Can. J. Math., Vol. XXIII, No. 3, 1971, pp. 421–425

SOME THEOREMS ON THE STRUCTURE OF NEARLY EQUICONTINUOUS TRANSFORMATION GROUPS

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The purpose of this paper is to extend the theorems in [**3**; **7**] to uniform spaces and to prove some additional theorems. These results are related to [**4**; **5**]. Notation and definitions are as in the book [**2**]. For a general reference on nets see [**6**]. All topological spaces are assumed to be Hausdorff.

THEOREM 1. *Let* (X, T, Π) *be a transformation group, where* X *is a locally compact, locally connected, uniform space. Let* E *denote the set of all points at which* T *is equicontinuous and* $N = X - E$. *Let* N *be closed totally disconnected and each orbit closure in* N *be compact and let* E *be connected. Then* N *contains at most two minimal sets.* (*Note: We will assume that* $N \neq \emptyset$ *so that* N *will contain at least one minimal set.*)

Proof. Let $s = \{s_a | a \in D\}$ be a net in T. If $x \in X$, let $C(x, s)$ denote the set of accumulation points of $\{xs_a | a \in D\}$. We define

$$A = \{x \in X | C(x, s) \cap E \neq \emptyset\}.$$

We will prove that

(1) A is open-closed in E.

Choose $z \in A$ and let $x \in C(z, s) \cap E$. Choose $\alpha \in \mathscr{W}$ ($\mathscr{W}$ is a compatible uniformity for X) such that $\overline{x\alpha} \cap N = \emptyset$ and $\overline{x\alpha}$ is compact. Choose β such that $\beta^2 \subset \alpha$ and a neighbourhood U of z such that $y \in U \Rightarrow yt \in zt\beta$ for every $t \in T$, and $U \subset E$. Since $x \in C(z, s)$, we may select a subnet u of s such that $zu_a \in x\beta$ for every a. If $y \in U$, then $(x, yu_a) = (x, zu_a) \cdot (zu_a, yu_a) \in \beta \cdot \beta \subset \alpha$. Thus $yu_a \in \overline{x\alpha}$ for every $a \in D'$, hence $\{yu_a\}$ has a cluster point $w \in \overline{x\alpha}$. Then $w \in C(y, s) \cap E$ and $U \subset A$. We have shown A is open in E.

We will now prove that A is closed in E. Let $x \in \bar{A} \cap E$ and choose $\alpha \in \mathscr{W}$ such that

$$\overline{x\alpha} \cap N = \emptyset \text{ and } \overline{x\alpha} \text{ is compact.}$$

Choose β such that $\beta^3 \subset \alpha$, and choose a neighbourhood U of x such that $w \in U$ implies $wt \in xt\alpha$ for every t in T. Since $x \in \bar{A}$, we may choose $y \in U \cap A$, $z \in C(y, s) \cap E$ and γ such that $\gamma \cdot \gamma \subset \beta$ and a neighbourhood V of z such that $w \in V$ implies $wt \in zt\gamma$ for every t in T. Choose a subnet u of s such that $yu_a \in V$ for every $a \in D'$. Then

$$(y, yu_a u_{b'}) = (y, zu_b^{-1}) \cdot (zu_b^{-1}, yu_a u_{b'}) \in \gamma \cdot \gamma \subset \beta$$

Received July 16, 1970.

for all $a \in U$ such that $a > b$, b in some element of D'. Thus

$$(x, xu_a u_b^{-1}) = (x, y) \cdot (y, yu_a u_b^{-1}) \cdot (yu_a u_b^{-1}, xu_a u_b^{-1}) \in \beta \cdot \beta \cdot \beta \subset \alpha$$

for every a in u such that $a > b$, hence $xu_a u_b^{-1} \in \overline{x\alpha}$ for every a in u such that $a > b$. Therefore, $\{xu_a u_{b^*}^{-1}\}$ has an accumulation point w in $\overline{x\alpha}$ and wu_b is an accumulation point of $\{xu_a\}$. Since E is invariant under T we have

$$wu_1 \in C(x, s) \cap E$$

so that $x \in A$. We have shown that A is closed in E.

Now let $z \in N$. Since T is not equicontinuous, there is an $\alpha \in \mathscr{W}$ such that for every neighbourhood U of z, $Ut \not\subset xt\alpha$ for some t in T. Choose $\alpha, \beta \in \mathscr{W}$ such that $\beta \cdot \beta \subset \alpha$, and a finite cover $U_1, U_2, \ldots, U_n$ of $\overline{zT}$ such that $\overline{U_i}$ is compact, $U_i \subset y_i\beta$ for some y_i and $\mathrm{Fr}(U_i) \cap N = \emptyset$. Let $\{V_\gamma | \gamma \in U\}$ be a net of connected open neighbourhoods of z such that $V_\gamma \subset z\gamma$ and let $U^1 = U_1 \cup \ldots \cup U_n$. For each γ there is a t_γ such that $V_\gamma t_\gamma^{-1} \not\subset zt_\gamma^{-1}\alpha$; Note that $V_\gamma t_\gamma^{-1}$ meets some U_i but is not contained in any U_i. Since $V_\gamma t_\gamma^{-1}$ is connected, there is a x_γ such that $x_\gamma t_\gamma^{-1} \in \mathrm{Fr}(U_i)$ for some i. Since $\bigcup_{i=1}^n \mathrm{Fr}(U_i)$ is compact $\{x_\gamma t_\gamma^{-1}\}$ has an accumulation point x, and a subnet $\{x_\delta t_\delta^{-1} | \delta \in D'\}$ such that $\{x_\delta t_\delta^{-1}\} \to x$. We will prove that $\{xt_\delta\} \to z$. Let γ be given and find $\rho \in \mathscr{W}$ such that $\rho \cdot \rho \subset \gamma$. Since $x \in E$ there exists a neighbourhood W of x such that $y \in W \Rightarrow yt \in xt\rho$ for all $t \in T$; choose $b \in D'$ such that $x_\delta t_\delta^{-1} \in W$ for all $\delta \subset b$ and $b \subset \rho$. Now for all $\delta \subset b$ we have

$$(z, xt_\delta) = (z, x_\delta) \cdot (x_\delta, xt_\delta) \in \rho \cdot \rho \subset \gamma$$

This proves that $\{xt_\delta\} \to z$. Now if we let $t = \{t_\delta | \delta \in D'\}$ we get from (1) that

$$C(x, t) \cap E = \emptyset \text{ for every } x \in E. \tag{2}$$

Now let U be an open neighbourhood of z such that $\bar{U}$ is compact and $N \cap \mathrm{Fr}(U) = \emptyset$. Let $B = \{x \in E | \{xt_a | a \in D\}$ is eventually in $U\}$. We will prove that B is open and closed in E.

Let $x \in \bar{B} \cap E$ and assume that $x \notin B$. Then there exists a subnet $u = \{u_a | a \in D'\}$ of t such that $\{xu_a\} \notin U$ for every a in D'. Since T is equicontinuous at x, there is an x_α such that $(x_\alpha u_a, xu_a) \in \alpha$ for all a in D'. Since $\{x_\alpha U_a\}$ is eventually in U, we may choose a subnet $\mathscr{V} = \{v_\alpha | \alpha \in \mathscr{W}\}$ of U such that

$$(x_\alpha v_\alpha, xv_\alpha) \in \alpha \text{ for every } \alpha \in \mathscr{W} \text{ and } x_\alpha v_\alpha \in U. \tag{3}$$

Then $\{x_\alpha v_\alpha\}$ clusters at a point $y \in \bar{U}$. By (3), $y \in C(x, u)$, hence $y \in \mathrm{Fr}(U)$. Therefore, we have $y \in \mathrm{Fr}(U) \cap C(x, t)$, and this contradicts (2). We have shown that B is closed in E.

Let $x \in B$ and assume $x \notin \mathrm{int}\, B$. Then for each $\alpha \in \mathscr{W}$ choose an open neighbourhood $U_\alpha \subset E$ of x such that $y \in u_\alpha$ implies $(xs, ys) \in \alpha$ for every

$s \in T$. Since $U_\alpha - B \neq \emptyset$, there is a $y_\alpha \in U_\alpha$ such that given any $a \in D$ there is a $b > a$ such that $y_\alpha t_b \notin U$. Since $x \in B$ we may select a subnet $u = \{u_\alpha | \alpha \in \mathscr{V} \subset \mathscr{W}\}$ such that $y_\alpha u_\alpha \notin U$ and $x u_\alpha \in u$ for every $\alpha \in \mathscr{V}$. Then $\{x u_\alpha\}$ clusters at a point $w \in \bar{U}$. Since $(x u_\alpha, x_\alpha u_\alpha) \in \alpha$ for every α, $\{y_\alpha u_\alpha\}$ clusters at w also. Thus $w \in \operatorname{Fr}(U)$ and hence $C(x, t) \cap E \neq \emptyset$; this contradicts (2), and shows that B is open. Finally since $B \neq \emptyset$, we find that $B = E$. Since each point of N has a neighbourhood base consisting of open sets with compact closures whose boundaries do not intersect N, we have shown that

(4) if $z \in N$, there exists a net $t = \{t_a | a \in D\}$

in T such that $\{x t_a\} \to z$ for each $x \in E$.

We now show that N contains at most two minimal sets. Assume that x_1, x_2, and $x_3 \in N$ are such that $\overline{x_1 T}$, $\overline{x_2 T}$, and $\overline{x_3 T}$ are distinct and minimal. Since X is locally compact and $\overline{x_i T}$ is compact and N is totally disconnected, we can find an open neighbourhood W such that

$$W \supset \overline{x_1 T},\ \bar{W} \cap (\overline{x_2 T} \cup \overline{x_3 T}) = \emptyset,$$

and $N \cap \operatorname{Fr}(W) = \emptyset$. Choose connected open neighbourhoods of x_1 and x_2 respectively such that $V \subset W$, $\bar{U}$ is compact and $\bar{U} \cap \overline{x_3 T} = \emptyset$. Since E is dense in X, we may select $y \in U \cap E$. By (4) there is $s' \in T$ such that $ys' \in V$. Then $\bar{C} = \bar{V} \cup \bar{U}s'$ is a compact connected set such that $C \cap \overline{x_1 T} \neq \emptyset$, $C \cap \overline{x_2 T} \neq \emptyset$ and $C \cap \overline{x_3 T} = \emptyset$.

Now choose an open neighbourhood M of x_3 such that $\bar{M} \cap C = \emptyset$. Choose $\alpha \in \mathscr{W}$ such that $\bar{M} \cap \bigcup_{x \in C} x_\alpha = \emptyset$ and let $s = \{s_a | a \in D\}$ be any net in T. For each $a \in D$, we have that $\overline{x_3 T} \cap C s_a^{-1} = \emptyset$ and for $i = 1, 2$ $\overline{x_i T} \cap C s_a^{-1} \neq \emptyset$ and since $C s_a^{-1}$ is connected, we must have

$$C s_a^{-1} \cap \operatorname{Fr}(W) \neq \emptyset.$$

Thus for each $a \in D$ we have $y_a \in C$ such that $y_a s_a^{-1} \in \operatorname{Fr}(W)$. Since $\operatorname{Fr}(W)$ and C are compact, there is a subnet $\{u_a | a \in D'\}$ of s such that

$$\{y_a s_a^{-1}\} \to y \in \operatorname{Fr}(W)$$

and $\{y_a\} \to x \in C$. Choose β such that $\beta \cdot \beta \subset \alpha$. Since $y \in E$, we may select a neighbourhood U of y such that if $w \in U$ then $wt \in yt\beta$ for all t in T. Choose b such that $a > b$ implies $y_a s_a^{-1} \in U$ and $y_a \in x\beta$, so that

$$(x, y s_a) = (x, y_a) \cdot (y_a, y s_a) \in \beta \cdot \beta \subset \alpha.$$

This means $y s_a \in \bigcup_{x \in C} x\alpha$ for every $a > b$. Thus the net $\{y s_a\}$ cannot eventually be in M. Since s is an arbitrary net in T this contradicts (4) and completes the proof.

In the following corollaries the hypothesis of the theorem is assumed.

COROLLARY 1. *If X is not compact, then N contains exactly one minimal set.*

Proof. This follows from the main theorem in exactly the same way that the theorem in [**7**] follows from the theorem in [**3**].

COROLLARY 2. *If x_1, x_2, x_3 belong to X, then of the three orbit closures, at least two have nonempty intersections.*

Proof. Each orbit closure contains a minimal set of N. The result follows.

Note. The corollaries of [**3**; **7**] are valid here.

The results above are generalizations of previous results on metric spaces to uniform spaces. The following theorems are new results concerning when the minimal sets of the above theorem are points.

LEMMA 1. *Under the hypothesis of the theorem if $z \in N$ and $\overline{xT} \subset N$ is a minimal set of N such that $z \notin \overline{xT}$ and $t = \{t_a | a \in D\}$ is the net which carries all of E to z then there exist a subnet $s = \{t_a | a \in D'\}$ of t such that*

$$s^{-1} = \{t_a^{-1} | a \in D'\}$$

will carry all of E to x.

Proof. The net t can be induced by a subnet $\mathscr{V}$ of $\mathscr{W}$, such that if α is in $\mathscr{W}$ there is a δ in $\mathscr{V}$ such that $\delta \subset \alpha$. Let U be any neighbourhood of z such that $\bar{U}$ is compact and $\mathrm{Fr}(U) \cap N = \emptyset$. For each $\delta \in \mathscr{V}$ there is a $\delta' \subset \delta$ such that $V_\delta t_{\delta'} \cap U \neq \emptyset$ where $V_\delta \subset x\delta$ is a connected neighbourhood of x. For each δ choose a y_δ in $V_\delta t_\delta \cap \mathrm{Fr}(U)$. Since $\mathrm{Fr}(U)$ is compact, the net $\{x_\delta | \delta \in \mathscr{V}\}$ has an accumulation point y in $\mathrm{Fr}(U)$, and we can assume without loss of generality that $\{y_\delta | \delta \in \mathscr{V}\} \rightarrow y$. Since $y \in E$, for each $\beta \in \mathscr{V}$ there is a neighbourhood W of y such that w in W implies $wt \in yt\gamma$ for all t in T where $\gamma \cdot \gamma \subset \beta$, and $y_\delta \in W$ for all $\delta \subset b \in \mathscr{V}$.

$$(x, yt_{\delta'}^{-1}) = (x, y_\delta t_{\delta'}^{-1}) \cdot (y_\delta t_{\delta'}^{-1}, yt_{\delta'}^{-1}) \in \gamma \cdot \gamma \subset \beta$$

for all $\delta \subset b'$ where $b' \subset b$ and $b' \subset \gamma$. It is clear that $s = \{t_{\delta'} | \delta \in \mathscr{V}\}$ is the required subnet.

THEOREM 2. *If N contains two minimal sets both are points.*

Proof. Assume x_1, x_2, x_3 belong to N and x_1 is in one minimal set and x_2 is in another. From Lemma 1 there is a net $t = \{ta | a \in D\}$ such that t takes all of E to x_3 and $t^{-1} = \{t_a^{-1} | a \in D\}$ takes all of E to x_2. Again from Lemma 1, there exists a subnet $s = \{t | a \in D'\}$ of t such that $s^{-1} = \{t_a^{-1} | a \in D'\}$ takes all of E to x_1. This is a contradiction. Hence N contains at most two points.

COROLLARY 3. *If X is not compact then N consists of a single point.*

Proof. This follows from Theorem 2 and Corollary 1.

LEMMA 2. *If N consists of a finite number of points then N consists of two points.*

Proof. Let $z \in N$ and $t = \{t_a | a \in D\}$ be the net which takes all of E to z. Since N is finite, we can find a subnet which maps z onto a single point of N. Let this subnet be $s = \{t_a | a \in D'\}$ and consider $s' = \{t_b^{-1} t_a | a < b\}$ where a

is in D' and b is a fixed element of D'. Now s' is a net in T which leaves z fixed and still takes all of E to z. Now assume N contains more than two points. The same process used in the lemma can be repeated here to get a subnet of s' such that s'^{-1} takes E to two different points of N, which is a contradiction.

THEOREM 3. *If N contains a point which is isolated from the rest of N, then N contains at most two points.*

Proof. Let z be the isolated point and $\alpha \in U$. Let $\{U_i | i = 1, \ldots, n\}$ be a finite subcover of $\overline{zT}$ such that z belongs to one U_i which does not meet the others, and $U_i \subset z\alpha$ for some $z_i \in \overline{zT}$. Let $t = \{t_a | a \in D\}$ be the net which takes all of E to z and find a subnet s of t which maps z into U_k where U_k is one of the U_i's. Now if there are two points outside U_k, we can find a subnet s', of s such that s'^{-1} will take E into both points, which is a contradiction. This, plus the fact that U_k is arbitrarily small, implies $\overline{zT}$ can contain only three points. Now if there are points outside $\overline{zT}$, then N contains two minimal sets and the result follows from the main theorem; if not, the results follow from Lemma 2.

COROLLARY 4. *If N contains a point which has a countable neighbourhood in N, then N has at most two points.*

Proof. Apply [**1**; Theorem 10.3] to $\overline{zT}$ to get a point z isolated from the rest of $\overline{zT}$ and the result follows.

COROLLARY 5. *If N is countable, then N contains at most two points.*

Remark. I originally set out to prove that N has at most two points. I have shown that if N has more than two points, then X must be compact, T must be non-abelian and have an infinite number of components, and N must be minimal or have exactly one fixed point. In addition, every neighbourhood in N must contain an uncountable number of points.

Question. Does one such transformation group exist?

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https://doi.org/10.4153/CJM-1971-044-x Published online by Cambridge University Press