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A PROPERTY OF BERNSTEIN–SCHOENBERG SPLINE OPERATORS

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1. Introduction

Let $B_n(f;x)$ denote the Bernstein polynomial of degree n on $[0,1]$ for a function $f(x)$ defined on this interval. Among the many properties of Bernstein polynomials, we recall in particular that if $f(x)$ is convex in $[0,1]$ then (i) $B_n(f;x)$ is convex in $[0,1]$ and (ii) $B_n(f;x) \geqq B_{n+1}(f;x)$, $(n=1,2,\ldots)$. Recently these properties have been the subject of study for Bernstein polynomials over triangles [**1**].

Our object here is to consider these properties in relation to the Bernstein–Schoenberg spline operator first introduced by Schoenberg [**6**]. We shall denote by $V_n^T(f;x)$ the B–S spline of degree n with reference to a knot sequence T (not necessarily distinct) in the interval $(0,1)$. The operator $V_n^T(f;\cdot)$ shares many properties with Bernstein polynomials. Besides its convergence properties, it also has the variation-diminishing property which yields the fact that if $f(x)$ is convex, then so is $V_n^T(f;x)$.

We shall give here an analogue of property (ii) for B–S operators. We also find conditions for equality to be attained and derive, as a special case, a result of Freedman and Passow [**3**] for $B_n(f;x)$. In Section 2 we give the preliminaries and a statement of results, which are contained in Theorem 1 and 2. The first Theorem is proved in Section 3 and the second in Section 4.

2. Preliminaries

For given integers $n \geqq 1, k \geqq 0$, take a sequence of knots $\{t_i\}_{-n}^{k+n+1}$ in $[0,1]$ satisfying

$$0 = t_{-n} = \cdots = t_0 < t_1 \leqq t_2 \leqq \cdots \leqq t_k < t_{k+1} = \cdots = t_{k+n+1} = 1$$

$$t_{i-n} < t_{i+1} \qquad (i=0,1,\ldots,k+n).$$

For $i=0,1,\ldots,k+n$ let $N_{n,i}(x) = N(x \mid t_{i-n},\ldots,t_{i+1})$ denote the B-spline of degree n with knots $t_{i-n},\ldots,t_{i+1}$ normalized so that $\sum_{i=0}^{k+n} N_{n,i}(x) = 1$. Following Schoenberg [**6**], for any function f on $[0,1]$, we set

$$V_n^T(f;x) = \sum_{i=0}^{k+n} f(\xi_i) N_{n,i}(x), \tag{2.1}$$

where $\xi_i=(1/n)(t_{i-n+1}+\cdots+t_i)$ and $T=\{t_i,\ldots,t_k\}$ denotes the set of knots with multiplicities in the open interval $(0,1)$. The operator V_n^T reproduces linear functions and reduces to Bernstein polynomials of degree n when $k=0$.

We note that if f is convex, then $V_n^T(f;x)\geqq f(x)$ with equality if and only if f is linear. This is so since for $x\in[0,1]$,

$$V_n^T(f;x)=\sum_{i=0}^{k+n} f(\xi_i)N_{n,i}(x)\geqq f\left(\sum_{i=0}^{k+n}\xi_i N_{n,i}(x)\right)=f(x).$$

In this paper we consider two operators $V_n^T(f;x)$ and $V_m^S(f;x)$ such that the B-splines $\{N_{n,i}(x)\}$ for V_n^T lie in the linear span of the B-splines $\{N_{m,i}(x)\}$ for V_m^S and show that then $V_n^T(f;x)\geqq V_m^S(f;x)$ when $f(x)$ is convex in $[0,1]$. It is clearly sufficient to prove the result for the following two cases:

(A) Firstly suppose $m=n$ and that S comprises T together with one extra knot, i.e. $S=\{s_1,\ldots,s_{k+1}\}$ $T=\{t_1,\ldots,t_k\}$ and $s_i=t_i$ $(i=1,\ldots,l)$, $t_l\leqq s_{l+1}<t_{l+1}$ and $s_i=t_{i-1}$ $(i=l+2,\ldots,k+1)$. In this case we shall prove

Theorem 1. *Suppose $f(x)$ is convex in $[0,1]$ and S and T are as given in (A). Then*

$$V_n^T(f;x)\geqq V_n^S(f;x) \tag{2.2}$$

and equality occurs only if f is linear on $[\xi_{i-1},\xi_i]$ for $i=l+1,\ldots,p+n$ where $p=\max\{i\colon t_i<s_{l+1}\}$. Moreover if f is any function (not necessarily convex) which is linear on $[\xi_{i-1},\xi_i]$ for $i=l+1,\ldots,p+n$, then equality holds in (2.2).

Remark. When $t_l<s_{l+1}<t_{l+1}$, then $p=l$. If $s_{l+1}=t_l=t_{l-1}=\cdots=t_{l-\nu}>t_{l-\nu-1}$, for some ν, then $p=l-\nu-1$.

(B) Secondly we suppose $m=n+1$ and S comprises the same distinct knots as T but with the multiplicity of each element increased by 1. In this case, we have

Theorem 2. *Suppose $f(x)$ is a convex function in $[0,1]$ and S and T satisfy the conditions in (B). Then*

$$V_n^T(f;x)\geqq V_{n+1}^S(f;x) \tag{2.3}$$

and equality occurs only if f is piecewise linear with simple knots at those ξ_i for which $\{t_{i-n+1},\ldots,t_i\}$ comprises at most two distinct elements. Moreover if f is any piecewise linear function (not necessarily convex) with knots as above, then equality occurs in (2.3).

By putting $k=0$ *we have*

Corollary. *If $f(x)$ is a convex function on $[0,1]$, then*

$$B_n(f;x)\geqq B_{n+1}(f;x) \tag{2.4}$$

and equality occurs only if f is linear on $[i/n, i+1/n]$ for $i=0,1,\dots,n-1$. Moreover if f is any function which is linear on $[i/n, i+1/n]$ for $i=0,1,\dots,n-1$ then equality occurs in (2.4).

Remark. Take $n \geqq 1$ and suppose f is linear on $[i/n, i+1/n]$ for $i=0,1,\dots,n-1$. Then for any $m \geqq 1$, f is linear on $[i/mn, i+1/mn]$, $i=0,1,\dots,mn-1$, and the corollary tells us that $B_{mn}(f;x)=B_{mn+1}(f;x)$. This yields a result of Freedman and Passow [3].

3. Proof of Theorem 1

Take S and T as in Theorem 1. As before we let $N_{n,i}(x)=N(x \mid t_{i-n},\dots,t_{i+1})$ $(i=0,1,\dots,k+n)$ and set

$$\tilde{N}_{n,i}(x)=N(x \mid s_{i-n},\dots,s_{i+1}) \quad (i=0,1,\dots,k+n+1).$$

Now for $i=0,1,\dots,k+n$, $\{t_{i-n},\dots,t_{i+1}\} \subseteq \{s_{i-n},\dots,s_{i+2}\}$ and so there are numbers α_i, β_i such that

$$N_{n,i}(x)=\alpha_i \tilde{N}_{n,i}(x)+\beta_i \tilde{N}_{n,i+1}(x). \tag{3.1}$$

We claim that $\alpha_i \geqq 0$, $\beta_i \geqq 0$. For $i=0,1,\dots,l-1$, we have $\{t_{i-n},\dots,t_{i+1}\}=\{s_{i-n},\dots,s_{i+1}\}$ and so $\alpha_i=1$, $\beta_i=0$. For $i=p+n+1,\dots,k+n$, we have $t_{i-n} \geqq s_{l+1}$ so that $\{t_{i-n},\dots,t_{i+1}\}=\{s_{i-n+1},\dots,s_{i+1}\}$ and so $\alpha_i=0$, $\beta_i=1$. For $i=l,\dots,p+n$, we have $t_{i-n}<s_{l+1}<t_{i+1}$. Thus the support of $N_{n,i}(x)$ contains $\{s_{i-n},\dots,s_{i+2}\}$ and so $\alpha_i \neq 0 \neq \beta_i$. Indeed if t_{i-n} has multiplicity μ in $\{t_{i-n},\dots,t_{i+1}\}$, then

$$N_{n,i}^{(n-\mu+1)}(t_{i-n}^{+})>0, \; \tilde{N}_{n,i}^{(n-\mu+1)}(t_{i-n}^{+})>0$$

while $\tilde{N}_{n,i+1}^{(n-\mu+1)}(t_{i-n}^{+})=0$. So (3.1) gives $\alpha_i>0$. Similarly considerations near t_{i+1} give $\beta_i>0$, which proves the assertion. Now letting $\tau_i=1/n(s_{i-n+1}+\dots+s_i)$, we have

$$V_n^S(f;x)=\sum_{i=0}^{k+n+1} f(\tau_i)\tilde{N}_{n,i}(x). \tag{3.2}$$

Also from (3.1), we see that

$$\begin{aligned} V_n^T(f;x) &= \sum_{i=0}^{k+n} f(\xi_i)N_{n,i}(x) \\ &= \sum_{i=0}^{k+n+1} \{\alpha_i f(\xi_i)+\beta_{i-1} f(\xi_{i-1})\}\tilde{N}_{n,i}(x) \end{aligned} \tag{3.3}$$

where we have set $\alpha_{k+n+1}=0=\beta_{-1}$. Comparing (3.2) and (3.3) and putting $f(x)=1$ gives

$$\alpha_i+\beta_{i-1}=1 \quad (i=0,1,\dots,k+n+1). \tag{3.4}$$

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Similarly, putting $f(x)=x$ gives

$$\alpha_i\xi_i+\beta_{i-1}\xi_{i-1}=\tau_i \quad (i=0,1,\dots,k+n+1). \tag{3.5}$$

If f is convex, then from (3.4) and (3.5),

$$f(\tau_i)\leqq\alpha_i f(\xi_i)+\beta_{i-1}f(\xi_{i-1})$$

and so from (3.2) and (3.3), we get (2.1).

Equality occurs in (2.1) if and only if for $i=0,1,\dots,k+n+1$,

$$f(\alpha_i\xi_i+\beta_{i-1}\xi_{i-1})=\alpha_i f(\xi_i)+\beta_{i-1}f(\xi_{i-1}), \tag{3.6}$$

For $i=0,1,\dots,l$, we have $\beta_{i-1}=0$ and $\alpha_i=1$ and (3.6) is satisfied. For $i=p+n+1,\dots,k+n+1$, we have seen above that $\beta_{i-1}=0$, $\alpha_i=1$ and again (3.6) is satisfied. For $i=l+1,\dots,p+n$ we have $\alpha_i>0$, $\beta_{i-1}>0$ and so if f is convex, (3.6) is valid only if f is linear in $[\xi_{i-1},\xi_i]$. Moreover if f is any function which is linear on $[\xi_{i-1},\xi_i]$, then (3.6) holds. □

4. Proof of Theorem 2

Let T comprise distinct elements $x_1,\dots,x_l$ with multiplicites $\mu_1,\dots,\mu_l$ respectively, so that $\sum_1^l \mu_j=k$. Then S comprises the same distinct elements $x_1,\dots,x_l$ with multiplicities $\mu_1+1,\dots,\mu_l+1$ respectively. We define $\{s_i\}_{-n-1}^{k+l+n+2}$ so that

$$0=s_{-n-1}=\cdots=s_0<s_1\leqq s_1\leqq s_2\leqq\cdots\leqq s_{k+l}<s_{k+l+1}=\cdots=s_{k+l+n+2}=1$$

and $S=\{s_1,\dots,s_{k+l}\}$. As before we let $N_{n,i}(x)=N(x\,|\,t_{i-n},\dots,t_{i+1})$ $(i=0,1,\dots,n+k)$, and we set

$$M_{n+1,i}(x)=N(x\,|\,s_{i-n-1},\dots,s_{i+1}) \quad (i=0,1,\dots,n+k+l+1).$$

Lemma 1. *For any i $(0\leqq i\leqq n+k)$, let $\lambda=\lambda(i)$ denote the number of distinct elements of T in (t_{i-n},t_{i+1}). Then for some μ (depending on i), we have*

$$N_{n,i}(x)=\sum_{j=0}^{\lambda+1}a_{ij}M_{n+1,j+\mu}(x) \tag{4.1}$$

where $a_{i0}>0$, $a_{i,\lambda+1}>0$ and $a_{ij}\geqq 0$ for $1\leqq j\leqq\lambda$.

Proof. For $k=1$ the coefficients (a_{ij}) can be determined explicitly. However for $k>1$ this does not appear feasible and so for all $k\geqq 1$ we shall prove the coefficients are non-negative by using the concept of total positivity.

Suppose t_{i-n} has multiplicity v in $\{t_{i-n},\dots,t_{i+1}\}$, i.e., $t_{i-n}=\cdots=t_{i-n+v-1}<t_{i-n+v}$. Choose μ so that $t_{i-n}=s_{\mu-n-1}=\cdots=s_{\mu-n-1+v}<s_{\mu-n+v}$. Then clearly (4.1) holds for

some constants a_{ij} $(j=0,1,\dots,\lambda+1)$. Now

$$N_{n,i}^{(n-\nu+1)}(t_{i-n}^{+})>0, \quad M_{n+1,\mu}^{(n-\nu+1)}(t_{i-n}^{+})>0, \quad N_{n+1,j+\mu}^{(n-\nu+1)}(t_{i-n}^{+})=0 \quad (j=1,\dots,\lambda+1).$$

So from (4.1), $a_{i0}>0$. Similar reasoning near t_{i+1} gives $a_{i,\lambda+1}>0$.

It remains to show that $a_{ij}\geqq 0$ for $1\leqq j\leqq \lambda$. Let $v_0,\dots,v_{\lambda+1}$ denote the distinct elements of $\{t_{i-n},\dots,t_{i+1}\}$. For $j=0,1,\dots,\lambda$ choose any point σ_j in (v_j,v_{j+1}) and consider the system of $\lambda+2$ equations

$$\sum_{j=0}^{\lambda+1} B_j M_{n+1,j+\mu}^{(n+1)}(\sigma_k)=0 \quad (k=0,1,\dots,\lambda), \qquad B_{\lambda+1}=a_{i,\lambda+1}. \tag{4.2}$$

Differentiating (4.1) $(n+1)$ times shows that the system (4.2) has a unique solution $B_j=a_{ij}$ $(j=0,1,\dots,\lambda+1)$. So the matrix for the system (4.1) is non-singular and solving by Cramer's rule gives

$$a_{ij}=a_{i,\lambda+1}(-1)^{\lambda+j+1}C_jC_{\lambda+1}^{-1} \quad (j=0,1,\dots,\lambda+1), \tag{4.3}$$

where

$$C_j=\mathrm{Det}(M_{n+1,q+\mu}^{(n+1)}(\sigma_p))_{p=0,\,q=0,\,q\neq j}^{\lambda,\ \lambda+1} \tag{4.4}$$

Now we recall that a matrix is called *totally positive* if all its minors are non-negative. We shall call a matrix $M=(m_{jk})_{j=0,k=0}^{r\ \ s}$ *checkerboard* if the matrix

$$((-1)^{j+k}m_{jk})_{j,k=0}^{r\ s}$$

is *totally positive.*

For $m\leqq n$, we set

$$M_{m,i}(x)=N(x\,|\,s_{i-m},\dots,s_{i+1}) \quad (i=m-n-1,\dots,n+k+l+1)$$

where $M_{m,i}(x)\equiv 0$ when $s_{i-m}=s_{i+1}$. Then

$$\frac{1}{n+1}M'_{n+1,j+\mu}(x)=b_jM_{n,j+\mu-1}(x)-b_{j+1}M_{n,j+\mu}(x)$$

where

$$b_j=\begin{cases}\dfrac{1}{s_{j+\mu}-s_{j+\mu-n-1}}, & s_{j+\mu-n-1}<s_{j+\mu},\\ 0, & s_{j+\mu-n-1}=s_{j+\mu}.\end{cases}$$

Thus

$$M'_{n+1,j+\mu}(x)=\sum_{k=0}^{\lambda+2} a_{jk}^{[1]}M_{n,k+\mu-1}(x) \tag{4.5}$$

where

$$a^{[1]}_{jk}=(n+1)b_j\,\delta_{jk}-(n+1)b_{j+1}\,\delta_{j+1,k}.$$

It is easily seen that the matrix $(a^{[1]}_{jk})^{\lambda+1,\lambda+2}_{j=0,k=0}$ checkerboard. Similarly, we have

$$M'_{n,j+\mu-1}(x)=\sum_{k=0}^{\lambda+3}a^{[2]}_{jk}M_{n-1,k+\mu-2}(x) \tag{4.6}$$

where the matrix $a^{[2]\lambda+2,\lambda+3}_{jk\;j=0,k=0}$ is checkerboard. Differentiating (4.5) and applying (4.6) gives

$$M''_{n+1,j+\mu}(x)=\sum_{k=0}^{\lambda+2}\sum_{l=0}^{\lambda+3}a^{[1]}_{jk}a^{[2]}_{jk}M_{n-1,l+\mu-2}(x).$$

Continuing in this way and noting that the product of checkerboard matrices is checkerboard, we obtain

$$M^{(n+1)}_{n+1,j+\mu}(x)=\sum_{k=0}^{\lambda+n+2}M_{jk}m_{0,k+\mu-n-1}(x) \tag{4.7}$$

where the matrix $M=(M_{jk})^{\lambda+1,\lambda+n+2}_{j=0,k=0}$ is checkerboard. Now note that

$$M_{0,j}(x)=\begin{cases}1 & s_j<x<s_{j+1}\\ 0 & \text{elsewhere.}\end{cases}$$

Thus these are numbers $0<j_0<j_1<\cdots<j_\lambda<\lambda+n+2$ such that for $k=0,1,\ldots,\lambda$

$$M_{0,j+\mu-n-1}(\sigma_k)=\begin{cases}1, & j=j_k\\ 0, & \text{otherwise.}\end{cases}$$

Then from (4.7) we get

$$M^{(n+1)}_{n+1,j+\mu}(\sigma_k)=m_{j,j_k}. \tag{4.8}$$

Recalling (4.4) we see from (4.8) that since M is checkerboard

$$(-1)^{s+j}C_j\geqq 0\qquad (j=0,1,\ldots,\lambda+1), \tag{4.9}$$

where $s=j_0+\cdots+j_\lambda+\frac{1}{2}(\lambda+1)(\lambda+2)$.
Then (4.9) and (4.3) give $a_{ij}\geqq 0$ $(j=0,1,\ldots,\lambda+1)$. □

We now apply Lemma 1 to express V^T_n in the form

$$V^T_n(f;x)=\sum_{i=0}^{n+k+l+1}\left\{\sum_{j=0}^{n+k}D_{ij}f(\xi_j)\right\}M_{n+1,i}(x) \tag{4.10}$$

where $D_{ij}\geqq 0$ for all i,j.

Letting $\tau_i=(1/n+1)(s_{i-n}+\cdots+s_i)$, we have

$$V^S_{n+1}(f;x)=\sum_{i=0}^{n+k+l+1} f(\tau_i)M_{n+1,i}(x). \tag{4.11}$$

Putting $f(x)=1$ and comparing (4.10) and (4.11) gives

$$\sum_{j=0}^{n+k} D_{ij}=1 \qquad (i=0,1,\dots,n+k+l+1). \tag{4.12}$$

Similarly, putting $f(x)=x$ gives

$$\sum_{j=0}^{n+k} D_{ij}\xi_j=\tau_i \qquad (i=0,1,\dots,n+k+l+1). \tag{4.13}$$

If f is convex, then from (4.12) and (4.13),

$$f(\tau_i)\leqq \sum_{j=0}^{n+k} D_{ij}f(\xi_j) \qquad (i=0,1,\dots,n+k+l+1),$$

and so from (4.10) and (4.11) we get (2.3).

Equality occurs in (2.3) if and only if for $i=0,1,\dots,n+k+l+1$,

$$f\left(\sum_{j=0}^{n+k} D_{ij}\xi_j\right)=\sum_{j=0}^{n+k} D_{ij}f(\xi_j). \tag{4.14}$$

To see when this occurs, we must examine the constants D_{ij} more closely. Fix i $(0\leqq i\leqq n+k+l+1)$ and suppose s_{i-n-1} and s_{i+1} have multiplicities $\alpha=\alpha(i)$ and $\beta=\beta(i)$ respectively in $\{s_{i-n-1},\dots,s_{i+1}\}$. We choose $\gamma=\gamma(i)$ and $\delta=\delta(i)$ as follows. If $\beta(i)\geqq 2$, $t_{\gamma-\beta+2}<t_{\gamma-\beta+3}=\cdots=t_{\gamma+1}=s_{i+1}$. If $\beta(i)=1$, then $t_\gamma<t_{\gamma+1}=s_{i+1}$. If $\alpha(i)\geqq 2$, then $s_{i-n+1}=t_{\delta-n}=\cdots=t_{\delta-n+\alpha-2}<t_{\delta-n+\alpha-1}$. If $\alpha(i)=1$, then $s_{i-n-1}=t_{\delta-n}<t_{\delta-n+1}$. Clearly $\gamma\leqq\delta$ and as in Lemma 1, we can see that $D_{i\gamma}>0$, $D_{i\delta}>0$ and $D_{ij}=0$ for $j<\gamma$ and $j>\delta$.

If f is convex, then (4.14) holds only if f is linear on $[\xi_\gamma,\gamma_\delta]$. Moreover if f is any function which is linear on $[\xi_\gamma,\xi_\delta]$, then (4.14) holds. Thus if f is convex, equality holds in (2.2) only if f is piecewise linear and the possible knots are those points ξ_j which do not lie in any interval of the form $(\xi_{\gamma(i)},\xi_{\delta(i)})$ for $i=0,1,\dots,n+k+l+1$. This can happen if and only if for some i, $\xi_j=\xi_{\delta(i)}=\xi_{\gamma(i+1)}$. Checking all possible cases we see that this happens if and only if the set $\{t_{j-n+1},\dots,t_j\}$ contains at most two distinct elements. Similarly, if f is any piecewise linear function with knots at such points ξ_j, then equality holds in (2.2).

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