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THREE BIMODULES FOR MANSFIELD'S IMPRIMITIVITY THEOREM

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Abstract

For a maximal coaction δ of a discrete group G on a C^*-algebra A and a normal subgroup N of G, there are at least three natural $A \times_\delta G \times_{\hat{\delta}|} N - A \times_{\delta|} G/N$ imprimitivity bimodules: Mansfield's bimodule $Y^G_{G/N}(A)$; the bimodule assembled by Ng from Green's $A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N - A \times_\delta G \times_{\hat{\delta}|} N$ imprimitivity bimodule $X^G_N(A \times_\delta G)$ and Katayama duality; and the bimodule assembled from $X^G_N(A \times_\delta G)$ and the crossed-product Mansfield bimodule $Y^G_{G/G}(A) \times G/N$. We show that all three of these are isomorphic, so that the corresponding inducing maps on representations are identical. This can be interpreted as saying that Mansfield and Green induction are inverses of one another 'modulo Katayama duality'. These results pass to twisted coactions; dual results starting with an action are also given.

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1. Introduction

Ng has recently observed [12] that an abstract Morita equivalence between a restricted coaction crossed product $A \times_{\delta|} G/N$ and the iterated dual action crossed product $A \times_\delta G \times_{\hat{\delta}} N$ can be pieced together from Green's imprimitivity theorem [6, Theorem 6] and Katayama duality [9, Theorem 8], thus giving a relatively non-technical, nonconstructive proof of Mansfield's imprimitivity theorem [11, Theorem 27]. However, in applications—especially those concerning induced representations—it is often necessary to work with an explicit bimodule. Because Morita equivalence relations are composed with one another by tensoring the corresponding imprimitivity bimodules together, Ng's transitivity argument does implicitly provide a bimodule. Thus

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the natural question arises as to whether Ng's bimodule is in fact isomorphic to Mansfield's.

In more detail: Ng considers a nondegenerate reduced coaction δ of a locally compact group G on a C^*-algebra A, and a closed normal amenable subgroup N of G. An application of Green's theorem to the dual action $(A \times_\delta G, G, \hat{\delta})$ gives an $A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N - A \times_\delta G \times_{\hat{\delta}|} N$ imprimitivity bimodule $X_N^G(A \times_\delta G)$. Moreover, looking closely at Katayama's duality theorem, one can derive an isomorphism Θ of $A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N$ onto $(A \times_{\delta|} G/N) \otimes \mathcal{K}(L^2(G))$, and this latter algebra is Morita equivalent to $A \times_{\delta|} G/N$ via the bimodule $(A \times_{\delta|} G/N) \otimes L^2(G)$. Implicitly, then, Ng's $A \times_\delta G \times_{\hat{\delta}|} N - A \times_{\delta|} G/N$ imprimitivity bimodule is the tensor product

$$\widetilde{X_N^G}(A \times_\delta G) \otimes_\Theta \big((A \times_{\delta|} G/N) \otimes L^2(G)\big).$$

(Here the tilde denotes the reverse bimodule.) Let $Y_{G/N}^G(A)$ denote Mansfield's imprimitivity bimodule. Then the question in question is precisely whether

$$X_N^G(A \times_\delta G) \otimes_{A \times G \times N} Y_{G/N}^G(A) \cong (A \times_{\delta|} G/N) \otimes L^2(G) \tag{1.1}$$

as imprimitivity bimodules. In other words, modulo crossed-product duality, are Green and Mansfield induction inverses of one another?

This and related questions concerning actions, twisted actions, and twisted coactions are addressed in the present paper in the context of discrete groups and full coactions. Our approach is to exploit the natural equivariance of the Mansfield and Green bimodules. For instance, in Section 3, we consider a maximal discrete coaction (A, G, δ) and Mansfield's $A \times_\delta G \times_{\hat{\delta}} G - A$ imprimitivity bimodule $Y_{G/G}^G(A)$, which carries a $\hat{\hat{\delta}} - \delta$ compatible bimodule coaction δ^Y. If in addition N is a normal subgroup of G, Theorem 3.1 states that

$$X_N^G(A \times_\delta G) \otimes_{A \times G \times N} Y_{G/N}^G(A) \cong Y_{G/G}^G(A) \times_{\delta^Y|} G/N \tag{1.2}$$

as $A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N - A \times_{\delta|} G/N$ imprimitivity bimodules. The only difference between (1.1) and (1.2) is the bimodule on the right-hand sides; but these turn out to be isomorphic (see the proof of Theorem 7.1). Thus Mansfield induction of representations from Rep $A \times_{\delta|} G/N$ to Rep $A \times_\delta G \times_{\hat{\delta}} N$ via $Y_{G/N}^G(A)$ can be 'undone' by Green induction via $X_N^G(A \times_\delta G)$ followed by Katayama duality to get from Rep $A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N$ back to Rep $A \times_{\delta|} G/N$. In this sense we can usefully view Mansfield and Green induction as inverse to one another. In Section 4, we obtain results dual to those of Section 3, starting with an action instead of a coaction. In Section 5 we show that the results of Section 3 pass to twisted coactions, and in Section 6 we round out this square of ideas with a set of results for twisted actions.

In Section 7 we return to the comparison between Ng's bimodule and Mansfield's. This is done by first establishing (1.1) for full coactions and discrete groups, and then dropping it down to reduced coactions. In general, we feel that this approach—establishing results first for full coactions, and later passing to quotients if results for reduced coactions are desired—is more efficient and cleaner conceptually than working with reduced coactions directly. On the other hand, we work with discrete groups simply to avoid many of the technicalities associated with coactions of general locally compact groups; also, this is the only context in which we have induced algebras for coactions, which appear in Section 6. There is no reason to believe that the other results in this paper will not hold in the general case. In fact, Theorem 3.1 and Theorem 4.1 will appear in [2] for locally compact groups, but only as a product of the extensive machinery developed therein. The proofs here are considerably more direct, and more instructive.

2. Preliminaries

In this preliminary section we collect the formulae relevant to crossed products and imprimitivity theorems involving actions and coactions of *discrete* groups. Because the groups are discrete, the theory acquires quite an algebraic flavour, and to take full advantage of this we translate the standard machinery involving locally compact groups to our context.

Coactions. Let $\delta\colon A \to A \otimes C^*(G)$ be a coaction of a discrete group G on a C^*-algebra A. Because G is discrete, the spectral subspaces $\{A_s : s \in G\}$ of δ densely span A, and the union $\mathscr{A} := \bigcup_{s\in G} A_s$ (more precisely, the disjoint union, but this abuse will cause no harm, since the spectral subspaces are linearly independent) forms a Fell bundle over G. As shown in [1], the coaction on A sits 'between' a 'maximal' coaction on the full cross-sectional algebra $C^*(\mathscr{A})$ and a 'minimal' coaction on the reduced cross-sectional algebra $C^*_r(\mathscr{A})$. Although the crossed product $A \times_\delta G$ and the covariant representations of the coaction cannot distinguish among the various possibilities between the extremes $C^*(\mathscr{A})$ and $C^*_r(\mathscr{A})$, some other constructions can. In particular, the imprimitivity theorems we need require the full cross-sectional algebra. Therefore, we *assume throughout that* $A = C^*(\mathscr{A})$, and we call the coaction δ *maximal* in this case.

The crossed product $A \times_\delta G$ is densely spanned by the Cartesian product $\mathscr{A} \times G$, where $A_s \times \{t\}$ has the obvious vector space structure for all $s, t \in G$, and the multiplication and involution are given on the generators by

$$(a_r, s)(b_t, u) = (a_r b_t, u) \quad \text{if } s = tu \text{ (and 0 if not)}$$
$$(a_s, t)^* = (a_s^*, st).$$

Since the coaction δ is maximal, $A \times_\delta G$ is the enveloping C^*-algebra of the linear span of the generators; that is, any operation-preserving mapping of the generators

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into a C^*-algebra C extends uniquely to a homomorphism $A \times_\delta G \to C$. If (B, G, ϵ) is another coaction and $\phi\colon A \to B$ is an equivariant homomorphism (equivalently, $\phi(A_s) \subseteq B_s$ for each $s \in G$), then we can 'integrate up' to get a homomorphism $\phi \times G\colon A \times_\delta G \to B \times_\epsilon G$ defined on the generators by

$$(\phi \times G)(a_s, t) = (\phi(a_s), t).$$

The dual action $\hat{\delta}$ of G on $A \times_\delta G$ is given on the generators by

$$\hat{\delta}_t(a_r, s) = (a_r, st^{-1}).$$

If N is a normal subgroup of G, the coaction δ restricts to a maximal coaction $\delta|$ of G/N on A, and the crossed product $A \times_{\delta|} G/N$ is densely spanned by $\mathscr{A} \times G/N$, with operations

$$(a_r, sN)(b_t, uN) = (a_r b_t, uN) \quad \text{if } sN = tuN \text{ (and 0 if not)},$$
$$(a_s, tN)^* = (a_s^*, stN),$$

and maps nondegenerately into $M(A \times_\delta G)$ by

$$(a_s, tN) \mapsto \sum_{n \in N} (a_s, tn) \qquad \text{(strictly convergent)}.$$

There is a *decomposition* coaction δ^{dec} of G on the restricted crossed product $A \times_{\delta|} G/N$, given on the generators by

$$\delta^{\mathrm{dec}}(a_s, tN) = (a_s, tN) \otimes s.$$

A coaction (A, G, δ) is *twisted* over G/N (see [13]) if there is an orthogonal family $\{p_{tN} : tN \in G/N\}$ of projections in $M(A_e)$ which sum strictly to 1 in $M(A_e)$, and such that

$$a_s p_{tN} = p_{stN} a_s \qquad \text{for all } s, t \in G.$$

The *twisted crossed product* $A \times_{\delta, G/N} G$ is the quotient of $A \times_\delta G$ by the ideal generated by differences of the form

$$(a_s, t) - (a_s p_{tN}, t).$$

We denote the quotient map by $q\colon A \times_\delta G \to A \times_{\delta, G/N} G$, and we write

$$[a_s, t] := q(a_s, t).$$

The ideal $\ker q$ is invariant under the restriction $\hat{\delta}|_N$, and we denote the corresponding action of N on $A \times_{\delta, G/N} G$ by $\tilde{\delta}$. We also define the 'restriction' $q|$ by the commutative diagram

$$\begin{array}{ccc} A \times_{\delta|} G/N & \longrightarrow & M(A \times_{\delta} G) \\ & \searrow^{q|} & \downarrow q \\ & & M(A \times_{\delta, G/N} G) \end{array}$$

and we write $[a_s, tN] := q|(a_s, tN)$. It is shown in [13] that $[a_s, tN] \mapsto a_s p_{tN}$ extends to an isomorphism $q(A \times_{\delta|} G/N) \cong A$.

Let ϵ be a maximal coaction of the quotient G/N on A. It is shown in [4] that there is an *induced* maximal coaction $(\operatorname{Ind} A, G, \operatorname{Ind} \epsilon)$ with spectral subspaces

$$(\operatorname{Ind} A)_s = A_{sN} \times \{s\},$$

and where the generators have the coordinate-wise operations

$$(a_{sN}, s)(b_{tN}, t) = (a_{sN} b_{tN}, st) \quad \text{and} \quad (a_{sN}, s)^* = (a_{sN}^*, s^{-1}).$$

Actions. Let $\alpha : G \to \operatorname{Aut} B$ be an action of the discrete group G on a C^*-algebra B. The crossed product $B \times_\alpha G$ is densely spanned by the Cartesian product $B \times G$, where $B \times \{s\}$ has the obvious vector space structure for all $s \in G$, and the multiplication and involution are given on the generators by

$$(a, r)(b, s) = (a\alpha_r(b), rs) \quad \text{and} \quad (a, r)^* = (\alpha_{r^{-1}}(a^*), r^{-1}).$$

Again, $B \times_\alpha G$ is the enveloping C^*-algebra of the span of the generators, and if (C, G, β) is another action and $\phi : B \to C$ is an equivariant homomorphism, then we can integrate up to get a homomorphism $\phi \times G : B \times_\alpha G \to C \times_\beta G$ defined on the generators by

$$(\phi \times G)(b, s) = (\phi(b), s).$$

The dual coaction $\hat{\alpha}$ of G on $B \times_\alpha G$ is given on the generators by

$$\hat{\alpha}(a, r) = (a, r) \otimes r.$$

The decomposition action α^{dec} of G on the restricted crossed product $B \times_{\alpha|} N$ is given on the generators by

$$\alpha_r^{\mathrm{dec}}(a, n) = (\alpha_r(a), rnr^{-1}).$$

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An action (B, G, α) is *twisted* over N (see [6]) if there is a unitary homomorphism $n \mapsto u_n : N \to M(B)$ such that

$$\alpha_s(u_n) = u_{sns^{-1}} \quad \text{and} \quad \alpha_n(b) = u_n b u_n^*.$$

The twisted crossed product $B \times_{\alpha,N} G$ is the quotient of $B \times_\alpha G$ by the ideal generated by differences of the form

$$(bu_n, s) - (b, ns).$$

We denote the quotient map by $q : B \times_\alpha G \to B \times_{\alpha,N} G$, and we write

$$[b, s] := q(b, s).$$

We denote the dual coaction of G/N on $B \times_{\alpha,N} G$ by $\tilde{\alpha}$. We also define the 'restriction' $q|$ by the commutative diagram

$$\begin{array}{ccc} B \times_{\alpha|} N & \longrightarrow & B \times_\alpha G \\ & \searrow^{q|} & \downarrow q \\ & & B \times_{\alpha,N} G \end{array}$$

and we write $[b, n] := q|(b, n)$. It is shown in [6] that $[b, n] \mapsto bu_n$ extends to an isomorphism $q(B \times_{\alpha|} N) \cong B$.

Let β be an action of the normal subgroup N on B. We identify the induced algebra $\operatorname{Ind} B$ as the c_0-section algebra of a C^*-bundle $G \times_N B$ over G/N. Specifically, N acts diagonally on the trivial C^*-bundle $G \times B \to G$ by $n(s, b) := (sn^{-1}, \beta_n(b))$, and the associated orbit space $G \times_N B$ has a natural C^*-bundle structure over G/N: we denote the orbit of a pair (s, b) by $[s, b]$, and the fiber over sN is $\{[s, b] : b \in B\}$. The induced action $(\operatorname{Ind} B, G, \operatorname{Ind} \beta)$ is given on the generators by $\operatorname{Ind} \beta_s([t, b]) = [st, b]$.

Commutative diagrams of bimodules. Most of this paper concerns imprimitivity bimodules, but a few times (in Section 5 and Section 6) we will need the following more general concept: a *right-Hilbert* $A - B$ bimodule is a Hilbert B-module X equipped with a left A-module action by adjointable maps (which are automatically bounded and B-linear). We will *assume throughout* that the right inner product is *full* and the left action is *nondegenerate*. For example, a surjective homomorphism $\phi : A \to B$ determines a right-Hilbert bimodule ${}_A B_B$ with the obvious right B-module action, left A-module action $a \cdot b := \phi(a)b$, and right inner product $\langle b, c \rangle_B := b^* c$. Moreover, in this situation any right-Hilbert $B - C$ bimodule X can also be regarded as a right-Hilbert $A - C$ bimodule with left A-module action $a \cdot x := \phi(a) \cdot x$.

When X is an $A - B$ imprimitivity bimodule, we use the more-or-less standard notation $\tilde{X}$ for the reverse bimodule; recall that this is a $B - A$ imprimitivity bimodule

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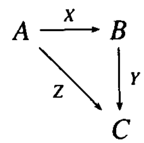
which coincides with X as a set, although for clarity the element of $\tilde{X}$ corresponding to an element $x \in X$ is denoted by $\tilde{x}$, and the module actions and inner products are given by

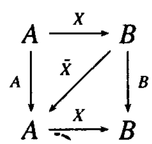
$$b \cdot \tilde{x} = \widetilde{x \cdot b^*}, \quad \tilde{x} \cdot a = \widetilde{a^* \cdot x}, \quad {}_B\langle \tilde{x}, \tilde{y} \rangle = \langle x, y \rangle_B, \quad \text{and} \quad \langle \tilde{x}, \tilde{y} \rangle_A = {}_A\langle x, y \rangle.$$

We have found it convenient to signify right-Hilbert bimodule isomorphisms using diagrams: given right-Hilbert bimodules ${}_AX_B$, ${}_BY_C$, and ${}_AZ_C$, when we say the diagram



commutes, we mean $X \otimes_B Y \cong Z$ as right-Hilbert $A - B$ bimodules, and similarly for rectangular diagrams, *etc.*

For example, when ${}_AX_B$ is an imprimitivity bimodule, the whole point of the reverse bimodule $\tilde{X}$ is that the diagram



commutes.

If ${}_AX_B$ and ${}_CY_D$ are right-Hilbert bimodules and $\phi\colon A \to C$ and $\psi\colon B \to D$ are C^*-homomorphisms, a linear map $\Phi\colon X \to Y$ is a *right-Hilbert bimodule homomorphism* with *coefficient maps* ϕ and ψ if it preserves the bimodule actions and the right inner product, that is,

(i) $\Phi(a \cdot x) = \phi(a) \cdot \Phi(x)$,
(ii) $\Phi(x \cdot b) = \Phi(x) \cdot \psi(b)$, and
(iii) $\langle \Phi(x), \Phi(y) \rangle_D = \psi(\langle x, y \rangle_B)$

for all $a \in A$, $b \in B$, and $x, y \in X$. If ϕ and ψ are isomorphisms and Φ has dense range, then Φ is also an isomorphism, and if X and Y are imprimitivity bimodules then Φ is an imprimitivity bimodule map.

An easy modification of [17, Lemma 2.2] shows that the B-linearity condition (ii) is redundant; in fact, if X_0 and A_0 are dense subspaces of X and A, respectively, and if $\Phi\colon X_0 \to Y$ is a linear map satisfying (i) and (iii) above for all $x, y \in X_0$ and $a \in A_0$, then Φ uniquely extends to a right-Hilbert bimodule homomorphism of ${}_AX_B$ into ${}_CY_D$. Indeed, if $S \subseteq X$ linearly spans X_0, $T \subseteq A$ linearly spans A_0, Φ satisfies (i) and (iii) on S and T and extends linearly to X_0, then the same conclusion holds. We will repeatedly use this fact without comment.

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Given an imprimitivity bimodule homomorphism ${}_AX_B \to {}_CY_D$ with dense range and surjective coefficient maps $\phi\colon A \to C$ and $\psi\colon B \to D$, by [8, Lemma 5.3] the diagram

$$\begin{array}{ccc} A & \xrightarrow{X} & B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ C & \xrightarrow[Y]{} & D \end{array}$$

commutes.

Bimodule crossed products. Let Z be an $A - B$ imprimitivity bimodule, and let $\eta : Z \to Z \otimes C^*(G)$ be a bimodule coaction which is compatible with coactions δ on A and ϵ on B (see [5]). Then the spectral subspaces $\{Z_s : s \in G\}$ densely span Z, and the bimodule crossed product $Z \times_\eta G$ is densely spanned by the pairs $\{(x_s, t) : s, t \in G, x_s \in Z_s\}$, and is an $A \times_\delta G - B \times_\epsilon G$ imprimitivity bimodule with operations given on the generators by

$$\begin{aligned} (a_r, s) \cdot (x_t, u) &= (a_r \cdot x_t, u) && \text{if } s = tu \text{ (and 0 if not),}\\ {}_{A\times G}\langle (x_r, s), (y_t, u)\rangle &= ({}_A\langle x_r, y_t\rangle, tu) && \text{if } s = u \text{ (and 0 if not),}\\ (x_r, s) \cdot (b_t, u) &= (x_r \cdot b_t, u) && \text{if } s = tu \text{ (and 0 if not),}\\ \langle (x_r, s), (y_t, u)\rangle_{B\times G} &= (\langle x_r, y_t\rangle_B, u) && \text{if } rs = tu \text{ (and 0 if not).} \end{aligned}$$

The dual action $\hat{\eta}$ of G on $Z \times_\eta G$ is given on the generators by

$$\hat{\eta}_t(x_r, s) = (x_r, st^{-1}).$$

Similarly, if γ is an action of G on Z which is compatible with actions α on A and β on B, then the crossed product $Z \times_\gamma G$ is densely spanned by the Cartesian product $Z \times G$, and is an $A \times_\alpha G - B \times_\beta G$ imprimitivity bimodule with operations given on the generators by

$$\begin{aligned} (a, r) \cdot (x, s) &= (a \cdot \alpha_r(x), rs), & {}_{A\times G}\langle (x, r), (y, s)\rangle &= ({}_A\langle x, \alpha_{rs^{-1}}(y)\rangle, rs^{-1}),\\ (x, r) \cdot (b, s) &= (x \cdot \alpha_r(b), rs), & \langle (x, r), (y, s)\rangle_{B\times G} &= (\alpha_{r^{-1}}(\langle x, y\rangle_B), r^{-1}s). \end{aligned}$$

The dual coaction $\hat{\gamma}$ of G on $Z \times_\gamma G$ is given on the generators by

$$\hat{\gamma}(x, r) = (x, r) \otimes r.$$

Imprimitivity theorems. Let (A, G, δ) be a maximal discrete coaction, and let N be a normal subgroup of G. By the version of Mansfield's imprimitivity theorem due to Echterhoff and the second author [3, Theorem 3.1], there exists an $A \times_\delta G \times_{\hat{\delta}|}$

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$N - A \times_{\delta|} G/N$ imprimitivity bimodule $Y^G_{G/N}(A)$. Mansfield's bimodule is densely spanned by the Cartesian product $\mathscr{A} \times G$, with operations given on the generators by

$$\begin{aligned}
(a_r, s, n) \cdot (b_t, u) &= (a_r b_t, un^{-1}) && \text{if } sn = tu \text{ (and 0 if not)},\\
{}_{A\times G\times N}\langle (a_r, s), (b_t, u)\rangle &= (a_r b_t^*, ts, s^{-1}u) && \text{if } sN = uN \text{ (and 0 if not)},\\
(a_r, s) \cdot (b_t, uN) &= (a_r b_t, t^{-1}s) && \text{if } sN = tuN \text{ (and 0 if not)},\\
\langle (a_r, s), (b_t, u)\rangle_{A\times G/N} &= (a_r^* b_t, uN) && \text{if } rs = tu \text{ (and 0 if not)}.
\end{aligned}$$

It is easy to see that $Y(A)$ is functorial in the sense that if (B, G, ϵ) is another coaction and $\phi\colon A \to B$ is an equivariant homomorphism then $(a_s, t) \mapsto (\phi(a_s), t)$ extends to an imprimitivity bimodule homomorphism $\Phi\colon Y(A) \to Y(B)$ with coefficient homomorphisms $\phi \times G \times N$ and $\phi \times G/N$. Moreover, Φ is surjective if ϕ is, and a similar comment applies to the imprimitivity bimodules $X(B)$, $U(A)$, and $V(B)$ described below.

Dually, for an action (B, G, α) and a normal subgroup N of G, Green's imprimitivity theorem [6, Proposition 3] provides a $B \times_\alpha G \times_{\hat{\alpha}|} G/N - B \times_{\alpha|} N$ imprimitivity bimodule $X^G_N(B)$. Green's bimodule is densely spanned by the Cartesian product $B \times G$, with operations given on the generators by

$$\begin{aligned}
(a, r, sN) \cdot (b, t) &= (a\alpha_r(b), rt) && \text{if } sN = tN \text{ (and 0 if not)},\\
{}_{B\times G\times G/N}\langle (a, r), (b, s)\rangle &= (a\alpha_{rs^{-1}}(b^*), rs^{-1}, sN), &&\\
(a, r) \cdot (b, n) &= (a\alpha_r(b), rn), &&\\
\langle (a, r), (b, s)\rangle_{B\times N} &= (\alpha_{r^{-1}}(a^* b), r^{-1}s) && \text{if } rN = sN \text{ (and 0 if not)}.
\end{aligned}$$

$X(B)$ is functorial in the sense that if (C, G, β) is another action and $\phi\colon B \to C$ is an equivariant homomorphism then $(b, s) \mapsto (\phi(b), s)$ extends to an imprimitivity bimodule homomorphism $X(B) \to X(C)$ with coefficient homomorphisms $\phi \times G \times G/N$ and $\phi \times N$.

If the coaction (A, G, δ) is twisted over G/N, then the quotient map $q\colon A \times_\delta G \to A \times_{\delta, G/N} G$ restricts to a surjection $q|\colon A \times_{\delta|} G/N \to A$, giving us an ideal $\ker q|$ of $A \times G/N$. Inducing across Mansfield's bimodule Y via the Rieffel correspondence gives an ideal Y-$\operatorname{Ind}(\ker q|)$ of $A \times_\delta G \times_{\hat{\delta}|} N$, and [13, Theorem 4.1] shows that this ideal is precisely $\ker(q \times N)$. Rieffel's theory thus gives an $A \times_{\delta, G/N} G \times_{\tilde{\delta}} N - A$ imprimitivity bimodule

$$Z^G_{G/N}(A) := Y/(Y \cdot \ker q|).$$

Moreover, the diagram

$$\begin{array}{ccc}
A \times_\delta G \times_{\hat{\delta}|} N & \xrightarrow{\;Y\;} & A \times_{\delta|} G/N \\
{\scriptstyle q\times N}\big\downarrow & & \big\downarrow{\scriptstyle q|} \\
A \times_{\delta, G/N} G \times_{\tilde{\delta}} N & \xrightarrow[\;Z\;]{} & A
\end{array}$$

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commutes.

Dually, if the action (B, G, α) is twisted over N, by [6, Corollary 5] we have a $B \times_{\alpha,N} G \times_{\hat{\alpha}|} G/N - B$ imprimitivity bimodule $W_N^G(B) := X/(X \cdot \ker q|)$, and we get a commutative diagram

$$\begin{array}{ccc} B \times_\alpha G \times_{\hat{\alpha}|} G/N & \xrightarrow{X} & B \times_{\alpha|} N \\ {\scriptstyle q \times G/N}\downarrow & & \downarrow{\scriptstyle q|} \\ B \times_{\alpha,N} G \times_{\tilde{\alpha}} G/N & \xrightarrow[W]{} & B. \end{array}$$

If ϵ is a maximal coaction of the quotient G/N on A, the recent imprimitivity theorem for induced coactions [4, Theorem 4.1] gives an $\operatorname{Ind} A \times_{\operatorname{Ind}\epsilon} G - A \times_\epsilon G/N$ imprimitivity bimodule $U_{G/N}^G(A)$ densely spanned by the subset

$$\{(a_{sN}, t) : sN \in G/N, a_{sN} \in A_{sN}, t \in G\}$$

of the Cartesian product $A \times G$, with operations given on the generators by

$$\begin{aligned} (a_{sN}, s, t)(b_{uN}, v) &= (a_{sN}b_{uN}, sv) && \text{if } t = v \text{ (and 0 if not)},\\ (a_{sN}, t)(b_{uN}, vN) &= (a_{sN}b_{uN}, t) && \text{if } tN = suvN \text{ (and 0 if not)},\\ {}_{\operatorname{Ind} A \times G}\langle (a_{sN}, t), (b_{uN}, v)\rangle &= (a_{sN}b_{uN}^*, tv^{-1}, v) && \text{if } s^{-1}tN = u^{-1}vN \text{ (and 0 if not)},\\ \langle (a_{sN}, t), (b_{uN}, v)\rangle_{A \times G/N} &= (a_{sN}^* b_{uN}, u^{-1}vN) && \text{if } t = v \text{ (and 0 if not)}. \end{aligned}$$

$U(A)$ is functorial in the sense that if $(B, G/N, \eta)$ is another coaction and $\phi\colon A \to B$ is an equivariant homomorphism then $(a_sN, t) \mapsto (\phi(a_{sN}), t)$ extends to an imprimitivity bimodule homomorphism $U(A) \to U(B)$ with coefficient homomorphisms $\operatorname{Ind}\phi \times G$ and $\phi \times G/N$.

Dually, if β is an action of the normal subgroup N on B, the imprimitivity theorem for induced actions (sometimes attributed to Green [6, Theorem 17]) gives an $\operatorname{Ind} B \times_{\operatorname{Ind}\beta} G - B \times_\beta N$ imprimitivity bimodule $V_N^G(B)$ densely spanned by the Cartesian product $B \times G$, with operations given on the generators by

$$\begin{aligned} ([tr, b], t)(c, r) &= (bc, tr), & (b, s)(c, n) &= (\beta_{n^{-1}}(bc), sn),\\ {}_{\operatorname{Ind} B \times G}\langle (b, s), (c, t)\rangle &= ([s, bc^*], st^{-1}), & \langle (b, s), (c, sh)\rangle_{B \times N} &= (b^*\beta_h(c), h). \end{aligned}$$

$V(B)$ is functorial in the sense that if (C, N, β) is another action and $\phi\colon B \to C$ is an equivariant homomorphism then $(b, s) \mapsto (\phi(b), s)$ extends to an imprimitivity bimodule homomorphism $V(B) \to V(C)$ with coefficient homomorphisms $\operatorname{Ind}\phi \times G$ and $\phi \times N$.

3. The Mansfield-Green triangle

In this section we show a curious duality between the Mansfield and Green imprimitivity theorems. Theorem 3.1 will show that, roughly speaking, and modulo crossed product duality, Mansfield and Green induction are inverse processes.

Let (A, G, δ) be a maximal discrete coaction, and let N be a normal subgroup of G. Not only do we have Mansfield's $A \times_\delta G \times_{\hat{\delta}|} N - A \times_{\delta|} G/N$ imprimitivity bimodule $Y^G_{G/N}(A)$, but also, replacing N by G, an $A \times_\delta G \times_{\hat{\delta}} G - A$ imprimitivity bimodule $Y^G_{G/G}(A)$. There is a $\hat{\hat{\delta}} - \delta$ compatible coaction δ_Y of G on $Y^G_{G/G}(A)$ ([3, Remark 3.2]) determined by

$$\delta_Y(a_r, s) = (a_r, s) \otimes s^{-1}.$$

THEOREM 3.1. *Let (A, G, δ) be a maximal coaction and let N be a normal subgroup of G. Then the diagram*

$$\begin{array}{ccc} A \times_\delta G \times_{\hat{\delta}|} N & \xrightarrow{\;Y^G_{G/N}(A)\;} & A \times_{\delta|} G/N \\ {\scriptstyle X^G_N(A\times G)}\Big\uparrow & \nearrow {\scriptstyle Y^G_{G/G}(A)\times G/N} & \\ A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N & & \end{array}$$

commutes.

PROOF. There is an imprimitivity bimodule isomorphism

$$\Phi\colon Y(A) \times G/N \otimes \widetilde{Y(A)} \xrightarrow{\cong} X(A \times G)$$

defined on the generators by

$$\Phi((a_r, s, tN) \otimes \widetilde{(b_u, v)}) = (a_r b_u^*, us, s^{-1}v) \quad \text{if } tN = vN \text{ (and 0 if not)},$$

since straightforward calculations verify that the above mapping on generators preserves the left action and the right inner product. □

REMARK 3.2. To motivate the formula for Φ, note that

$$\Phi((a_r, s, tN) \otimes \widetilde{(c_u, v)}) = {}_{A\times G\times G}\langle (a_r, s), (c_u, v)\rangle \quad \text{if } tN = vN \text{ (and 0 if not)},$$

where (a_r, s) and (c_u, v) are viewed as elements of $Y^G_{G/G}(A)$, and the inner product is viewed as taking values in $X^G_N(A \times_\delta G)$.

The next two results will not be needed until Section 5; we include them here because they don't involve twists, and are of general interest.

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PROPOSITION 3.3. *Let (A, G, δ) be a coaction, and let N be a normal subgroup of G. Then $A \times_\delta G \times_{\hat\delta} G \times_{\hat\delta|} G/N \cong A \times_{\delta|} G/N \times_{\delta^{\mathrm{dec}}} G \times_{\widehat{\delta^{\mathrm{dec}}}} G$.*

PROOF. Straightforward calculations verify that the mapping

$$(a_r, s, t, uN) \mapsto (a_r, stuN, s, t)$$

on the generators preserves the operations, hence extends to a homomorphism $\phi: A \times_\delta G \times_{\hat\delta} G \times_{\hat\delta|} G/N \to A \times_{\delta|} G/N \times_{\delta^{\mathrm{dec}}} G \times_{\widehat{\delta^{\mathrm{dec}}}} G$, since the crossed product is the enveloping C^*-algebra of the linear span of the generators. On the other hand, the mapping $(a_r, uN, s, t) \mapsto (a_r, s, t, t^{-1}s^{-1}uN)$ is the inverse of ϕ on generators, hence also preserves operations, and hence extends to a homomorphism $\psi: A \times_{\delta|} G/N \times_{\delta^{\mathrm{dec}}} G \times_{\widehat{\delta^{\mathrm{dec}}}} G \to A \times_\delta G \times_{\hat\delta} G \times_{\hat\delta|} G/N$. Now $\psi \circ \phi$ is the identity map on generators, so $\psi \circ \phi = \mathrm{id}$, by uniqueness of extensions to enveloping algebras. Similarly, $\phi \circ \psi = \mathrm{id}$, so that $\psi = \phi^{-1}$. Therefore, ϕ is an isomorphism of $A \times_\delta G \times_{\hat\delta} G \times_{\hat\delta|} G/N$ onto $A \times_{\delta|} G/N \times_{\delta^{\mathrm{dec}}} G \times_{\widehat{\delta^{\mathrm{dec}}}} G$. □

PROPOSITION 3.4. *Let (A, G, δ) be a coaction and N a normal subgroup of G. Then the diagram*

$$\begin{array}{ccc} & & A \times_{\delta|} G/N \\ & \nearrow^{Y^G_{G/G}(A)\times G/N} & \uparrow {\scriptstyle Y^G_{G/G}(A\times G/N)} \\ A \times_\delta G \times_{\hat\delta} G \times_{\hat\delta|} G/N & \xrightarrow{\cong} & A \times_{\delta|} G/N \times_{\delta^{\mathrm{dec}}} G \times_{\widehat{\delta^{\mathrm{dec}}}} G \end{array}$$

commutes, where the isomorphism is that of Proposition 3.3.

PROOF. There is an imprimitivity bimodule isomorphism

$$\Theta : Y(A) \times G/N \xrightarrow{\cong} Y(A \times G/N)$$

defined on the generators by $\Theta(a_r, s, tN) = (a_r, tN, s)$, since straightforward calculations verify that the above mapping on generators preserves the left action and the right inner product. □

4. The dual triangle

The results of this section are dual to those of the previous section, in the sense that actions correspond to coactions, Green bimodules correspond to Mansfield bimodules, and subgroups correspond to quotient groups. The only additional apparatus we need is to observe that if (B, G, α) is a discrete action then there is an $\hat\alpha - \alpha$ compatible action α^X of G on $X_e^G(B)$ given on the generators by

$$\alpha^X_r(a, s) = (a, sr^{-1}).$$

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THEOREM 4.1. *Let (B, G, α) be an action an let N be a normal subgroup of G. Then the diagram*

$$\begin{array}{ccc} B \times_\alpha G \times_{\hat{\alpha}} G \times_{\hat{\hat{\alpha}}|} N & \xrightarrow{Y^G_{G/N}(B \times G)} & B \times_\alpha G \times_{\hat{\alpha}|} G/N \\ & \searrow^{X^G_e(B) \times N} & \downarrow X^G_N(B) \\ & & B \times_{\alpha|} N \end{array}$$

commutes.

PROOF. There is an imprimitivity bimodule isomorphism

$$\Phi \colon (X^G_e(B) \times N) \otimes \widetilde{X^G_N}(B) \xrightarrow{\cong} Y^G_{G/N}(B \times G)$$

defined on the generators by

$$\Phi((a, r, n) \otimes \widetilde{(b, s)}) = (a\alpha_{rns^{-1}}(b^*), rns^{-1}, sn^{-1}),$$

since straightforward calculations verify that the above mapping on generators preserves the left action and the right inner product. □

REMARK 4.2. To motivate the formula for Φ, note that

$$\Phi((a, r, n) \otimes \widetilde{(b, s)}) = {}_{B \times G \times G}\langle (a, r), \alpha^X_n(b, s) \rangle,$$

where (a, r) and (b, s) are viewed as elements of $X^G_e(B)$, and the inner product is viewed as taking values in $Y^G_{G/N}(B \times_\alpha G)$.

In analogy with the previous section, the next two results will not be needed until Section 6; they are presented here for convenience and general interest.

PROPOSITION 4.3. *Let (B, G, α) be an action and let N be a normal subgroup of G. Then $B \times_\alpha G \times_{\hat{\alpha}} G \times_{\hat{\hat{\alpha}}|} N \cong B \times_{\alpha|} N \times_{\alpha^{\mathrm{dec}}} G \times_{\widehat{\alpha^{\mathrm{dec}}}} G$.*

PROOF. Straightforward calculations verify that the mapping

$$(a, r, s, n) \mapsto (a, rsns^{-1}r^{-1}, rsn^{-1}s^{-1}, sn)$$

on the generators preserves the operations and is invertible on generators, and therefore extends to a C^*-isomorphism by the same argument used in the proof of Proposition 3.3. □

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PROPOSITION 4.4. *Let (B, G, α) be an action and let N be a normal subgroup of G. Then the diagram*

$$\begin{array}{ccc} B \times_\alpha G \times_{\hat\alpha} G \times_{\hat{\hat\alpha}|} N & & \\ \cong \big\downarrow & \searrow^{X_e^G(B)\times N} & \\ B \times_{\alpha|} N \times_{\alpha^{\mathrm{dec}}} G \times_{\widehat{\alpha^{\mathrm{dec}}}} G & \xrightarrow[X_e^G(B\times N)]{} & B \times_{\alpha|} N \end{array}$$

commutes.

PROOF. There is an imprimitivity bimodule isomorphism

$$\Theta\colon X(B) \times N \xrightarrow{\cong} X(B \times N)$$

defined on the generators by $\Theta(a, r, n) = (a, rnr^{-1}, r)$, since straightforward calculations verify that the above mapping on generators preserves the left action and the right inner product. □

5. The twisted Mansfield-Green square

Let (A, G, δ) be a maximal discrete coaction, and let N be a normal subgroup of G. Combining Theorem 3.1 and Corollary 3.4, we get a commutative rectangle

$$(5.1)\qquad \begin{array}{ccc} A \times_\delta G \times_{\hat\delta|} N & \xrightarrow{Y^G_{G/N}(A)} & A \times_{\delta|} G/N \\ {\scriptstyle X^G_N(A\times G)}\big\uparrow & & \big\uparrow{\scriptstyle Y^G_{G/G}(A\times G/N)} \\ A \times_\delta G \times_{\hat\delta} G \times_{\hat{\hat\delta}|} G/N & \xrightarrow[\Upsilon]{\cong} & A \times_{\delta|} G/N \times_{\delta^{\mathrm{dec}}} G \times_{\widehat{\delta^{\mathrm{dec}}}} G. \end{array}$$

Now suppose the coaction δ is twisted over G/N. Then the top arrow of Diagram (5.1) has

$$A \times_{\delta, G/N} G \times_{\tilde\delta} N \xrightarrow{W} A$$

as a quotient. The imprimitivity bimodules in (5.1) above determine corresponding ideals of the bottom corners, and we can form a quotient commutative rectangle with upper right corner A. What happens to the rest of the diagram? We will answer this question in the present section.

However, we first modify the lower left corner of Diagram (5.1); the action $\tilde\delta$ of N on $A \times_{G/N} G$ does not extend to G, so the Green bimodule X^G_N on the left edge of (5.1) will not pass to a Green bimodule in the quotient. Rather, it will be more appropriate to use the bimodule arising from the imprimitivity theorem for induced actions. The action $\tilde\delta$ of N on $A\times_{G/N} G$ induces to an action $\operatorname{Ind}\tilde\delta$ of G on the induced algebra $\operatorname{Ind}(A\times_{G/N} G)$, and we have an $\operatorname{Ind}(A\times_{G/N} G)\times_{\operatorname{Ind}\tilde\delta} G - (A\times_{G/N} G)\times_{\tilde\delta} N$ imprimitivity bimodule $V_N^G(A\times_{G/N} G)$.

It is shown in [15, Theorem 4.4] that

$$(a_s p_{tN}, r)\mapsto \left[r^{-1}t, [a_s, t]\right]$$

extends to an isomorphism $A\times_\delta G\cong \operatorname{Ind}(A\times_{\delta,G/N} G)$ which is equivariant for the actions $\hat\delta$ and $\operatorname{Ind}\tilde\delta$ of G. Turning this around and integrating up, we get an isomorphism

$$\operatorname{Ind}(A\times_{\delta,G/N} G)\times_{\operatorname{Ind}\tilde\delta} G\cong A\times_\delta G\times_{\hat\delta} G.$$

THEOREM 5.1. *If (A, G, δ) is a maximal discrete coaction which is twisted over G/N, the diagram*

$$\begin{array}{ccc} A\times_{\delta,G/N} G\times_{\tilde\delta} N & \xrightarrow{Z^G_{G/N}(A)} & A \\ {\scriptstyle V^G_N(A\times_{G/N}G)}\Big\uparrow & & \Big\uparrow{\scriptstyle Y^G_{G/G}(A)} \\ \operatorname{Ind}(A\times_{\delta,G/N}G)\times_{\operatorname{Ind}\tilde\delta} G & \xrightarrow[\cong]{} & A\times_\delta G\times_{\hat\delta} G\end{array}$$

commutes.

PROOF. The desired diagram is the inner rectangle of the diagram

(5.2)
$$\begin{array}{ccccc} A\times G\times N & & \xrightarrow{\quad Y^G_{G/N}(A)\quad} & & A\times G/N \\ \Big\uparrow & \searrow{\scriptstyle q\times N} & & {\scriptstyle q|}\swarrow & \Big\uparrow \\ \Big| & A\times_{G/N}G\times N & \xrightarrow{Z^G_{G/N}(A)} & A & \Big| \\ {\scriptstyle V^G_N(A\times G)} & \uparrow{\scriptstyle V^G_N(A\times_{G/N}G)} & & \uparrow{\scriptstyle Y^G_{G/G}(A)} & {\scriptstyle Y^G_{G/G}(A\times G/N)} \\ \Big| & \operatorname{Ind}(A\times_{G/N}G)\times G & \xrightarrow[\cong]{} & A\times G\times G & \Big| \\ \Big| & \nearrow{\scriptstyle \operatorname{Ind}q\times G} & & \nwarrow{\scriptstyle q|\times G\times G} & \Big| \\ \operatorname{Ind}(A\times G)\times G & & \overset{}{\underset{\cong}{\dashrightarrow}} & & A\times G/N\times G\times G. \end{array}$$

(Here and in Diagram (5.3) the action and coaction symbols have been omitted for clarity.) We will show how to fill in the bottom arrow so that each of the outer rectangle and the top, bottom, left, and right quadrilaterals commute. Since $\operatorname{Ind}q\times G$ is surjective, the result will then follow from standard bimodule techniques.

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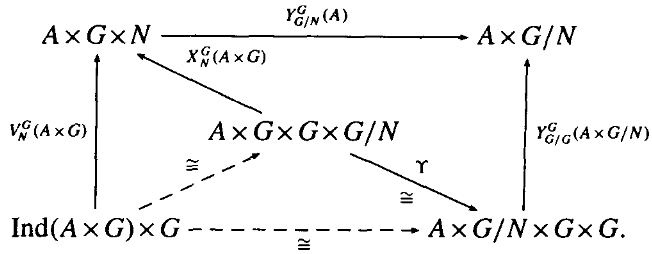
Consider the diagram

(5.3)



Since the action $\hat{\delta}|$ of N extends to G, it follows from standard facts concerning induced actions that

$$([s,a_t,r],u)\mapsto(a_t,rs^{-1},u,u^{-1}sN)$$

extends to an isomorphism $\mathrm{Ind}(A\times_\delta G)\times_{\mathrm{Ind}\,\hat{\delta}|}G\cong A\times_\delta G\times_{\hat{\delta}}G\times_{\hat{\delta}|}G/N$. Then an easy check on the generators shows that $(a_s,t,r)\mapsto(a_s,tr^{-1},r)$ extends to an isomorphism $V(A\times_\delta G)\cong X(A\times_\delta G)$ of $\mathrm{Ind}(A\times_\delta G)\times_{\mathrm{Ind}\,\hat{\delta}|}G - A\times_\delta G\times_{\hat{\delta}|}N$ imprimitivity bimodules. This shows the left triangle of Diagram (5.3) commutes. The inner quadrilateral in (5.3) is the commutative diagram (5.1). We *define* the isomorphism $\mathrm{Ind}(A\times_\delta G)\times_{\mathrm{Ind}\,\hat{\delta}|}G\cong A\times_{\delta|}G/N\times_{\delta^{\mathrm{dec}}}G\times_{\widehat{\delta^{\mathrm{dec}}}}G$ at the bottom arrow of (5.3) so that the bottom triangle commutes. On the generators, this isomorphism is given by

$$([s,a_t,r],u)\mapsto(a_t,rN,rs^{-1},u).$$

Thus the outer rectangle in Diagram (5.2) commutes.

We noticed in Section 2 that the top quadrilateral in (5.2) commutes.

For the right quadrilateral, $Y(B)$ is functorial in B, so the homomorphism $q|:A\times_{\delta|}G/N\to A$ yields an imprimitivity bimodule homomorphism $Y(q|)\colon Y(A\times_{\delta|}G/N)\to Y(A)$ with the desired coefficient homomorphisms. As we mentioned in Section 2, by [8, Lemma 5.3] this implies the quadrilateral commutes.

Similarly, the left quadrilateral commutes by functoriality of $V(B)$: the homomorphism $q\colon A\times_\delta G\to A\times_{\delta,G/N}G$ yields an imprimitivity bimodule homomorphism $V(q)\colon V(A\times_\delta G)\to V(A\times_{\delta,G/N}G)$ with the desired coefficient homomorphisms.

Finally, the bottom quadrilateral in (5.2) commutes by a routine computation on the generators. □

6. The twisted dual square

In this section we introduce a twist into Theorem 4.1, just as in the preceding section we threw a twist into Theorem 3.1; unsurprisingly, the development will closely parallel that of Section 5.

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Let (B, G, α) be a discrete action which is twisted over a normal subgroup N in the sense of [6]. Theorem 4.1 and Corollary 4.4 together give a commutative rectangle

$$
(6.1)\qquad
\begin{array}{ccc}
B \times_\alpha G \times_{\hat{\alpha}|} G/N & \xrightarrow{X^G_N(B)} & B \times_{\alpha|} N \\
\uparrow {\scriptstyle Y^G_{G/N}(B\times G)} & & \uparrow {\scriptstyle X^G_{\{e\}}(B\times N)} \\
B \times_\alpha G \times_{\hat{\alpha}} G \times_{\hat{\hat{\alpha}}|} N & \xrightarrow[\Upsilon]{\cong} & B \times_{\alpha|} N \times_{\alpha^{\rm dec}} G \times_{\widehat{\alpha^{\rm dec}}} G.
\end{array}
$$

As in the preceding section, in order to form a suitable quotient diagram we need to replace the lower left corner by an induced algebra.

The dual coaction $\tilde{\alpha}$ of G/N on the twisted crossed product $B \times_{\alpha,N} G$ induces to a coaction $\operatorname{Ind}\tilde{\alpha}$ of G on the induced algebra $\operatorname{Ind}(B \times_{\alpha,N} G)$, and we have an $\operatorname{Ind}(B\times_{\alpha,N} G)\times_{\operatorname{Ind}\tilde{\alpha}} G - (B\times_{\alpha,N} G)\times_{\tilde{\alpha}} G/N$ imprimitivity bimodule $U^G_{G/N}(B\times_{\alpha,N} G)$.

It is shown in [4, Theorem 5.6] that

$$(b, s) \mapsto ([b, s], s)$$

extends to an isomorphism $B \times_\alpha G \cong \operatorname{Ind}(B \times_{\alpha,N} G)$ which is equivariant for the coactions $\hat{\alpha}$ and $\operatorname{Ind}\tilde{\alpha}$ of G. Turning this around and integrating up, we get an isomorphism

$$\operatorname{Ind}(B \times_{\alpha,N} G) \times_{\operatorname{Ind}\tilde{\alpha}} G \cong B \times_\alpha G \times_{\hat{\alpha}} G.$$

THEOREM 6.1. *If (B, G, α) is a discrete action which is twisted over N, the diagram*

$$
\begin{array}{ccc}
B \times_{\alpha,N} G \times_{\tilde{\alpha}} G/N & \xrightarrow{W^G_N(B)} & B \\
\uparrow {\scriptstyle U^G_{G/N}(B\times_{\alpha,N} G)} & & \uparrow {\scriptstyle X^G_{\{e\}}(B)} \\
\operatorname{Ind}(B \times_{\alpha,N} G) \times_{\operatorname{Ind}\tilde{\alpha}} G & \xrightarrow[\cong]{} & B \times_\alpha G \times_{\hat{\alpha}} G
\end{array}
$$

commutes.

PROOF. The desired diagram is the inner rectangle of the diagram

$$
(6.2)\qquad
\begin{array}{ccccc}
B\times G\times G/N & & \xrightarrow{\qquad X^G_N(B) \qquad} & & B\times N \\
 & \searrow^{q\times G/N} & & \swarrow^{q|} & \\
\uparrow & B\times_N G\times G/N & \xrightarrow{W^G_N(B)} & B & \uparrow \\
{\scriptstyle Y^G_{G/N}(B\times G)} & \uparrow {\scriptstyle U^G_N(B\times_N G)} & & \uparrow {\scriptstyle X^G_{\{e\}}(B)} & {\scriptstyle X^G_{\{e\}}(B\times N)} \\
 & \operatorname{Ind}(B\times_N G)\times G & \xrightarrow[\cong]{} & B\times G\times G & \\
 & \nearrow_{\operatorname{Ind} q\times G} & & \nwarrow_{q|\times G\times G} & \\
\operatorname{Ind}(B\times G)\times G & & \dashrightarrow_{\cong} & & B\times N\times G\times G.
\end{array}
$$

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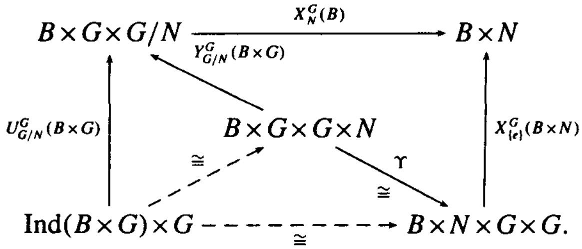
Consider the diagram

(6.3)



It follows from [3, Remark 3.3] that the map

$$(b, s, sn, t) \mapsto (b, s, nt, t^{-1}n^{-1}t)$$

extends to an isomorphism $\operatorname{Ind}(B \times_\alpha G) \times_{\operatorname{Ind}\hat{\alpha}|} G \cong B \times_\alpha G \times_{\hat{\alpha}} G \times_{\hat{\hat{\alpha}}|} N$, and this serves as the left-hand coefficient map for an isomorphism $U(B \times_\alpha G) \cong Y(B \times_\alpha G)$, hence the left triangle of the diagram (6.3) commutes. The inner quadrilateral is the commutative diagram (6.1). We define the isomorphism $\operatorname{Ind}(B \times_\alpha G) \times_{\operatorname{Ind}\hat{\alpha}|} G \cong B \times_{\alpha|} N \times_{\alpha^{\text{dec}}} G \times_{\widehat{\alpha^{\text{dec}}}} G$ at the bottom arrow of (6.3) so that the bottom triangle commutes. On the generators, this isomorphism is given by

$$(b, s, t, r) \mapsto (b, st^{-1}, t, r).$$

Thus the outer rectangle in the diagram (6.2) commutes.

We noticed in Section 2 that the top quadrilateral in (6.2) commutes.

$X(A)$ is functorial in A, so the homomorphism $q|: B \times_{\alpha|} N \to B$ yields an imprimitivity bimodule homomorphism $X(q|): X(B \times_{\alpha|} N) \to X(B)$ with the desired coefficient homomorphisms, so the right quadrilateral commutes.

Similarly, the left quadrilateral commutes because by functoriality of $U(A)$ the homomorphism $q: B \times_\alpha G \to B \times_{\alpha,N} G$ yields an imprimitivity bimodule homomorphism $U(q): U(B \times_\alpha G) \to U(B \times_{\alpha,N} G)$ with the desired coefficient homomorphisms.

Finally, the bottom quadrilateral in (6.2) commutes by a routine computation on the generators. □

7. Ng's Bimodule

We now return to the comparison between Ng's bimodule and Mansfield's, beginning with maximal coactions and full crossed products. In this context, by 'Ng's bimodule' we mean the bimodule gotten from the lower three sides of Diagram (7.1); the map Θ will be defined in the proof of Theorem 7.1 by a construction parallel to Ng's.

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THEOREM 7.1. *If (A, G, δ) is a maximal coaction of a discrete group G and N is a normal subgroup of G, then Ng's bimodule is isomorphic to Mansfield's; that is, the diagram*

$$
\begin{array}{ccc}
A \times_\delta G \times_{\hat\delta|} N & \xrightarrow{\;Y^G_{G/N}(A)\;} & A \times_{\delta|} G/N \\
\Big\uparrow{\scriptstyle X^G_N(A\times G)} & & \Big\uparrow{\scriptstyle (A\times G/N)\otimes\ell^2(G)} \\
A \times_\delta G \times_{\hat\delta} G \times_{\hat{\hat\delta}|} G/N & \xrightarrow[\Theta]{\;\cong\;} & (A \times_{\delta|} G/N) \otimes \mathcal{K}(\ell^2(G))
\end{array}
\tag{7.1}
$$

commutes.

PROOF. The desired diagram is the outer rectangle of

$$
\begin{array}{ccc}
A \times_\delta G \times_{\hat\delta|} N & \xrightarrow{\;Y^G_{G/N}(A)\;} & A \times_{\delta|} G/N \\
\Big\uparrow{\scriptstyle X^G_N(A\times G)} & \nearrow{\scriptstyle Y^G_{G/G}(A)\times G/N} & \Big\uparrow{\scriptstyle (A\times G/N)\otimes\ell^2(G)} \\
A \times_\delta G \times_{\hat\delta} G \times_{\hat{\hat\delta}|} G/N & \xrightarrow[\Theta]{} & (A \times_{\delta|} G/N) \otimes \mathcal{K}(\ell^2(G)).
\end{array}
$$

The upper left triangle commutes by Theorem 3.1, so we must show the lower right triangle commutes.

We construct the isomorphism Θ as a composition

$$
\begin{aligned}
A \times_\delta G \times_{\hat\delta} G \times_{\hat{\hat\delta}|} G/N &\xrightarrow{\Theta_1} (A \otimes \mathcal{K}) \times_{\epsilon_1} G/N \\
&\xrightarrow{\Theta_2} (A \otimes \mathcal{K}) \times_{\epsilon_2} G/N \xrightarrow{\Theta_3} (A \times_{\delta|} G/N) \otimes \mathcal{K}.
\end{aligned}
\tag{7.2}
$$

Here, ϵ_2 is the coaction

$$\epsilon_2 := (\mathrm{id} \otimes \sigma) \circ (\delta| \otimes \mathrm{id})$$

of G/N on $A \otimes \mathcal{K}$, where $\sigma : C^*(G/N) \otimes \mathcal{K} \to \mathcal{K} \otimes C^*(G/N)$ is the flip isomorphism. It follows from [14, Lemma 1.16 (b)] (see also [16]) that there is an isomorphism Θ_3 of $(A \otimes \mathcal{K}) \times_{\epsilon_2} G/N$ onto $(A \times_{\delta|} G/N) \otimes \mathcal{K}$ defined on the generators by

$$\Theta_3(a_{sN} \otimes b, tN) = (a_{sN}, tN) \otimes b.$$

Perpetuating our perverse numbering scheme, we use ϵ_2 to define the coaction

$$\epsilon_1 := \operatorname{Ad} u \circ \epsilon_2$$

of G/N on $A \otimes \mathcal{K}$, where u is the unitary element of $M(A \otimes \mathcal{K} \otimes C^*(G/N))$ given by the strictly convergent series

$$u = \sum_{s\in G} (1_{M(A)} \otimes M_{\chi_s} \otimes s^{-1}N),$$

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where χ_s denotes the characteristic function of the singleton $\{s\}$ and M_{χ_s} the associated multiplication operator on $\ell^2(G)$. It is easy to see that u is an ϵ_2-cocycle (more precisely, the obvious analogue for full coactions of the more usual cocycles for reduced coactions—see [10]). It follows from [10, Theorem 2.9] (also see [15, Proposition 2.8]) that there is an isomorphism Θ_2 of $(A \otimes \mathcal{K}) \times_{\epsilon_1} G/N$ onto $(A \otimes \mathcal{K}) \times_{\epsilon_2} G/N$ defined by

$$\Pi_2 \circ \Theta_2 = \mathrm{Ad}(\mathrm{id}_{A\otimes\mathcal{K}} \otimes \lambda_{G/N})(u^*) \circ \Pi_1,$$

where Π_i is the regular representation of $(A \otimes \mathcal{K}) \times_{\epsilon_i} G/N$ on $\mathcal{H} \otimes \ell^2(G) \otimes \ell^2(G/N)$ for $i = 1, 2$ (and A is faithfully represented on a Hilbert space $\mathcal{H}$).

Finally, from [3, Equation (5.1) and Proposition 5.3] we have the isomorphism $\Phi\colon A \times_\delta G \times_{\hat{\delta}} G \to A \otimes \mathcal{K}$ given on generators by

$$\Phi(a_s, t, r) = a_s \otimes \lambda_s M_{\chi_t} \rho_r,$$

where λ and ρ are the left and right regular representations of G. (Φ is the isomorphism of Katayama's duality theorem [9, Theorem 8], but for maximal coactions rather than reduced ones.) The arguments of [9] (or in this case an easy calculation with the generators), adapted to our context, show that Φ is equivariant for the coactions $\hat{\hat{\delta}}|$ and ϵ_1 of G/N; we define $\Theta_1 = \Phi \times G/N$ to be the corresponding isomorphism of the crossed products.

Careful study of the isomorphisms Θ_1, Θ_2, and Θ_3 now shows that the composition $\Theta\colon A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N \xrightarrow{\cong} (A \times_{\delta|} G/N) \otimes \mathcal{K}$ is given on the generators by

$$\Theta(a_s, t, r, qN) = (a_s, trqN) \otimes \lambda_s M_{\chi_t} \rho_r.$$

Using this, straightforward calculations show that there is an $A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N$ – $A \times_{\delta|} G/N$ imprimitivity bimodule isomorphism

$$\Upsilon\colon Y^G_{G/G}(A) \times_{\delta_Y} G/N \xrightarrow{\cong} (A \times_{\delta|} G/N) \otimes \ell^2(G)$$

defined on the generators by $\Upsilon(a_s, t, rN) = (a_s, rN) \otimes \chi_{st}$. □

REMARK 7.2. Taking $N = G$ in Theorem 7.1 shows that Katayama's bimodule (by which we mean the bottom and right-hand sides of that rectangle, taken together) is isomorphic to Mansfield's in this special case. This justifies the idea that Mansfield's theorem 'reduces to Katayama's' when $N = G$, a fact which is well known to the cognoscenti, but to our knowledge has not explicitly appeared in the literature.

To complete the connection with Ng's theorem, we need to pass to *reduced* coactions and *amenable* subgroups in Diagram 7.1. Here we use $Y(A, \delta)$ to denote the $A \times_\delta G \times_{\hat{\delta}|}$

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$N - A \times_{\delta|} G/N$ imprimitivity bimodule provided by the original form of Mansfield's Imprimitivity Theorem [11, Theorem 27]. The isomorphism $\Theta_r \colon A \times_\delta G \times_{\hat\delta} G \times_{\hat\delta|} G/N \to (A \times_{\delta|} G/N) \otimes \mathcal{K}(\ell^2)$ is constructed as in Equation (7.2).

COROLLARY 7.3. *If (A, G, δ) is a reduced coaction of a discrete group G and N is an amenable normal subgroup of G, then Ng's bimodule is isomorphic to Mansfield's; that is, the diagram*

$$\begin{array}{ccc} A \times_\delta G \times_{\hat\delta|} N & \xrightarrow{\;Y(A,\delta)\;} & A \times_{\delta|} G/N \\ {\scriptstyle X(A \times_\delta G)} \Big\uparrow & & \Big\uparrow {\scriptstyle (A \times_{\delta|} G/N) \otimes \ell^2} \\ A \times_\delta G \times_{\hat\delta} G \times_{\hat\delta|} G/N & \underset{\Theta_r}{\xrightarrow{\;\cong\;}} & (A \times_{\delta|} G/N) \otimes \mathcal{K}(\ell^2) \end{array}$$

commutes.

PROOF. Since G is discrete, the coaction δ is automatically nondegenerate, so by [14, Theorem 4.7] there is a unique full coaction δ^f of G on A whose reduction coincides with δ, and then [14, Proposition 3.8] gives an isomorphism $A \times_\delta G \xrightarrow{\cong} A \times_{\delta^f} G$; it is easy to see that this isomorphism is equivariant for the dual actions. Then [3, Proposition 5.3] applies, giving a maximal coaction (A^m, G, δ^m) (the 'maximalization' of δ^f) and an equivariant surjection $\Psi \colon A^m \to A$ whose integrated form $\Psi \times G \colon A^m \times_{\delta^m} G \to A \times_{\delta^f} G$ is an isomorphism which is equivariant for the dual actions. Then Ψ is also equivariant for the restricted coactions $\delta^m|$ and $\delta^f|$, hence certainly gives a surjection

$$\Psi \times G/N \colon A^m \times_{\delta^m|} G/N \to A \times_{\delta^f|} G/N.$$

Since δ^f is the normalization of δ^m, [3, Theorem 3.4] tells us that, if $I = \ker \Psi \times G/N$, then the ideal of $A^m \times_{\delta^m} G \times_{\widehat{\delta^m}|} N$ induced from I via the Mansfield imprimitivity bimodule $Y(A^m)$ coincides with the kernel of the regular representation

$$A^m \times_{\delta^m} G \times_{\widehat{\delta^m}|} N \to A^m \times_{\delta^m} G \times_{\widehat{\delta^m}|,r} N,$$

so that $Y/(Y \cdot I)$ is canonically an $A^m \times_{\delta^m} G \times_{\widehat{\delta^m}|,r} N - A \times_{\delta^f|} G/N$ imprimitivity bimodule. But N is amenable, so the regular representation of $A^m \times_{\delta^m} G \times_{\widehat{\delta^m}|} N$ is faithful. Hence we must have $I = \{0\}$, so $\Psi \times G/N$ is actually an isomorphism of $A^m \times_{\delta^m|} G/N$ onto $A \times_{\delta^f|} G/N$.

It is now clear from the constructions that the identity map on the ordered pairs $\{(a_s, t) : s, t \in G\}$ extends to an isomorphism

$$Y(A^m) \xrightarrow{\cong} Y(A, \delta^f)$$

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of the Mansfield imprimitivity bimodule $Y(A^m)$ used in the present paper onto the version of the Mansfield bimodule associated to the normal coaction δ^f in [7]. Thus the diagram

$$\begin{array}{ccc}
A^m \times_{\delta^m} G \times_{\widehat{\delta^m}|} N & \xrightarrow{Y(A^m)} & A^m \times_{\delta^m|} G/N \\
\cong \downarrow & & \downarrow \cong \\
A \times_{\delta^f} G \times_{\widehat{\delta^f}|} N & \xrightarrow[Y(A,\delta^f)]{} & A \times_{\delta^f|} G/N
\end{array}$$

commutes. On the other hand, one of the main points of [7] is that the diagram

$$\begin{array}{ccc}
A \times_{\delta^f} G \times_{\widehat{\delta^f}|} N & \xrightarrow{Y(A,\delta^f)} & A \times_{\delta^f|} G/N \\
\cong \downarrow & & \downarrow \cong \\
A \times_{\delta} G \times_{\hat{\delta}|} N & \xrightarrow[Y(A,\delta)]{} & A \times_{\delta|} G/N
\end{array}$$

commutes; combining these shows that the top quadrilateral of the diagram

$$\begin{array}{ccccc}
A^m \times_{\delta^m} G \times_{\widehat{\delta^m}|} N & & \xrightarrow{\quad Y(A^m) \quad} & & A^m \times_{\delta^m|} G/N \\
\uparrow & \searrow \cong & & \cong \swarrow & \uparrow \\
 & A \times_\delta G \times_{\hat{\delta}|} N & \xrightarrow{Y(A,\delta)} & A \times_{\delta|} G/N & \\
X(A^m \times_{\delta^m} G) & \uparrow X(A \times_\delta G) & & \uparrow (A \times_{\delta|} G/N) \otimes \ell^2 & (A^m \times_{\delta^m|} G/N) \otimes \ell^2 \\
 & A \times_\delta G \times_{\hat{\delta}} G \times_{\hat{\hat{\delta}}|} G/N & \xrightarrow[\Theta_r]{\cong} & (A \times_{\delta|} G/N) \otimes \mathcal{K} & \\
| & \nearrow \cong & & \cong \nwarrow & | \\
A^m \times_{\delta^m} G \times_{\widehat{\delta^m}} G \times_{\hat{\hat{\delta}}|} G/N & & \xrightarrow[\Theta]{\quad \cong \quad} & & (A^m \times_{\delta^m|} G/N) \otimes \mathcal{K}
\end{array}$$

commutes. Since the outer rectangle commutes by Theorem 7.1, and the left, right, and bottom quadrilaterals are easily seen to commute, we conclude that the inner rectangle commutes as well. □

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