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SEQUENCES OF CONTRACTIONS IN A GENERALIZED METRIC SPACE

BY

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The main aim of this paper is to study the convergence of a sequence of contractions in a generalized metric space. More specifically, we investigate the following question:

"If a sequence of contractions $\{f_r\}$ with fixed points u_r $(r=1, 2, \ldots)$ converges to a mapping f with a fixed point u, under what conditions will the sequence u_r converge to u?"

A partial answer to the above question has been given in metric spaces by Bonsall [1]. This result has since been improved by Russell and Singh [6]. Further results will now be given in a generalized metric space.

During the course of our investigations we shall make use of two fixed point theorems of Luxemburg ([3], [4]), and also a more general fixed point theorem of Margolis [5].

The generalized metric space, first introduced by W. A. J. Luxemburg, we define as follows:

DEFINITION. *A generalized metric space* (X, d) *is a pair composed of a non-empty set* X *and a distance function* $d\colon X\times X\to[0, \infty]$ *satisfying the usual axioms for a metric space:*

(a) $d(x, y) = 0$ *if and only if* $x = y$.

(b) $d(x, y) = d(y, x)$.

(c) $d(x, y) \leqq d(x, z)+d(z, y)$.

If further:

(d) $\lim_{n, m\to\infty} d(x_n, x_m) = 0 \Rightarrow \lim_{n\to\infty} d(x, x_n) = 0$,

where $x_n \in X (n=1, 2, \ldots)$ *and* x *is unique, then* (X, d) *is called a generalized complete metric space.*

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THEOREM 1. *Suppose* f_r $(r=1, 2, \cdots)$ *is a sequence of self-mappings of a generalized complete metric space X satisfying the following:*

(1) $d(f_r x, f_r y) \leq \rho d(x, y)$ $(0 \leq \rho < 1)$ *for all* (x, y) *in* X *with* $d(x, y) < \infty$.
(2) *The family of contractions* f_r, *have fixed points* u_r $(r=1, 2, \ldots)$.
(3) $\lim_{r\to\infty} f_r x = fx$ *for all* $x \in X$, *where f is any self-mapping of X.*
(4) *Let* $x_0 \in X$ *be arbitrary and define* $x_n = fx_{n-1}$. *Then there exists an index* $N(x_0)$ *such that* $d(x_N, x_{N+l}) < \infty$, $l=1, 2, \ldots$.

Then $\lim_{r\to\infty} u_r = u$, *and u is a fixed point of f.*

Proof. Since $\rho < 1$ is the same Lipschitz constant for all f_r, we get

$$d(fx, fy) = \lim_{r\to\infty} d(f_r x, f_r y) \leq \rho d(x, y)$$

for all (x, y) in X with $d(x, y) < \infty$. Hence f is a contraction on X. Using property (4) we can show that f has a fixed point u say. By an inequality of Luxemburg [3], we have, for each $r=1, 2, \ldots,$

$$d(u_r, f_r^n x_0) < \frac{\rho^{n-N}}{1-\rho} d(f_r^N x_0, f_r^{N+1} x_0),$$

where $N(x_0)$ is an index and $n \geq N$.

Put $n=N=0$ and $x_0=u$. Then

$$d(u_r, u) \leq \frac{1}{1-\rho} d(u, f_r u) = \frac{1}{1-\rho} d(fu, f_r u).$$

But $r \to \infty$, $d(fu, f_r u) \to 0$. Hence

$$\lim_{r\to\infty} d(u_r, u) = 0.$$

Example 1. Let $X=\{1, 2, 3, \ldots, n, \ldots\}$ and let

$$d(i,j) = \begin{cases} \infty, & i \neq j \\ 0, & i = j \end{cases}.$$

Let f_r, for each $r=1, 2, \ldots,$ as well as f be the identity mapping on X, i.e., $f_r i = i = fi$ for each r and each i. Let $u_r = r$ for each $r=1, 2, \ldots$. Now all the conditions of the above theorem are fulfilled. Also $\lim_{r\to\infty} u_r = \infty$, and ∞ is a fixed point for f.

Remark. A "local" version of the above theorem can be similiarly proved by using Luxemburg's "local" theorem [4].

THEOREM 2. *Suppose* $f(r=1, 2, \ldots)$ *is a family of self-mappings of a generalized complete metric space X satisfying the following:*

(1) $d(f_r x, f_r y) \leq \rho d(x, y)$, $(0 \leq \rho < 1)$ *for all* x, y *in* X *with* $d(x, y) \leq C$, $C > 0$.
(2) *The family of local contractions* f_r *have fixed points* u_r $(r=1, 2, \ldots)$.

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(3) $\lim_{r\to\infty} f_r x = fx$ *for all* $x \in X$ *where* f *is any self-mapping of* X.

(4) *Let* $x_0 \in X$ *be arbitrary and define* $x_n = fx_{n-1}$. *Then there exists an index* $N(x_0)$ *such that* $d(x_n, x_{n+l}) \le C$ *for all* $n \ge N$ *and* $l = 1, 2, \ldots$.

Then $\lim_{r\to\infty} u_r = u$, *and* u *is a fixed point of* f.

Proof. Adopt the procedure of Theorem 1 and use the inequality of Luxemburg's "local" theorem [4].

Example 2. Let X be the extended reals with the ordinary Euclidian metric.

Let f_r: $X \to X$ be defined by $f_r x = (x+1)/(r+1)$. Now $\lim_{r\to\infty} f_r x = 0 = fx$. The fixed points of f_r are given by $u_r = 1/r$. All the conditions of the above theorem are satisfied for any $c > 0$. Clearly $\lim_{r\to\infty} u_r = 0$ where 0 is a fixed point of f.

Remark. If all of the family $\{f_r\}$ commute, then f and f_r share a common fixed point. Using Luxemburg's extra condition (C3), we can show that f and $\{f_r\}$ share a common unique fixed point in both of the above theorems.

We now prove a fixed point theorem "of the alternative" for a sequence of contractions in a generalized complete metric space.

THEOREM 3. *Suppose* (X, d) *is a generalized complete metric space and* f_r: $X \to X$ *is a sequence of contractions in the sense that* $d(x, y) < \infty \Rightarrow d(f_r x, f_r y) \le \rho d(x, y)$, $(0 < \rho < 1)$ *and* $(r = 1, 2, \ldots)$. *Let* $\lim_{r\to\infty} f_r x = fx$ *for all* $x \in X$, *where* f *is any self-mapping of* X. *Suppose* $x_0 \in X$ *and consider the sequences of successive approximations with initial element* x_0; $x_0, f_r x_0, f_r^2 x_0, \ldots, f_r^l x_0, \ldots$, *where* $l = 0, 1, 2, \ldots$. *Then the following alternative holds:*

Either (a) for every $l = 0, 1, 2, \ldots$ *one has* $d(f_r^l x_0, f_r^{l+1} x_0) = \infty$, *or* (b) *the sequences of successive approximations are d-convergent to* u_r $(r = 1, 2, \ldots)$, *the fixed points of* f_r *and* $\lim_{r\to\infty} u_r = u$, *a fixed point of* f.

Proof. There are two mutually exclusive possibilities: Either (1) for every $l = 0, 1, 2, \ldots$ one had $d(f_r^l x_0, f_r^{l+1} x_0) = \infty$ which is precisely alternative (a), or (2) $d(f_r^l x_0, f_r^{l+1} x_0) < \infty$.

Assume that (2) holds. Follow the proof of Theorem 2 of [2] and we get the following inequality:

$$d(f_r^n x_0, f_r^{n+1} x_0) < \rho^{n-N} \cdot \frac{1-\rho^l}{1-\rho} \cdot d(f_r^N x_0, f_r^{N+1} x_0)$$

whenever $n \ge N(x_0)$.

It can be shown using the method of the above theorem that for each $r = 1, 2, \ldots$, there exists an element u_r in X such that $\lim_{r\to\infty} d(f_r^n x_0, u_r) = 0$.

One can now show that each map f_r has fixed point u_r $(r = 1, 2, \ldots)$. I.e., $f_r(u_r) = u$ for each r.

Since $d(x, y) < \infty$ we now have

$$d(f_r x, f_r y) \le \rho d(x, y),$$

and
$$\lim_{r\to\infty} d(f_r x, f_r y) \le \rho d(x, y)$$

i.e.,
$$d(fx, fy) \le \rho d(x, y).$$

Hence f is a contraction on X and by Diaz and Margolis [2], has fixed point u, say.

Referring to the above inequality we have for each $r = 1, 2, \ldots,$

$$d(u_r, f_r^n x_0) < \frac{\rho^{n-N}}{1-\rho} d(f_r^N x_0, f_r^{N+1} x_0)$$

whenever $n \ge N(x_0)$.

Using the same procedure as in Theorem 1, we can now show that $\lim_{r\to\infty} u_r = u$.

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