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CHARACTERIZING RINGS BY A DIRECT DECOMPOSITION PROPERTY OF THEIR MODULES

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Abstract

A module M is said to satisfy the condition $(\wp^*)$ if M is a direct sum of a projective module and a quasi-continuous module. In an earlier paper, we described the structure of rings over which every (countably generated) right module satisfies $(\wp^*)$, and it was shown that such a ring is right artinian. In this note some additional properties of these rings are obtained. Among other results, we show that a ring over which all right modules satisfy $(\wp^*)$ is also left artinian, but the property $(\wp^*)$ is not left-right symmetric.

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1. Introduction

Throughout this note, all rings are associative with identity, and all modules are unitary modules. Let M be a right module over a ring R. The Jacobson radical and the injective hull of M are denoted respectively by $J(M)$ and $E(M)$. For a module M consider the following conditions:

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand.

(C_3) If A, B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

A module M is defined to be a *CS module* (or an *extending module*) if M satisfies condition (C_1). If M satisfies (C_1) and (C_2), then M is said to be a *continuous module*. A module M is called *quasi-continuous* if it satisfies (C_1) and (C_3).

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Let M be a module. A module N is called *M-injective* if every homomorphism of any submodule $L \subseteq M$ to N can be extended to a homomorphism of M to N. A module N is called *quasi-injective* (or *self-injective*), if N is N-injective.

If M is a module of finite composition length, we denote its length by $l(M)$.

Following [5], a module M is said to satisfy the condition $(\wp^*)$ if M is a direct sum of a projective module and a quasi-continuous module. A ring R is called a *right $\wp^*$-semisimple ring*, if every right R-module satisfies $(\wp^*)$. Rings whose countably generated right modules satisfy $(\wp^*)$ were characterized in [5, Theorem 7]. These rings are exactly right artinian rings over which every finitely generated right module is a direct sum of a projective module and a quasi-injective module (and in particular, are also right $\wp^*$-semisimple). In this note we improve this result by showing:

(1) Every right $\wp^*$-semisimple ring is left artinian.

(2) A right $\wp^*$-semisimple ring is not necessarily left $\wp^*$-semisimple.

(3) In general, the direct sum decomposition of R in [5, Theorem 7 (III)] is not a ring-direct sum decomposition.

(4) Finally we give a correction that the right ideal B of R is not necessarily a CS right R-module as claimed in [5, Theorem 7 (III) (ii) and Lemma 11].

Thus, combining with [5, Theorem 7], we describe the structure of right $\wp^*$-semisimple rings in the following theorem.

THEOREM 1.1. *For a ring R, the following conditions are equivalent*:

(I) *Every countably generated right R-module satisfies $(\wp^*)$.*

(II) *R is right artinian and every finitely generated right R-module satisfies $(\wp^*)$.*

(III) *R is a right and left artinian ring with Jacobson radical square zero*; *$R_R = A \oplus B \oplus C$, where $(B \oplus C)A = BC = CB = 0$, and B_R and C_R are nonsingular right ideals of R. In general, this direct sum is not a ring-direct sum. Moreover,*

(i) *$A_R = A_1 \oplus \cdots \oplus A_l$, where each A_i is uniform, $E(A_i)$ is projective, and $l(E(A_i)) \leqslant 2$.*

(ii) *$B_R = B_1 \oplus \cdots \oplus B_m$, where each B_j is a uniform module of length one or two*; *the injective hull $E(S)$ of each minimal submodule S of B_R has length three. Moreover, $E(S)/S$ is a direct sum of two simple modules, in particular $E(S) = xR + yR$ for some $x,\ y \in E(S)$. If $B \neq 0$, then there exist at least two (uniform) direct summands B_j and $B_{j'}$ of B with $l(B_j) = 1$, $l(B_{j'}) = 2$ and $B_j \cong \mathrm{Soc}(B_{j'})$. Furthermore, B_R is not necessarily CS and has the structure described in Proposition* 3.2.

(iii) *$C_R = C_1 \oplus \cdots \oplus C_q$, where each C_k is an indecomposable module of length one or three*; *the injective hull of each minimal submodule of C_R is of length two and not projective. If $C \neq 0$, there exist at least two C_k, say C_1, C_2 with $l(C_1) = 1$, $l(C_2) = 3$ and C_1 is embedable in $\mathrm{Soc}(C_2)$.*

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(IV) *Every right R-module is a direct sum of a projective module and a quasi-injective module. In particular, R is right $\wp^*$-semisimple.*

In general, right $\wp^*$-semisimple rings need not be left $\wp^*$-semisimple.

2. The proof of Theorem 1.1

We refer to [5, Theorem 7] for the stucture of a right $\wp^*$-semisimple ring. Hence, in addition to [5, Theorem 7], for a right $\wp^*$-semisimple ring R we need to prove:

(1) R is left artinian.
(2) The direct sum decomposition $R_R = A \oplus B \oplus C$ in [5, Theorem 7 (III)] is not necessarily a ring-direct sum decomposition.
(3) R is not necessarily left $\wp^*$-semisimple.
(4) In general, B_R in (ii) of [5, Theorem 7 (III)] is not CS.

PROOF. (1) By [5, Theorem 7], R is right artinian, and for any right R-module M, $M = P \oplus Q$, where P_R is projective, and Q_R is quasi-injective. By [1, Theorem 27.11], P is a direct sum of cyclic modules, each of which is isomorphic to some eR with a primitive idempotent $e^2 = e \in R$. As R is right artinian, $Q = \oplus_{i \in I} U_i$, where each U_i is uniform and isomorphic to the quasi-injective hull of some simple right R-module (compare with [7]). By [5, Theorem 7], each $E(S_i)$ is 2-generated. But, as a right artinian ring, R has only finitely many non-isomorphic simple right R-modules, and finitely many non-isomorphic indecomposable projective right R-modules. It follows that R has only finitely many non-isomorphic indecomposable right R-modules, or in other words, R is a ring of finite representation type. Thus it is well-known that R is left artinian.

(2) We consider the following example.

Let $\mathbb{C}$ and $\mathbb{R}$ be the fields of complex numbers and real numbers, respectively. Let $V = \left\{\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right) \mid y \in \mathbb{C}\right\} \subset \left\{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \mid x,\ y \in \mathbb{C}\right\}$, $K = \left\{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \mid x \in \mathbb{C}\right\}$, and $F = \left\{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \mid x \in \mathbb{R}\right\}$. Then V is a K-bialgebra with $\dim(V_K) = \dim({}_K V) = 1$, $\dim(V_F) = \dim({}_F V) = 2$, $V^2 = 0$, and $KV = VK = FV = VF = V$. Notice that $K \cong \mathbb{C}$, $F \cong \mathbb{R}$, and F is a subfield of K with $\dim(K_F) = 2$. We consider the ring

$$R = \begin{pmatrix} K & V & 0 \\ 0 & K & V \\ 0 & 0 & F \end{pmatrix}$$

and aim to show first that R is a right $\wp^*$-semisimple ring.

Matrix rings of this type are very useful in describing the structure of some other interesting classes of rings, see [6].

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Let

$$L_1 = \begin{pmatrix} K & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & V & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & V \\ 0 & 0 & F \end{pmatrix}.$$

Then ${}_R R = L_1 \oplus L_2 \oplus L_3$, a direct sum of three local left ideals with $l(L_1) = 1$, $l(L_2) = l(L_3) = 2$. In particular, R is left serial. Moreover, $R/J(R) \cong K \oplus K \oplus F$, that is, commutative. Hence by [3, Theorem 3.2], the injective hull of every simple right R-module is uniserial, that is, its lattice of submodules is linearly ordered by inclusion. Let S be a simple right R-module, and let $E(S)$ be the injective hull of S. As $J(R)^2 = 0$, we have $E(S)J(R) \subseteq S$. This shows that the uniserial module $E(S)/S$ is semisimple, hence it is zero or simple. Therefore $l(E(S)) \leqslant 2$.

Set

$$A_1 = \begin{pmatrix} K & V & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K & V \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix}.$$

Then A_1 is injective, because $l(A_1) = 2$. Moreover, C_1 and C_2 are nonsingular right ideals, C_1 has length 3, and uniform dimension 2. Each simple submodule of C_1 is isomorphic to C_2.

Write $\mathrm{Soc}(C_1) = S \oplus T$ where S, T are minimal right ideals. Let T^* be a maximal essential extension of T in C_1, that is, T^* is a closure of T in C_1. If $l(T^*) > 1$, then $T^* \oplus S = C_1$, a contradiction. Hence $l(T^*) = 1$, or equivalently, $T^* = T$, that is T is a closed submodule of C_1. Therefore, C_1/T is uniform (compare with [2, Section 5.10 (1)]), and it has length 2. Whence C_1/T must be injective, and since S embeds in C_1/T, we have $E(S) \cong C_1/T$. Moreover, C_1/T is not projective, because otherwise T would split in C_1. A similar consideration yields that C_1/S is injective, uniform, not projective, $l(C_1/S) = 2$ and $E(T) \cong C_1/S$. It follows that $E(C_2)$ is also not projective, and $l(E(C_2)) = 2$.

Set $A = A_1$, $C = C_1 \oplus C_2$. Then $R = A \oplus C$ and $CA = 0$. Thus R is a ring of Theorem 1.1 with $B = 0$, but $AC \neq 0$. This proves (2).

(3) We consider the left side of the above right $\wp^*$-semisimple ring R. Let L_i be as before. It is easy to check that L_3 is a two-sided ideal of R, for which we have

$$R/L_3 \cong \begin{pmatrix} K & V \\ 0 & K \end{pmatrix} \cong \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}.$$

It follows that $\left(\begin{smallmatrix} 0 & V \\ 0 & K \end{smallmatrix}\right)$ is an injective left ideal of R/L_3. Hence $(L_2 + L_3)/L_3$ ($\cong L_2$) is an injective left R/L_3-module. Therefore L_2 is a quasi-injective left ideal of R.

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We aim to show that it is even an injective left R-module. It is obvious that L_2 is L_1-injective. Let

$$T = \begin{pmatrix} K & V & 0 \\ 0 & 0 & V \\ 0 & 0 & 0 \end{pmatrix}.$$

Then T is an essential left ideal of R. As $V^2 = 0$, it is clear that $T\,\mathrm{Soc}(L_3) = 0$. This means that $\mathrm{Soc}(L_3)$ is a singular left ideal of R. As L_2 is a nonsingular left ideal of R, there is no nonzero map from submodules of L_3 to L_2. This shows that L_2 is L_3-injective. Thus by [1, Section 16.13 (2)], L_2 is $(L_1 \oplus L_2 \oplus L_3 = R)$-injective, as claimed.

Now if R is left $\wp^*$-semisimple, so applying [5, Theorem 7] for left $\wp^*$-semisimple rings we see that L_3 must be injective. This means that R is a direct sum of a simple left ideal and two injective uniform left ideals of length 2. By [2, Section 13.5 (e), (g)], R must be right serial also. However, this is impossible because the local right ideal C_1 defined in the proof of (2) is not uniform. Thus R is not left $\wp^*$-semisimple, completing the proof of (3).

We prove (4) by giving a more general observation on CS modules in the next section. In particular, in Proposition 3.2, we will give more information on the structure of the right ideal $B \subseteq R$ of Theorem 1.1. □

3. A correction

The conclusion in (ii) of [5, Theorem 7 (III)] and [5, Lemma 11], that B_R is CS, is unfortunately incorrect. This mistake arose from an incorrect conclusion in the proof of [5, Theorem 7] on page 144, line 6, that '$\mathrm{ann}_R(w) = \mathrm{ann}_R(ur) \cap \mathrm{ann}_R(vs)$ if $w = ur + vs$'. Fortunately, this mistake does not affect the correctness of other parts of Theorem 1.1, because the CS conclusion for B_R was not used anywhere in the remainder of the proof of [5, Theorem 7]. In (a) of the proof of [5, Lemma 13] the fact that B_R is a direct sum of uniform modules was used, but this property follows from the definition of B_R and not because B_R was CS.

For the purpose of showing that the right ideal B of R in [5, Theorem 7] is, in general, not a CS right R-module, we first prove a general result, which might also be of interest on its own.

For a module M_R over a ring R we denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules. For $N \in \sigma[M]$, the injective hull of N in $\sigma[M]$ is denoted by $E_M(N)$. It is known that $E_M(N)$ is M-injective and for each nonzero proper submodule T of $E_M(N)$, T is not M-injective. This fact is used in the proof of Lemma 3.1 below. For more on basic properties of $E_M(N)$ we refer to [8, Section 15].

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LEMMA 3.1. *For a right module M_R over a ring R, let $M_R = M_1 \oplus \cdots \oplus M_t \oplus M_{t+1} \oplus \cdots \oplus M_n$, such that each M_i is uniform, $l(M_1) = \cdots = l(M_t) = 2$, $(t \geqslant 1)$, and $l(M_{t+1}) = \cdots = l(M_n) = 1$. Assume further that* $\mathrm{Soc}(M_i) \cong \mathrm{Soc}(M_j)$, *and $l(E_M(M_i)) > 2$ for all $i,\ j = 1, 2, \ldots, n$. Then M is a CS module if and only if $t = 1$.*

PROOF. Let $t = 1$ and let V be a closed submodule of M. If $M_1 \cap V = 0$, then by modularity we have $M_1 \oplus V = M_1 \oplus V'$ where $V' = (M_1 \oplus V) \cap (M_2 \oplus \cdots \oplus M_n)$. Since V' is a direct summand of $M_2 \oplus \cdots \oplus M_n$, it is clear that $M_1 \oplus V$ is a direct summand of M. It follows that V is a direct summand of M. Now we consider the case $U = M \cap V \neq 0$. If $U = M_1$, then by modularity, we conclude that V is a direct summand of M. If $U \neq M_1$, then U is a minimal submodule of M_1. Let S^* be the closure of U in V. As V is closed in M, S^* must be closed in M (see, for example, [2, Section 1.10 (4)]). Hence $l(S^*)$ is at least 2. Since $S^* \cap (M_2 \oplus \cdots \oplus M_n) = 0$, we have $S^* \oplus (M_2 \oplus \cdots \oplus M_n) = M$. From here we conclude as before that V is a direct summand of M. Thus M is CS.

Conversely, assume that $t > 1$. We use an idea in the proof of [4, Theorem 6] to show that M is not CS. Suppose on the contrary that M_R is CS. Then for $j = 2, \ldots, t$, $M_1 \oplus M_j$ is a CS module. Hence by [2, Section 7.3 (ii)], M_1 is M_j-injective. Let S_i be the socle of M_i. Then M_1 is S_i-injective for any i.

Let $\varphi : S_1 \to S_j$ be an isomorphism, and let $L = \{x + \varphi(x) \mid x \in S_1\}$. Then L is a minimal submodule of $M_1 \oplus M_j$. There are two possibilities:

(a) L is closed in $M_1 \oplus M_j$. Hence L is a direct summand of $M_1 \oplus M_j$. This is impossible by the Krull-Schmidt Theorem (compare with [1, Section 12.9]).

(b) L is not closed in $M_1 \oplus M_j$. Then the closure L' of L in $M_1 \oplus M_j$ has length at least 2. As $l(M_1 \oplus M_j) = 4$, we have $M_1 \oplus M_j = L' \oplus M_j = M_1 \oplus L'$. It follows $M_1 \cong M_j$. Thus by [1, Section 16.13 (2)], M_1 is $(M_1 \oplus \cdots \oplus M_t \oplus M_{t+1} \oplus \cdots \oplus M_n = M)$-injective, a contradiction to the assumption that $l(E_M(M_1)) > 2$. □

The following example shows the existence of a ring R $(= B)$ of Theorem 1.1 with $A = C = 0$, but R is not right CS.

EXAMPLE 1 (compare with [4, Example 3.2]). Let

$$R = \begin{pmatrix} \mathbb{C} & 0 & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix}.$$

Then R is a right (and left) SI ring, that is a ring over which every singular right (left) R-module is injective (see [3, Chapter 3]). Let

$$e_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Then R can be written in the form $R = e_{11}R \oplus e_{22}R \oplus e_{33}R$. It is clear that

$$e_{33}R \cong \mathrm{Soc}(e_{11}R) \cong \mathrm{Soc}(e_{22}R), \quad l(e_{11}R) = l(e_{22}R) = 2.$$

Moreover,

$$E(e_{11}R) = \begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and hence $l(E(e_{11}R)) = 3$. Thus R is a ring of Theorem 1.1 with $A = C = 0$. However, by Lemma 3.1, R is not right CS.

In light of Lemma 3.1, we can give some more information on the structure of the right ideal B of R in Theorem 1.1.

PROPOSITION 3.2. *Let R be a ring of Theorem* 1.1 *and B be a right ideal of R described in* III (ii). *Then $B_R = V_1 \oplus \cdots \oplus V_k \oplus V_{k+1} \oplus \cdots \oplus V_n$, where each V_i has a homogeneous socle such that for $i \neq j$,* $\mathrm{Soc}(V_i) \ncong \mathrm{Soc}(V_j)$. *Moreover, $V_1, \ldots, V_k$ are CS, and $V_{k+1}, \ldots, V_n$ are not CS.*

PROOF. As in Theorem 1.1 part III (ii), $B = B_1 \oplus \cdots \oplus B_m$ where each B_i is uniform of length 1 or 2, and the injective hull of each simple submodule of B_R has length 3. We can renumber the B_i's such that $B_1, \ldots, B_k$ has length 2, each pair of these B_i's do not have isomorphic socles, and no $B_j \in \{B_{k+1}, \ldots, B_m\}$ is of length 2 and has a socle isomorphic to the socle of one of the B_i's for $i = 1, \ldots, k$. The next $B_{k+1}, \ldots, B_n$ $(n \leqslant m)$ have the property that each pair of them do not have isomorphic socles, each B_j, $k+1 \leqslant j \leqslant n$, has length 2 and for each of them there is at least one more $B_{t_j} \in \{B_n, \ldots, B_m\}$ such that $l(B_{t_j}) = 2$ and $\mathrm{Soc}(B_{t_j}) \cong \mathrm{Soc}(B_j)$. The socle of each $B_t \in \{B_{n+1}, \ldots, B_m\}$ is isomorphic to either the socle of some B_i, $1 \leqslant i \leqslant k$, or the socle of some B_j with $k+1 \leqslant j \leqslant n$.

Now let $[B_i]$ be the direct sum of all $B_{i'} \in \{B_1, \ldots, B_m\}$ with $\mathrm{Soc}(B_{i'}) \cong \mathrm{Soc}(B_i)$. By the structure of the right ideals A, B, C of R in Theorem 1.1, there is no nonzero homomorphism of any submodule of A_R and respectively, of any submodule of C_R to B_R. This implies that every submodule of B is A- and C-injective. Hence any B-injective submodule of B is injective. Thus we can apply Lemma 3.1 to see that $[B_1], \ldots, [B_k]$ are CS modules, and $[B_{k+1}], \ldots, [B_n]$ are not CS, proving Proposition 3.2. □

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