omega}_{m',n'} \in F^1 H_{\mathbb{R}}.$$

Define the regulator as

$$R_{m,n} := \int_{\Delta_0} \omega_{m,n} = \frac{B(1-\alpha, \beta)}{l(\beta - \alpha + \mu)} \cdot F\left(\begin{matrix} 1-\alpha, \beta, \beta - \alpha + \mu \\ 1 - \alpha + \beta, \beta - \alpha + \mu + 1 \end{matrix} ; 1 \right).$$

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By Theorem 6.5, for any $\gamma \in H_1(Z_1, \mathbb{Q})$ corresponding to $x \in K$ as in Theorem 6.5 we have

$$\begin{aligned}\rho'_{\mathbb{R}}(\gamma)(\widetilde{\omega}_{m,n} + \widetilde{\omega}_{m',n'}) &= \operatorname{Im}\left(\chi_{m,n}(x)\Omega^{-1}_{m,n}R_{m,n} + \overline{\chi_{m,n}(x)}\Omega^{-1}_{m',n'}R_{m',n'}\right)\\ &= \operatorname{Im}(\chi_{m,n}(x))\left(\Omega^{-1}_{m,n}R_{m,n} - \Omega^{-1}_{m',n'}R_{m',n'}\right).\end{aligned}$$

Since $\Omega_{m,n}\Omega_{m',n'} < 0$ and $R_{m,n}, R_{m',n'} > 0$, the above does not vanish for $x \in K \setminus \mathbb{R}$. Hence $\rho_{\mathbb{R}}$ is non-trivial. Since $\rho_{\mathbb{R}}$ is K-linear, the second assertion follows. ■

The non-vanishing of ρ is a more subtle problem. For the case $a + b = p$, we have the following criterion.

Proposition 6.7 *Let p, l be distinct prime numbers and suppose that $a + b = p$. If ρ is trivial, then there exists an $x \in K$ such that $R_{m,n} = \chi_{m,n}(x)\Omega_{m,n}$, for any $n = 1, \dots, p-1$, and $m \in I^1_n$ such that $\frac{m}{l} > \{\frac{na}{p}\} - \{\frac{nb}{p}\}$.*

Proof Let $\gamma = \partial\Delta_0$ and suppose that $\rho(\gamma) = 0$. Since $H_B^\vee$ is generated by Δ_1 over K, there exists an $x \in K$ such that $\rho(\gamma)$ is represented by the functional $\int_{x_*\Delta_1}$. If m, n are as in the statement, then $\int_{x_*\Delta_1}\omega_{m,n} = \int_{\Delta_1} x^*\omega_{m,n} = \chi_{m,n}(x)\Omega_{m,n}$ by the definition. Hence the proposition follows. ■

Example 6.8 If $p = 2$, then $\alpha = \beta = 1/2$ and Y is nothing but the Legendre family of elliptic curves. By Proposition 4.8, we have $\operatorname{Gr}^1_F H_{\mathrm{dR}} = 0$ and the Deligne cohomology with $\mathbb{R}$-coefficients is trivial. Since the condition $\frac{m}{l} > \{\frac{na}{p}\} - \{\frac{nb}{p}\}$ $(= 0)$ is automatically satisfied, Proposition 6.7 is, in fact, an equivalence. If, for example, $l = 3$, then ρ is trivial if and only if

$$\sqrt{3}\left(\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}\right)^2 \cdot F\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{3}\\ 1, \frac{4}{3}\end{matrix}; 1\right) \in \mathbb{Q}.$$

Here we used $\mathbb{Q}(\zeta_3) \cap i\mathbb{R} = \sqrt{3}i\mathbb{Q}$.

A Appendix: (M. Asakura) Fibration of Curves and Extension of Motives

In this appendix, we develop a technique that was used in the proof of the regulator formula (Theorem 6.5) to compute regulators for a fibration of curves and motivic elements constructed from degenerating fibers [3].

A.1 Relative Cohomology

Let V be a quasi-projective smooth surface over $\mathbb{C}$. Let $D \subset V$ be a chain of curves. Let $\pi\colon \widetilde{D} \to D$ be the normalization and $\Sigma \subset D$ be the set of singular points. Let $s\colon \widetilde{\Sigma} := \pi^{-1}(\Sigma) \hookrightarrow \widetilde{D}$ be the inclusion. There is an exact sequence

$$0 \longrightarrow \mathscr{O}_D \xrightarrow{\pi^*} \mathscr{O}_{\widetilde{D}} \xrightarrow{s^*} \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_{\Sigma} \longrightarrow 0,$$

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where $\mathbb{C}_{\widetilde{\Sigma}} = \mathrm{Maps}(\widetilde{\Sigma}, \mathbb{C}) = \mathrm{Hom}(\mathbb{Z}\widetilde{\Sigma}, \mathbb{C})$ and π^*, s^* are the pull-backs. For a smooth manifold M, let $\mathscr{A}^q(M)$ denote the space of smooth differential q-forms on M with coefficients in $\mathbb{C}$. We define $\mathscr{A}^\bullet(D)$ to be the mapping fiber of $s^*: \mathscr{A}^\bullet(\widetilde{D}) \to \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma$:

$$\mathscr{A}^0(\widetilde{D}) \xrightarrow{s^* \oplus d} \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \oplus \mathscr{A}^1(\widetilde{D}) \xrightarrow{0 \oplus d} \mathscr{A}^2(\widetilde{D}),$$

where the first term is placed in degree 0. Then $H^q_{\mathrm{dR}}(D) = H^q(\mathscr{A}^\bullet(D))$ is the de Rham cohomology of D, which fits into the exact sequence

$$\cdots \longrightarrow H^0_{\mathrm{dR}}(\widetilde{D}) \longrightarrow \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \longrightarrow H^1_{\mathrm{dR}}(D) \longrightarrow H^1_{\mathrm{dR}}(\widetilde{D}) \longrightarrow \cdots.$$

We have the natural pairing

$$\langle \cdot, \cdot \rangle_D: H_1(D, \mathbb{Z}) \otimes H^1_{\mathrm{dR}}(D) \longrightarrow \mathbb{C}, \quad \gamma \otimes z \longmapsto \int_\gamma \eta - c(\partial(\pi^{-1}\gamma)),$$

where z is represented by $(c, \eta) \in \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \oplus \mathscr{A}^1(\widetilde{D})$ with $d\eta = 0$ and ∂ denotes the boundary of homology cycles.

We define $\mathscr{A}^\bullet(V, D)$ to be the mapping fiber of $\widetilde{i}^*: \mathscr{A}^\bullet(V) \to \mathscr{A}^\bullet(\widetilde{D})$, the pull-back by $\widetilde{i}: \widetilde{D} \to V$:

$$\mathscr{A}^0(V) \xrightarrow{\mathscr{D}_0} \mathscr{A}^0(\widetilde{D}) \oplus \mathscr{A}^1(V) \xrightarrow{\mathscr{D}_1} \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \oplus \mathscr{A}^1(\widetilde{D}) \oplus \mathscr{A}^2(V) \xrightarrow{\mathscr{D}_2} \cdots.$$

Then the relative de Rham cohomology is defined by $H^q_{\mathrm{dR}}(V, D) = H^q(\mathscr{A}^\bullet(V, D))$ and fits into the exact sequence

$$\text{(A.1)} \qquad \cdots \longrightarrow H^{q-1}_{\mathrm{dR}}(D) \longrightarrow H^q_{\mathrm{dR}}(V, D) \longrightarrow H^q_{\mathrm{dR}}(V) \longrightarrow H^q_{\mathrm{dR}}(D) \longrightarrow \cdots.$$

An element of $H^2_{\mathrm{dR}}(V, D)$ is represented by

$$\text{(A.2)} \qquad (c, \eta, \omega) \in \mathbb{C}_{\widetilde{\Sigma}}/\mathbb{C}_\Sigma \oplus \mathscr{A}^1(\widetilde{D}) \oplus \mathscr{A}^2(V)$$

that satisfies $\widetilde{i}^*\omega = d\eta$ and $d\omega = 0$. The natural pairing

$$\langle\ ,\ \rangle_{V,D}: H_2(V, D; \mathbb{Z}) \otimes H^2_{\mathrm{dR}}(V, D) \longrightarrow \mathbb{C}$$

is given by

$$\langle \Gamma, z \rangle_{V,D} = \int_\Gamma \omega - \langle \partial\Gamma, (c, \eta) \rangle_D = \int_\Gamma \omega - \int_{\partial\Gamma} \eta + c(\partial(\pi^{-1}(\partial\Gamma))).$$

The complexes $\mathscr{A}^\bullet(V)$ and $\mathscr{A}^\bullet(D)$ are canonically equipped with Hodge and weight filtrations; then $(\mathbb{Q}_V, \mathscr{A}^\bullet(V), F^\bullet, W_\bullet)$ and $(\mathbb{Q}_D, \mathscr{A}^\bullet(D), F^\bullet, W_\bullet)$ become cohomological mixed Hodge complexes in the sense of [10, (8.1.2)]. The Hodge and weight filtrations on $\mathscr{A}^\bullet(V, D)$ are induced from them and the data

$$(\mathbb{Q}_{V,D}, \mathscr{A}^\bullet(V, D), F^\bullet, W_\bullet)$$

becomes a cohomological mixed Hodge complex as well. Hence we have an exact sequence

$$\cdots \longrightarrow H^{q-1}(D, \mathbb{Q}) \longrightarrow H^q(V, D; \mathbb{Q}) \longrightarrow H^q(V, \mathbb{Q}) \longrightarrow H^q(D, \mathbb{Q}) \longrightarrow \cdots$$

of mixed Hodge structures which is compatible with (A.1). Taking its dual, we obtain an exact sequence

$$0 \longrightarrow H_2(V, \mathbb{Q})/H_2(D) \longrightarrow H_2(V, D; \mathbb{Q}) \xrightarrow{\partial} H_1(D, \mathbb{Q}) \longrightarrow H_1(V, \mathbb{Q}).$$

Since $H_1(V,\mathbb{Q}) \cap H^{0,0} = 0$, we obtain the coboundary map

$$\rho_{V,D}\colon H_1(D,\mathbb{Q}) \cap H^{0,0} \longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_2(V,\mathbb{Q})/H_2(D)) \tag{A.3}$$

to the extension group of mixed Hodge structures. If we put

$$H^2_{\mathrm{dR}}(V)_D := \mathrm{Ker}[H^2_{\mathrm{dR}}(V) \longrightarrow H^2_{\mathrm{dR}}(D)],$$

then we have the Carlson isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H_2(V,\mathbb{Q})/H_2(D)) \simeq \mathrm{Coker}\left[H_2(V,\mathbb{Q}) \longrightarrow (F^1 H^2_{\mathrm{dR}}(V)_D)^*\right], \tag{A.4}$$

where $*$ denotes the $\mathbb{C}$-linear dual and the map is the natural pairing. Under this identification, the map $\rho_{V,D}$ is described as follows. For $\gamma \in H_1(D,\mathbb{Q}) \cap H^{0,0}$, take a $\Gamma \in H_2(V,D\,;\mathbb{Q})$ such that $\partial(\Gamma) = \gamma$. Then we have

$$\rho_{V,D}(\gamma) = [\omega \longmapsto \langle \Gamma, \omega_{V,D}\rangle_{V,D}], \tag{A.5}$$

where $\omega_{V,D} \in F^1 H^2_{\mathrm{dR}}(V,D)$ is a lifting of ω, on which the pairing does not depend.

A.2 Rational Forms

For a given ω, it is usually complicated to compute an analytic lifting $\omega_{V,D}$ explicitly. In the following situation, we shall be able to associate a *rational* 2-form via Deligne's canonical extension, which gives a simple expression of $\rho_{V,D}$.

Let C be a projective smooth curve over $\mathbb{C}$ and $f\colon X \to C$ be a fibration of curves with connected general fiber that admits a section $e\colon C \to X$. Henceforth, we use the algebraic de Rham cohomology groups [14] and identify them with the analytic ones in the previous paragraph. For a Zariski open set $S \subset C$, let $V = f^{-1}(S)$ and put

$$H^2_{\mathrm{dR}}(V)_0 = \mathrm{Ker}\left[H^2_{\mathrm{dR}}(V) \to \prod_{s\in S} H^2_{\mathrm{dR}}(f^{-1}(s)) \times H^2_{\mathrm{dR}}(e(S))\right],$$
$$H^2_{\mathrm{dR}}(V,D)_0 = \mathrm{Ker}\left[H^2_{\mathrm{dR}}(V,D) \to H^2_{\mathrm{dR}}(V)/H^2_{\mathrm{dR}}(V)_0\right].$$

Then we have an exact sequence of mixed Hodge structures

$$H^1_{\mathrm{dR}}(V) \longrightarrow H^1_{\mathrm{dR}}(D) \longrightarrow H^2_{\mathrm{dR}}(V,D)_0 \longrightarrow H^2_{\mathrm{dR}}(V)_0 \longrightarrow 0. \tag{A.6}$$

The arrows are strictly compatible with the Hodge and weight filtrations. In particular, $F^1 H^2_{\mathrm{dR}}(V,D)_0 \to F^1 H^2_{\mathrm{dR}}(V)_0$ is surjective. Later, we shall use the following.

Lemma A.1 *Let $g\colon V' \to V$ be a birational transformation that is an isomorphism outside D and put $D' = g^{-1}(D)$. Then the pull-back g^* induces isomorphisms*

$$H^2_{\mathrm{dR}}(V)_0 \simeq H^2_{\mathrm{dR}}(V')_0 \quad \textit{and} \quad H^2_{\mathrm{dR}}(V,D)_0 \simeq H^2_{\mathrm{dR}}(V',D')_0.$$

Proof By (A.6) it is enough to show isomorphisms

$$H^1_{\mathrm{dR}}(V) \simeq H^1_{\mathrm{dR}}(V'), \quad H^1_{\mathrm{dR}}(D) \simeq H^1_{\mathrm{dR}}(D'), \quad H^2_{\mathrm{dR}}(V)_0 \simeq H^2_{\mathrm{dR}}(V')_0.$$

The first one is an easy exercise. Let X' be a smooth compactification of V' such that $X' \smallsetminus D' \simeq X \smallsetminus D$ and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
H^2_{\mathrm{dR}}(X') & \xrightarrow{a^2} & H^2_{\mathrm{dR}}(X' \smallsetminus D') & \longrightarrow & H_1^{\mathrm{dR}}(D') & \longrightarrow & H^3_{\mathrm{dR}}(X') & \xrightarrow{a^3} & H^3_{\mathrm{dR}}(X' \smallsetminus D') \\
{\scriptstyle g_*}\downarrow & & \| & & {\scriptstyle g_*}\downarrow & & {\scriptstyle g_*}\downarrow{\scriptstyle \simeq} & & \| \\
H^2_{\mathrm{dR}}(X) & \xrightarrow{b^2} & H^2_{\mathrm{dR}}(X \smallsetminus D) & \longrightarrow & H_1^{\mathrm{dR}}(D) & \longrightarrow & H^3_{\mathrm{dR}}(X) & \xrightarrow{b^3} & H^3_{\mathrm{dR}}(X \smallsetminus D).
\end{array}$$

The second isomorphism follows from the fact that

$$\mathrm{Image}(a^n) = \mathrm{Image}(b^n) = W_n H^n_{\mathrm{dR}}(X \smallsetminus D).$$

The last isomorphism follows from the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^2_{\mathrm{dR}}(V)_0 & \longrightarrow & H^2_{\mathrm{dR}}(V \smallsetminus D)_0 & \longrightarrow & H_1^{\mathrm{dR}}(D) \\
 & & {\scriptstyle g^*}\downarrow & & \| & & {\scriptstyle g^*}\downarrow{\scriptstyle \simeq} \\
0 & \longrightarrow & H^2_{\mathrm{dR}}(V')_0 & \longrightarrow & H^2_{\mathrm{dR}}(V' \smallsetminus D')_0 & \longrightarrow & H_1^{\mathrm{dR}}(D')
\end{array}$$

with exact rows. ■

Now fix a Zariski open set $S \subset C$ such that $U := f^{-1}(S) \to S$ is smooth. Put $T = C \smallsetminus S$ and $Z = X \smallsetminus U$. Let $\nabla\colon \mathscr{H}_e \to \Omega^1_C(\log T) \otimes \mathscr{H}_e$ be the Deligne canonical extension of the Gauss–Manin connection $(\mathscr{H} = R^1 f_* \Omega^\bullet_{U/S}, \nabla)$. Put $F^1\mathscr{H}_e = j_* F^1\mathscr{H} \cap \mathscr{H}_e$, where $j\colon S \hookrightarrow C$ and $\mathrm{Gr}^0_F \mathscr{H}_e = \mathscr{H}_e / F^1\mathscr{H}_e$. Let $\overline{\nabla}\colon F^1\mathscr{H}_e \to \Omega^1_C(\log T) \otimes \mathrm{Gr}^0_F \mathscr{H}_e$ be the $\mathscr{O}_C$-linear map induced from ∇. In what follows, we assume the following.

$(*)$ The map $\overline{\nabla}$ is generically bijective.

Let $C^\circ \subset C$ be a Zariski open set on which $\overline{\nabla}$ is bijective and put $X^\circ := f^{-1}(C^\circ)$. Note that $S \not\subset C^\circ$ in general and $X^\circ \to C^\circ$ is not necessarily smooth. Then the commutative diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & \Omega^1_C(\log T) \otimes F^1\mathscr{H}_e \\
 & & \downarrow \\
F^1\mathscr{H}_e & \xrightarrow{\nabla} & \Omega^1_C(\log T) \otimes \mathscr{H}_e \\
{\scriptstyle =}\downarrow & & \downarrow \\
F^1\mathscr{H}_e & \xrightarrow{\overline{\nabla}} & \Omega^1_C(\log T) \otimes \mathrm{Gr}^0_F \mathscr{H}_e \\
 & & \downarrow \\
 & & 0
\end{array}$$

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induces an isomorphism

$$\Lambda^\circ := \Gamma(C^\circ, \Omega^1_C(\log T)\otimes F^1\mathscr{H}_e) \xrightarrow{\simeq} H^1(C^\circ, F^1\mathscr{H}_e \longrightarrow \Omega^1_C(\log T)\otimes \mathscr{H}_e).$$

Note that $\Lambda^\circ \subset \Gamma(X^\circ, \Omega^2_X(\log Z))$.

Lemma A.2 *There are natural injections $F^1H^2_{\mathrm{dR}}(X)_0 \hookrightarrow F^1H^2_{\mathrm{dR}}(U)_0 \hookrightarrow \Lambda^\circ$.*

Proof The first injectivity follows from Zariski's lemma [6, III, (8.2)]. Since

$$H^2_{\mathrm{dR}}(U)_0 \simeq H^1(S, \mathscr{H} \to \Omega^1_S \otimes \mathscr{H}) \simeq H^1(C, \mathscr{H}_e \to \Omega^1_C(\log T)\otimes\mathscr{H}_e)$$

and

$$F^1H^1(S,\mathscr{H}\to\Omega^1_S\otimes\mathscr{H}) = H^1(C, F^1\mathscr{H}_e \to \Omega^1_C(\log T)\otimes\mathscr{H}_e)$$

[26, §5], the second injectivity follows from that of $F^1H^2_{\mathrm{dR}}(U)_0 \to F^1H^2_{\mathrm{dR}}(U\cap X^\circ)_0$. ∎

Define $\Lambda(X)\subset\Lambda(U)\subset\Lambda^\circ$ to be the images of $F^1H^2_{\mathrm{dR}}(X)_0$, $F^1H^2_{\mathrm{dR}}(U)_0$, respectively. By the commutative diagram

$$\begin{array}{ccccc} F^1H^2_{\mathrm{dR}}(X)_0 & \longrightarrow & F^1H^2_{\mathrm{dR}}(U)_0 & \longrightarrow & H_1^{\mathrm{dR}}(Z) \\ \simeq\downarrow & & \simeq\downarrow & & \downarrow \\ \Lambda(X) & \longrightarrow & \Lambda(U) & \longrightarrow & H^0(X^\circ, \Omega^2_X(\log Z)/\Omega^2_X) \end{array}$$

we have $\Lambda(X)\subset\Gamma(X^\circ,\Omega^2_X)$. For any cohomology class $\omega\in F^1H^2_{\mathrm{dR}}(X)_0$, let $\omega^\circ\in\Lambda(X)$ denote the corresponding rational 2-form.

A.3 Main Result

Now let $D\subset X^\circ$ be a finite union of fibers. We give a description of the map

$$\rho_{X,D}\colon H_1(D,\mathbb{Q})\cap H^{0,0} \longrightarrow \mathrm{Coker}[H_2(X,\mathbb{Q})\to(F^1H^2_{\mathrm{dR}}(X)_0)^*]$$

induced from (A.3), (A.4), and the inclusion $F^1H^2_{\mathrm{dR}}(X)_0 \subset F^1H_{\mathrm{dR}}(X)_D$. Note that this factors through $\rho_{X^\circ,D}$. We regard an element $\eta\in\Lambda^\circ$ as an element of $\mathscr{A}^2(X^\circ)$. For the dimension reasons, we have $\widetilde{i}^*\eta = 0$ and $d\eta=0$. Hence $(0,0,\eta)$ as in (A.2) defines a cohomology class $\widehat{\eta}\in H^2_{\mathrm{dR}}(X^\circ,D)$. Note that $\widehat{\eta}$ does not necessarily belong to F^1. For any $\omega\in F^1H^2_{\mathrm{dR}}(X)_0$, write $\widehat{\omega}$ instead of $\widehat{\omega^\circ}$.

Theorem A.3

(i) *For any $\omega\in F^1H^2_{\mathrm{dR}}(X)_0$, we have $\widehat{\omega}\in F^1H^2_{\mathrm{dR}}(X^\circ,D)_0$ and it lifts $\omega|_{X^\circ}$.*

(ii) *For any $\gamma\in H_1(D,\mathbb{Q})\cap H^{0,0}$, choose $\Gamma\in H_2(X^\circ,D)$ such that $\partial(\Gamma)=\gamma$. Then we have $\rho_{X,D}(\gamma) = [\omega\mapsto\int_\Gamma\omega^\circ]$.*

Proof By (A.5), assertion (ii) follows immediately from (i). By Lemma A.1, we may assume that D_{red} and Z_{red} are divisors with normal crossings. It suffices to prove the

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case where $D = f^{-1}(P)$, $P \in C^\circ$. For a Zariski sheaf $\mathscr{F}$, let $(\check{C}^\bullet(\mathscr{F}), \delta)$ denote its Čech complex. First, $H^2_{\mathrm{dR}}(X)$ is given by the cohomology in the middle of the complex

$$\check{C}^1(\mathscr{O}_X) \times \check{C}^0(\Omega^1_X) \xrightarrow{\mathscr{D}_1} \check{C}^2(\mathscr{O}_X) \times \check{C}^1(\Omega^1_X) \times \check{C}^0(\Omega^2_X) \xrightarrow{\mathscr{D}_2} \check{C}^3(\mathscr{O}_X) \times \check{C}^2(\Omega^1_X) \times \check{C}^1(\Omega^2_X).$$

A description of $H^2_{\mathrm{dR}}(U) = H^2(X, \Omega^\bullet_X(\log Z))$ is given similarly. Finally, $H^2_{\mathrm{dR}}(X, D)$ is given by the complex

$$\check{C}^1(\mathscr{O}_X) \times \check{C}^0(\mathscr{O}_{\widetilde{D}} \oplus \Omega^1_X) \xrightarrow{\mathscr{D}_3} \check{C}^2(\mathscr{O}_X) \times \check{C}^1(\mathscr{O}_{\widetilde{D}} \oplus \Omega^1_X) \times \check{C}^0(\mathscr{O}_{\widetilde{\Sigma}}/\mathscr{O}_\Sigma \oplus \Omega^1_{\widetilde{D}} \oplus \Omega^2_X) \xrightarrow{\mathscr{D}_4} \check{C}^3(\mathscr{O}_X) \times \check{C}^2(\mathscr{O}_{\widetilde{D}} \oplus \Omega^1_X) \times \check{C}^1(\mathscr{O}_{\widetilde{\Sigma}}/\mathscr{O}_\Sigma \oplus \Omega^1_{\widetilde{D}} \oplus \Omega^2_X).$$

Let $\omega \in F^1H^2_{\mathrm{dR}}(X)_0$ and take its representative $z = (0) \times (\alpha_{ij}) \times (\beta_i) \in \mathrm{Ker}(\mathscr{D}_2)$. Since $\omega \in F^1H^2_{\mathrm{dR}}(X)_D$, there exists $(\epsilon_i) \in \check{C}^0(\Omega^1_{\widetilde{D}})$ such that $\alpha_{ij}|_{\widetilde{D}} = \epsilon_j - \epsilon_i$. If we put $z_{X,D} = (0) \times (0, \alpha_{ij}) \times (0, \epsilon_i, \beta_i)$, then $z_{X,D} \in \mathrm{Ker}(\mathscr{D}_4)$. By the definition of the Hodge filtration, it represents a class $\omega_{X,D} \in F^1H^2_{\mathrm{dR}}(X, D)$ that lifts ω. Let $\omega_{X,D}|_{X^\circ}$ be its image in $H^2_{\mathrm{dR}}(X^\circ, D)$.

Let $\widehat{\omega} \in H^2_{\mathrm{dR}}(X^\circ, D)$ be the class of the Čech cocycle $\widehat{z} := (0) \times (0, 0) \times (0, 0, \omega^\circ)$. The group $H^1(C^\circ, F^1\mathscr{H}_e \to \Omega^1_C(\log T) \otimes \mathscr{H}_e)$ is given by the complex

$$\check{C}^0(F^1\mathscr{H}_e|_{C^\circ}) \xrightarrow{\mathscr{D}_5} \check{C}^1(F^1\mathscr{H}_e|_{C^\circ}) \times \check{C}^0(\Omega^1_C(\log T) \otimes \mathscr{H}_e|_{C^\circ}) \xrightarrow{\mathscr{D}_6} \check{C}^2(F^1\mathscr{H}_e|_{C^\circ}) \times \check{C}^1(\Omega^1_C(\log T) \otimes \mathscr{H}_e|_{C^\circ}).$$

By the definition of ω°, there exists $y = (\nu_i) \in \check{C}^0(F^1\mathscr{H}_e|_{C^\circ})$ such that $\mathscr{D}_5(y) = (\alpha_{ij}) \times (\beta_i) - (0) \times (\omega^\circ)$, *i.e.*, $\nu_j - \nu_i = \alpha_{ij}$, $d\nu_i = \beta_i - \omega^\circ$. Hence we have

$$z_{X,D}|_{X^\circ} - \widehat{z} = (0) \times (0, \nu_j - \nu_i) \times (0, \varepsilon_i, d\nu_i).$$

It is clear that this vanishes in $H^2_{\mathrm{dR}}(X^\circ)$, hence $\widehat{\omega}$ lifts $\omega|_{X^\circ}$.

We are left to show that the class of $\widehat{\omega}$ lies in F^1. Let V be a sufficiently small neighborhood of D so that we have an exact sequence

$$0 \longrightarrow \Omega^1_V \longrightarrow \Omega^1_V(\log D) \xrightarrow{\mathrm{Res}} \widetilde{i}_*\mathscr{O}_{\widetilde{D}} \longrightarrow 0.$$

Since $H^2_{\mathrm{dR}}(X^\circ, D)/F^1 \to H^2_{\mathrm{dR}}(V, D)/F^1$ is injective, it suffices to show the claim after restricting to V. Since $\mathrm{Res}(\nu_j) - \mathrm{Res}(\nu_i) = \mathrm{Res}(\alpha_{ij}) = 0$, $(\mathrm{Res}(\nu_i))$ defines a class $e \in H^0(\widetilde{D}, \mathscr{O}_{\widetilde{D}})$. Consider the composite

$$H^0(\widetilde{D}, \mathscr{O}_{\widetilde{D}}) \xrightarrow{\delta} H^1(V, \Omega^1_V) \xrightarrow{\widetilde{i}^*} H^1(\widetilde{D}, \Omega^1_{\widetilde{D}}) \simeq H^2_{\mathrm{dR}}(\widetilde{D}),$$

where δ is the connecting map. Then $(\widetilde{i}^* \circ \delta)(e)$ is represented by $(\alpha_{ij}|_{\widetilde{D}}) \in \check{C}^1(\Omega_{\widetilde{D}})$. Therefore, under the above isomorphism, $(\widetilde{i}^* \circ \delta)(e)$ corresponds to $\widetilde{i}^*(\omega) = 0$. Let $t \in \mathscr{O}_{C,P}$ be a uniformizer at P. By Zariski's lemma [6, III, (8.2)], $\mathrm{Ker}(\widetilde{i}^* \circ \delta)$ is one-dimensional and generated by $\mathrm{Res}(\frac{dt}{t})$. Hence there exists a constant c such that $\theta_i := \nu_i - c\frac{dt}{t}$ has no pole along D. By replacing ν_i with θ_i and taking $\varepsilon_i = \theta_i|_{\widetilde{D}}$, we see that $\omega_{X,D}|_V - \widehat{\omega}|_V$ is in the image of $F^1H^1_{\mathrm{dR}}(V) \to H^2_{\mathrm{dR}}(V, D)$. Hence we obtain $\widehat{\omega} \in F^1$ and the proof is complete. ∎

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