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1-REGULAR CAYLEY GRAPHS OF VALENCY 7

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Abstract

A graph Γ is called 1-regular if $\mathsf{Aut}\Gamma$ acts regularly on its arcs. In this paper, a classification of 1-regular Cayley graphs of valency 7 is given; in particular, it is proved that there is only one core-free graph up to isomorphism.

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1. Introduction

Throughout this paper, all graphs are finite, simple and undirected.

Let Γ be a graph. We denote the vertex set, edge set, arc set and full automorphism group by $V(\Gamma)$, $E(\Gamma)\mathrm{Arc}(\Gamma)$ and $\mathsf{Aut}\Gamma$, respectively. We say Γ is X-*vertex-transitive*, X-*edge-transitive* and X-*arc-transitive* if X acts transitively on $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Arc}(\Gamma)$ respectively, where $X \leq \mathsf{Aut}\Gamma$. We simply call Γ *vertex-transitive*, *edge-transitive* and *arc-transitive* for the case where $X = \mathsf{Aut}\Gamma$. In particular, Γ is called $(X, 1)$-*regular* if $X \leq \mathsf{Aut}\Gamma$ acts regularly on its arcs, and 1-*regular* when $X = \mathsf{Aut}\Gamma$. Let G be a finite group with identity 1. We call Γ a *Cayley graph* of G, denoted by $\Gamma = \mathsf{Cay}(G, S)$, if there is a subset S of G with $1 \notin S$ and $S = S^{-1} := \{s^{-1} \mid s \in S\}$ such that $V(\Gamma) = G$ and $E(\Gamma) = \{(g, sg) \mid g \in G, s \in S\}$. It is easy to see that $\mathsf{Cay}(G, S)$ has valency $|S|$. Moreover, $\mathsf{Cay}(G, S)$ is connected if and only if $\langle S \rangle = G$.

For a Cayley graph $\Gamma = \mathsf{Cay}(G, S)$, G can be viewed as a regular subgroup of $\mathsf{Aut}\Gamma$ by right multiplication action on $V(\Gamma)$. For convenience, we still denote this regular subgroup by G. Then a Cayley graph is vertex-transitive. If G is a normal subgroup of $\mathsf{Aut}\Gamma$, then Γ is called a *normal Cayley graph* of G. While for a nonnormal Cayley graph Γ, if $\mathsf{Aut}\Gamma$ contains a normal subgroup N that is semi-regularly on $V(\Gamma)$ and has exactly two orbits, then Γ is called a *bi-normal* Cayley graph. Both normal and bi-normal Cayley graphs have nice properties (see, for example, [5, 6, 7, 11]).

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And $\mathrm{Cay}(G, S)$ is called *core-free* (with respect to G) if G is core-free in some $X \le \mathrm{Aut}(\mathrm{Cay}(G, S))$, that is, $\mathrm{Core}_X(G) := \bigcap_{x \in X} G^x = 1$.

Li proved in [6] that there are only a finite number of core-free s-transitive Cayley graphs of given valency k for $s \in \{2, 3, 4, 5, 7\}$ and $k \ge 3$, and that, with the exceptions $s = 2$ and $(s, k) = (3, 7)$, every s-transitive Cayley graph is a normal cover of a core-free one. Li and Lu gave a classification of cubic s-transitive Cayley graphs for $s \ge 2$ in [8]. What about the case where $s = 1$? Until now, the results on 1-regular graphs have mainly focused on constructing examples. For example, Frucht gave the first example of cubic 1-regular graphs in [4]. Conder and Praeger then constructed two infinite families of cubic 1-regular graphs in [2]. Marušič [9] and Malnič [10] constructed two infinite families of tetravalent 1-regular graphs. Classification of such graphs has received great interest in recent years. Motivated by the above results and problem, we consider 1-regular Cayley graphs in this paper. We present the following theorem.

Theorem 1.1. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a 1-regular Cayley graph with valency 7, and let $N = \mathrm{Core}_A(G)$. Then Γ is connected, and one of the following holds:*

(1) *$G = N$ and Γ is a normal Cayley graph;*
(2) *$|G : N| = 2$, Γ is a bi-normal Cayley graph;*
(3) *Γ is a normal cover of a core-free graph (up to isomorphism):* $(A, G) = (S_7, S_6)$.

Remark 1.2. For more information on the core-free graph in (3) of Theorem 1.1, the readers can refer to Lemmas 2.2 and 2.3.

2. Core-free case

In this section, we will consider the core-free 1-regular Cayley graphs of valency 7. First, we will list all the $(X, 1)$-regular graphs with automorphism group X containing the regular subgroup. For an $(X, 1)$-regular graph, one does not expect it also to be 1-regular. So it is an important task for us to determine whether an $(X, 1)$-regular graph is 1-regular.

Let X be an arbitrary finite group with a core-free subgroup H. For an element $g \in X \setminus H$ such that $g^2 \in H$, the coset graph $\mathrm{Cos}(X, H, g)$ is the graph with vertex set $[X : H]$, and two vertices Hx, Hy are adjacent if and only if $y^{-1}x \in HgH$. By [8], we have the following simple proposition.

Proposition 2.1. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected $(X, 1)$-regular Cayley graph of valency 7 with $G \le X \le \mathrm{Aut}\Gamma$. Let H be the stabiliser of $1 \in V(\Gamma)$ in X. Then there exists an involution τ in S such that $\tau \in X \setminus \mathrm{N}_X(H)$, $\Gamma \cong \mathrm{Cos}(X, H, \tau)$, $\langle \tau, H \rangle = X$, $S = G \cap H\tau H$ and $G = \langle S \rangle$.*

Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected $(X, 1)$-regular core-free Cayley graph of valency 7, where $G \le X \le \mathrm{Aut}\Gamma$. For convenience, we let $\Sigma = \{1, 2, \ldots, 7\}$. By considering the right multiplication action of X on the right cosets of G in X, we may view X as a subgroup of the symmetric group S_7, which contains a regular subgroup (of S_7) isomorphic to a stabiliser of X acting on $V(\Gamma)$. And in this way, G is a stabiliser of X

acting on Σ by marking $[X : G]$ as Σ. Without loss of generality, we may assume that G fixes 1.

In the following, we will construct all possible connected core-free $(X, 1)$-regular Cayley graphs of valency 7 with a given stabiliser $H \cong \mathbb{Z}_7$. Without loss of generality we let $H = \langle\sigma\rangle$ with $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7)$. Clearly H acts regularly on Σ; then H is a regular subgroup of S_7. By Proposition 2.1, we may take an involution $\tau \in \mathrm{S}_7 \setminus \mathrm{N}_{\mathrm{S}_7}(H)$ such that $1^\tau = 1$. Let $X = \langle\tau, H\rangle$, $S = \{\sigma \in H\tau H \mid 1^\sigma = 1\}$; then

$$G = \langle S \rangle = \{\sigma \in X \mid 1^\sigma = 1\}$$

is a complement subgroup of H in X since G is a stabiliser and H is a regular subgroup of X. Thus G acts regularly on $[X : H]$, and it follows that $\mathsf{Cos}(X, H, \tau) \cong \mathsf{Cay}(G, S)$ is a connected core-free $(X, 1)$-regular Cayley graph of G. Note that all isomorphic regular subgroups of S_7 are conjugate in S_7 (see, for example, [12]), and $\mathsf{Cos}(X, H, \tau)$ is independent of the choice of H up to isomorphism. It is well known that $\mathsf{Cos}(X, H, \tau) \cong \mathsf{Cos}(X^\sigma, H, \tau^\sigma)$ for any $\sigma \in \mathrm{N}_{\mathrm{S}_7}(H)$. With the help of GAP, we see that there are in total 74 such τ's, which are conjugate under $\mathrm{N}_{\mathrm{S}_7}(H)$ to one of the following nine permutations:

$$\tau_{7,1} = (2\ 7), \quad \tau_{7,2} = (2\ 4)(3\ 7), \quad \tau_{7,3} = (2\ 5)(3\ 7),$$
$$\tau_{7,4} = (2\ 6)(3\ 7), \quad \tau_{7,5} = (2\ 3)(5\ 7), \quad \tau_{7,6} = (2\ 6)(3\ 7)(4\ 5),$$
$$\tau_{7,7} = (2\ 6)(3\ 4)(5\ 7), \quad \tau_{7,8} = (2\ 3)(4\ 6)(5\ 7), \quad \tau_{7,9} = (2\ 7)(3\ 5)(4\ 6).$$

For $i \in \{1, 2, \ldots, 9\}$, we let $\Gamma_{7,i} = \mathsf{Cos}(X_{7,i}, H, \tau_{7,i})$ and $G_{7,i} = \{\sigma \in X_{7,i} \mid 1^\sigma = 1\}$, where $X_{7,i} = \langle\tau_{7,i}, \sigma\rangle$. Then $\Gamma_{7,i} \cong \mathsf{Cay}(G_{7,i}, S_{7,i})$ with $S_{7,i} = G_{7,i} \cap H\tau_{7,i}H$ and $G_{7,i} = \langle S_{7,i}\rangle$.

Lemma 2.2. *We have* $(G_{7,2}, X_{7,2}) \cong (\mathrm{S}_4, \mathrm{PSL}(3, 2))$, $(G_{7,i}, X_{7,i}) \cong (\mathrm{A}_6, \mathrm{A}_7)$ *and* $(G_{7,j}, X_{7,j}) \cong (\mathrm{S}_6, \mathrm{S}_7)$, *where* $i \in \{3, 4, 5\}$ *and* $j \in \{1, 6, 7, 8, 9\}$.

Proof. Let $i \in \{3, 4, 5\}$, $j \in \{1, 6, 7, 8, 9\}$ and

$$\pi_1 = (\tau_{7,1}\sigma^{-1})^4\tau_{7,1}, \quad \beta_1 = \tau_{7,1},$$
$$\pi_6 = (\tau_{7,6}\sigma^3)^2(\tau_{7,6}\sigma)^3\sigma^{-3}, \quad \beta_6 = \tau_{7,6}\sigma^2(\tau_{7,6}\sigma^{-3})^2\tau_{7,6}\sigma^2\tau_{7,6},$$
$$\pi_7 = \tau_{7,7}\sigma^3\tau_{7,7}\sigma^{-2}\tau_{7,7}, \quad \beta_7 = \sigma\tau_{7,7}\sigma^{-3}(\tau_{7,7}\sigma)^2,$$
$$\pi_8 = (\sigma^{-1}\tau_{7,8})^2\sigma(\sigma\tau_{7,8})^2\sigma^{-2}\tau_{7,8}, \quad \beta_8 = (\tau_{7,8}\sigma^{-2})^2\tau_{7,8}\sigma^{-1}\tau_{7,8}\sigma^2\tau_{7,8}\sigma,$$
$$\pi_9 = (\tau_{7,9}\sigma^{-1})^4\tau_{7,9}, \quad \beta_9 = \sigma^2\tau_{7,9}\sigma^{-2}\tau_{7,9}\sigma^{-1}\tau_{7,9}\sigma^2(\tau_{7,9}\sigma)^2.$$

Then $\pi_j = (2\ 3\ 4\ 5\ 6\ 7)$ and $\beta_j = (2\ 7)$. Note that π_j acts transitively on $\Sigma \setminus \{1\}$, and $X_{7,j}$ acts 2-transitively on Σ. On the other hand, $X_{7,j}$ contains a 2-cycle $(2\ 7)$, which leads to $X_{7,j} \cong \mathrm{S}_7$ by [3, Theorem 3.3A]. Furthermore, $G_{7,j} \cong \mathrm{S}_6$.

Let

$$\pi_3 = \sigma^{-1}\tau_{7,3}\sigma^{-2}, \quad \beta_3 = \sigma^2(\tau_{7,3}\sigma)^2,$$
$$\pi_4 = \sigma^{-1}\tau_{7,4}\sigma^{-2}, \quad \beta_4 = (\tau_{7,4}\sigma^2)^2,$$
$$\pi_5 = \sigma^{-1}\tau_{7,5}\sigma^3, \quad \beta_5 = (\sigma\tau_{7,5}\sigma^{-1}\tau_{7,5})^2.$$

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Then $\pi_3 = (2\,6\,7\,4\,5)$, $\pi_4 = (2\,6\,3\,4\,5)$, $\pi_5 = (2\,4\,5\,7)(3\,6)$ and $\beta_i = (1\,2\,3)$. Noticing that $\langle \tau_{7,i}, \pi_i \rangle$ acts transitively on $\Sigma \setminus \{1\}$, $X_{7,i}$ acts 2-transitively on Σ. However, $X_{7,i}$ contains a 3-cycle $(1\ 2\ 3)$ and all generators of $X_{7,i}$ are even permutations. Thus $X_{7,i} \cong \mathrm{A}_7$ by [3, Theorem 3.3A], and, moreover, $G_{7,i} \cong \mathrm{A}_6$.

Let $\mu = (\sigma^2\tau_{7,2})^2 = (1\ 4)(6\ 7)$; then $\tau_{7,2} = \mu\sigma^{-2}\mu\sigma^2\mu$ and $X_{7,2} = \langle \tau_{7,2}, \sigma \rangle = \langle \mu, \sigma \rangle$. Note that $\sigma^7 = (\sigma^4\mu)^4 = (\sigma\mu)^3 = \mu^2 = 1$, $X_{7,2} \cong \mathrm{PSL}(3, 2)$ and $G_{7,2} \cong \mathrm{S}_4$ by [1]. □

In the rest of this section, we let

$$
\begin{aligned}
c_1 &= (1\,2\,3\,4\,5\,6\,7)(8\,9\,10\,11\,12\,13\,14),\\
c_2 &= (1\,8)(2\,14)(3\,13)(4\,12)(5\,11)(6\,10)(7\,9),\\
d_1 &= (1\,2\,3\,4\,5\,6\,7)(8\,9\,10\,11\,12\,13\,14)(15\,16\,17\,18\,19\,20\,21),\\
d_2 &= (1\,8\,15)(2\,10\,19)(3\,12\,16)(4\,14\,20)(5\,9\,17)(6\,11\,21)(7\,13\,18),\\
\bar\tau_{7,1} &= (3\,5)(11\,13), \quad \bar\tau_{7,2} = (3\,4)(5\,7)(9\,11)(12\,13),\\
\bar\tau_{7,3} &= (3\,4)(6\,7)(9\,10)(12\,13), \quad \bar\tau_{7,4} = (4\,5)(6\,7)(9\,10)(11\,12),\\
\bar\tau_{7,5} &= (2\,7)(4\,5)(9\,14)(11\,12), \quad \bar\tau_{7,6} = (2\,7)(3\,4)(5\,6)(9\,14)(10\,11)(12\,13),\\
\bar\tau_{7,7} &= (2\,6)(3\,4)(5\,7)(9\,13)(10\,11)(12\,14)(16\,20)(17\,18)(19\,21),\\
\bar\tau_{7,9} &= (2\,7)(3\,5)(4\,6)(9\,14)(10\,12)(11\,13).
\end{aligned}
$$

Let $\bar H_1 = \langle c_1, c_2 \rangle$, $\bar X_{7,i} = \langle c_1, c_2, \bar\tau_{7,i} \rangle$, $\bar\Gamma_{7,i} = \mathrm{Cos}(\bar X_{7,i}, \bar H_1, \bar\tau_{7,i})$, $\bar S_{7,i} = \{\sigma \in \bar H_1 \bar\tau_{7,i} \bar H_1 \mid 1^\sigma = 1\}$ and $\bar G_{7,i} = \langle \bar S_{7,i} \rangle$, with $i = 1, 2, 3, 4, 5, 6, 9$. Since $c_1^7 = c_2^2 = 1$ and $c_1^{c_2} = c_1^{-1}$, $\bar H_1 \cong \mathrm{D}_{14}$. Write $\bar\Sigma_1 = \{1, 2, \ldots, 14\}$, $\bar\Sigma_2 = \{1, 2, \ldots, 21\}$, $\bar H_2 = \langle d_1, d_2 \rangle$, $\bar X_{7,7} = \langle d_1, d_2, \bar\tau_{7,7} \rangle$, $\bar\Gamma_{7,7} = \mathrm{Cos}(\bar X_{7,7}, \bar H_2, \bar\tau_{7,7})$, $\bar S_{7,7} = \{\sigma \in \bar H_2 \bar\tau_{7,7} \bar H_2 \mid 1^\sigma = 1\}$ and $\bar G_{7,7} = \langle \bar S_{7,7} \rangle$. Since $d_1^7 = d_2^3 = 1$ and $d_1^{d_2} = d_1^2$, $\bar H_2 \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

Lemma 2.3. *If $i \in \{1, 2, 3, 4, 5, 6, 7, 9\}$, then $\Gamma_{7,i}$ is not a core-free 7-valent 1-regular Cayley graph.*

Proof. For convenience, we denote $\bar S_{7,i} = \{\bar s \mid s \in S_{7,i}\}$ with $i \in \{1, 2, 3, 4, 5, 6, 7, 9\}$. According to our calculations,

$$
\begin{aligned}
S_{7,1} &= \{a_{7,1}, b_{7,1}, b_{7,1}^{-1}, b_{7,1}^2 a_{7,1} b_{7,1}^{-2}, b_{7,1}^3 a_{7,1} b_{7,1}^3, b_{7,1}^{-2} a_{7,1} b_{7,1}^2, b_{7,1} a_{7,1} b_{7,1}^{-1}\},\\
S_{7,2} &= \{\tau_{7,2}, a_{7,2}, b_{7,2}, a_{7,2}^{-1}, b_{7,2}^{-1}, \tau_{7,2} b_{7,2}, a_{7,2}^{-1} b_{7,2}\},\\
S_{7,3} &= \{\tau_{7,3}, a_{7,3}, b_{7,3}, a_{7,3}^{-1}, b_{7,3}^{-1}, \tau_{7,3} a_{7,3} b_{7,3} \tau_{7,3} a_{7,3}^{-1}, \tau_{7,3} a_{7,3}^2 b_{7,3}^{-2}\},\\
S_{7,4} &= \{a_{7,4}, b_{7,4}, c_{7,4}, b_{7,4}^{-1}, c_{7,4}^{-1}, b_{7,4} a_{7,4} b_{7,4}^{-1}, c_{7,4} b_{7,4} a_{7,4} b_{7,4}^{-1} c_{7,4}^{-1}\},\\
S_{7,5} &= \{a_{7,5}, b_{7,5}, c_{7,5}, b_{7,5}^{-1}, c_{7,5}^{-1}, b_{7,5}^{-1} c_{7,5} a_{7,5} c_{7,5}^{-1} b_{7,5}, b_{7,5} c_{7,5}^{-1} a_{7,5} c_{7,5} b_{7,5}^{-1}\},\\
S_{7,6} &= \{\tau_{7,6}, a_{7,6}, b_{7,6}, a_{7,6}^{-1}, b_{7,6}^{-1}, a_{7,6}^2 b_{7,6}^{-1} a_{7,6}^{-1} b_{7,6}, \tau_{7,6} b_{7,6}^{-2} a_{7,6} b_{7,6}\},\\
S_{7,7} &= \{\tau_{7,7}, a_{7,7}, b_{7,7}, a_{7,7}^{-1}, b_{7,7}^{-1}, \tau_{7,7} b_{7,7}^{-1} \tau_{7,7} a_{7,7}^{-1} \tau_{7,7}, \tau_{7,7} a_{7,7} \tau_{7,7} b_{7,7} \tau_{7,7}\},\\
S_{7,9} &= \{\tau_{7,9}, a_{7,9}, b_{7,9}, a_{7,9}^{-1}, b_{7,9}^{-1}, a_{7,9} b_{7,9}^{-1} \tau_{7,9}, \tau_{7,9} b_{7,9} a_{7,9}^{-1}\},
\end{aligned}
$$

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where $a_{7,1} = (3\ 5)$, $\bar{a}_{7,1} = \bar{\tau}_{7,1}$, $a_{7,4} = (3\ 6)(4\ 7)$, $\bar{a}_{7,4} = \bar{\tau}_{7,4}$, $a_{7,5} = (2\ 7)(4\ 5)$, $\bar{a}_{7,5} = \bar{\tau}_{7,5}$ and

$$b_{7,1} = (2\ 7\ 5\ 3\ 6\ 4), \quad \bar{b}_{7,1} = (2\ 7\ 5\ 3\ 6\ 4)(8\ 13\ 11\ 14\ 12\ 10),$$
$$a_{7,2} = (2\ 6\ 3)(4\ 5\ 7), \quad \bar{a}_{7,2} = (2\ 7\ 4)(3\ 6\ 5)(8\ 14\ 10)(9\ 13\ 11),$$
$$b_{7,2} = (2\ 4)(3\ 5\ 7\ 6), \quad \bar{b}_{7,2} = (2\ 7\ 6\ 5)(3\ 4)(8\ 14)(9\ 13\ 12\ 11),$$
$$a_{7,3} = (2\ 6\ 7\ 4\ 5), \quad \bar{a}_{7,3} = (2\ 7\ 6\ 5\ 3)(8\ 13\ 12\ 11\ 9),$$
$$b_{7,3} = (2\ 5\ 4\ 7\ 3), \quad \bar{b}_{7,3} = (2\ 3\ 4\ 6\ 7)(8\ 9\ 10\ 12\ 13),$$
$$b_{7,4} = (2\ 6\ 3\ 4\ 5), \quad \bar{b}_{7,4} = (2\ 3\ 4\ 5\ 7)(8\ 9\ 10\ 12\ 14),$$
$$c_{7,4} = (3\ 7\ 4\ 5\ 6), \quad \bar{c}_{7,4} = (2\ 4\ 5\ 6\ 7)(9\ 10\ 11\ 12\ 14),$$
$$b_{7,5} = (2\ 4\ 5\ 7)(3\ 6), \quad \bar{b}_{7,5} = (2\ 7\ 5\ 4)(3\ 6)(9\ 14\ 12\ 11)(10\ 13),$$
$$c_{7,5} = (2\ 3\ 6\ 7)(4\ 5), \quad \bar{c}_{7,5} = (2\ 7\ 6\ 3)(4\ 5)(9\ 14\ 13\ 10)(11\ 12),$$
$$a_{7,6} = (2\ 6)(3\ 4\ 5), \quad \bar{a}_{7,6} = (2\ 7\ 4)(5\ 6)(9\ 14\ 11)(12\ 13),$$
$$b_{7,6} = (2\ 6\ 4\ 5\ 3\ 7), \quad \bar{b}_{7,6} = (2\ 7\ 4\ 3\ 6\ 5)(9\ 14\ 11\ 10\ 13\ 12),$$
$$a_{7,9} = (2\ 7\ 3\ 6), \quad \bar{a}_{7,9} = (2\ 6\ 3\ 7)(9\ 13\ 10\ 14),$$
$$b_{7,9} = (2\ 6\ 3\ 5), \quad \bar{b}_{7,9} = (3\ 6\ 4\ 7)(10\ 13\ 11\ 14),$$
$$a_{7,7} = (2\ 4\ 7\ 5\ 6\ 3), \quad b_{7,7} = (2\ 4\ 5\ 3\ 7\ 6),$$
$$\bar{a}_{7,7} = (2\ 5\ 6\ 4\ 3\ 7)(9\ 12\ 13\ 11\ 10\ 14)(16\ 19\ 20\ 18\ 17\ 21),$$
$$\bar{b}_{7,7} = (2\ 4\ 7\ 5\ 6\ 3)(9\ 11\ 14\ 12\ 13\ 10)(16\ 18\ 21\ 19\ 20\ 17).$$

Note that $\bar{\tau}_{7,i} \in \bar{S}_{7,i}$, $\bar{\tau}_{7,i} \in \bar{G}_{7,i}$, and it follows that $\bar{X}_{7,i} = \bar{G}_{7,i}\bar{H}_1$. It is easy to see that $\bar{H}_1$ acts regularly on $\bar{\Sigma}_1$ and $1^{\bar{G}_{7,i}} = 1$. It follows that $\bar{G}_{7,i} \cap \bar{H}_1 = 1$ and $\bar{G}_{7,i}$ is a complement subgroup of $\bar{H}_1$ in $\bar{X}_{7,i}$. Hence, we have $\bar{\Gamma}_{7,i} \cong \mathsf{Cay}(\bar{G}_{7,i}, \bar{S}_{7,i})$ for $i \in \{1, 2, 3, 4, 5, 6, 9\}$. With a similar argument, we get $\bar{\Gamma}_{7,7} \cong \mathsf{Cay}(\bar{G}_{7,7}, \bar{S}_{7,7})$. Let $\Phi_{7,1} : a_{7,1} \longrightarrow \bar{a}_{7,1}, b_{7,1} \longrightarrow \bar{b}_{7,1}$; $\Phi_{7,i} : \tau_{7,i} \longrightarrow \bar{\tau}_{7,i}, a_{7,i} \longrightarrow \bar{a}_{7,i}, b_{7,i} \longrightarrow \bar{b}_{7,i}$; $\Phi_{7,j} : a_{7,j} \longrightarrow \bar{a}_{7,j}, b_{7,j} \longrightarrow \bar{b}_{7,j}, c_{7,j} \longrightarrow \bar{c}_{7,j}$ with $i \in \{2, 3, 6, 7, 9\}$ and $j \in \{4, 5\}$.

According to the proof in the above paragraph, $\langle a_{7,1}, b_{7,1}\rangle \cong \mathrm{S}_6$, so $\langle \bar{a}_{7,1}, \bar{b}_{7,1}\rangle$ is also isomorphic to S_6 by replacing $a_{7,1}$ and $b_{7,1}$ with $\bar{a}_{7,1}$ and $\bar{b}_{7,1}$, respectively. Thus $G_{7,1} \cong \bar{G}_{7,1}$, that is, $\Phi_{7,1}$ is an isomorphism of $G_{7,1}$ to $\bar{G}_{7,1}$ denoted by $G_{7,1}^{\Phi_{7,1}} = \bar{G}_{7,1}$. Furthermore, $S_{7,1}^{\Phi_{7,1}} = \bar{S}_{7,1}$. With similar arguments, we have $G_{7,i}^{\Phi_{7,i}} = \bar{G}_{7,i}$ and $S_{7,i}^{\Phi_{7,i}} = \bar{S}_{7,i}$. Then $\mathsf{Cay}(G_{7,i}, S_{7,i}) \cong \mathsf{Cay}(\bar{G}_{7,i}, \bar{S}_{7,i})$, that is, $\Gamma_{7,i} \cong \bar{\Gamma}_{7,i}$. Since $\bar{H}_1 \cong \mathrm{D}_{14}$ and $\bar{H}_2 \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $\bar{\Gamma}_{7,i}$ is not 1-regular and so $\Gamma_{7,i}$ for $i \in \{1, 2, 3, 4, 5, 6, 7, 9\}$. □

3. The proof of Theorem 1.1

In this section, we will prove our main result. First, we need some definitions and properties.

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Assume that Γ is an X-vertex-transitive graph. Let N be a normal subgroup of X. Denote the set of N-orbits in $V(\Gamma)$ by V_N. The normal quotient Γ_N of Γ induced by N is defined as the graph with vertex set V_N, and two vertices $B, C \in V_N$ are adjacent if there exist $u \in B$ and $v \in C$ such that they are adjacent in Γ. It is easy to show that X/N acts transitively on the vertex set of Γ_N. Assume further that Γ is X-edge-transitive. Then X/N acts transitively on the edge set of Γ_N, and the valency $\mathsf{val}(\Gamma) = m\mathsf{val}(\Gamma_N)$ for some positive integer m. If $m = 1$, then Γ is called a normal cover of Γ_N.

We are now in a position to prove Theorem 1.1. Let $\Gamma = \mathsf{Cay}(G, S)$ be a 1-regular Cayley graph of valency 7. Then it is trivial to see that Γ is connected. Let $\mathrm{A} = \mathsf{Aut}\Gamma$ and $N = \mathrm{Core}_A(G)$ be the core of G in A. Assume that N is not trivial. Then either $G = N$ or $|G : N| \geq 2$. The former implies $G \trianglelefteq A$, that is, Γ is a normal Cayley graph with respect to G. For the case where $|G : N| = 2$, it is easy to see that Γ is a bi-normal Cayley graph. Suppose that $|G : N| > 2$, namely, N has at least three orbits on $V(\Gamma)$. Consider the normal quotient Γ_N; we have that Γ_N is a Cayley graph of G/N, $G/N \leq A/N \lesssim \mathsf{Aut}\Gamma_N$ and Γ_N is core-free with respect to G/N. Clearly Γ_N is an A/N-arc-transitive Cayley graph of G/N if Γ is A-arc-transitive, and under this assumption, Γ is a normal cover of Γ_N. Now suppose that N is trivial; then Γ is core-free. According to Lemmas 2.2 and 2.3, there is only one possible core-free 1-regular Cayley graph of valency 7 (up to isomorphism): $(G, A) \cong (\mathrm{S}_6, \mathrm{S}_7)$. The proof of Theorem 1.1 is complete.

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