On compound permutation matrices

By A. C. Aitken.

(*Received 8th January*, 1946. *Read 1st February*, 1946.)

Introduction

1. In an earlier paper[1] the author investigated the relation existing between the induced matrices of a group of permutation matrices and the table of group characters of the irreducible representations of the corresponding symmetric group. It was found that the traces of a particular set of induced matrices sufficed to give, by a relatively simple transformation, the complete table of characters. It was remarked also that for $n > 4$ the set of compound matrices of permutation matrices, on the other hand, could at most provide only part of the table; for in fact the number of compounds, $n + 1$, is then less than $P(n)$, the number of partitions of n. For this reason the subject was not pursued into further detail.

It is however of some interest to know what part of the table of group characters the compound matrices of permutation matrices actually provide, that is to say, what particular irreducible matrix representations of the symmetric group of order $n!$ are latent in these $n + 1$ compounds. It appears that there are in fact just n of these representations; that they occur simply combined, and are typified by the following partitions of n, namely,

$$[n],\ [n-1, 1],\ [n-2, 1^2],\ \ldots,\ [2, 1^{n-2}],\ [1^n], \tag{1}$$

and that not merely the traces but the representations themselves can be obtained directly. The distinctive feature of these partitions is evident when we set out the Ferrers-Sylvester diagrams, for example

$$\begin{array}{llll} \ast\ \ast\ \ast\ \ast & \ast\ \ast\ \ast & \ast\ \ast & \ast \\ & \ast & \ast & \ast \\ & & \ast & \ast \\ & & & \ast \end{array} \tag{2}$$

when $n = 4$. The diagrams are of a type that may be called *unicursal*; they form a subclass (with a slight change of convention)

---

[1] *Proc. Edin. Math. Soc.* (2), 5 (1937), 1-13.

of those used by MacMahon[1] to represent *compositions* of an integer n. In brief, the partitions concerned are at the same time compositions.

We shall prove that the k^{th} compounds $A^{(k)}$, $k = 0, 1, 2, \ldots, n$ of the permutation matrices A are equivalent respectively to the representations

$$[n],\ [n] \dotplus [n-1, 1],\ [n-1, 1] \dotplus [n-2, 1^2],\ \ldots,\ [2, 1^{n-2}] \dotplus [1^n]. \quad (3)$$

Hence, if the traces of these compounds are entered in rows corresponding to $k = 0, 1, 2, \ldots, n$ and the *first differences* of these rows are taken, we shall have a table of group characters of representations corresponding to unicursal partitions. Let this table or matrix be denoted by $\hat{G}$, the complete table of characters being denoted by G.

Preliminary Reduction

2. It is convenient to begin by removing from the permutation matrices the scalar or identity representation 1, 1, 1, ..., 1 contained in them. This can be done by subjecting all the permutation matrices to the operations $\text{row}_1 + \text{row}_2 + \ldots + \text{row}_n$, $\text{col}_2 - \text{col}_1$, $\text{col}_3 - \text{col}_1$, ..., $\text{col}_n - \text{col}_1$; or, without essential change, to the similar operations with rows and columns interchanged. These are operations of type HAH^{-1}, and they semi-isolate a leading unit element in each matrix. For example when $n = 3$ the permutation matrices

$$\begin{bmatrix}1&\cdot&\cdot\\ \cdot&1&\cdot\\ \cdot&\cdot&1\end{bmatrix}\begin{bmatrix}1&\cdot&\cdot\\ \cdot&\cdot&1\\ \cdot&1&\cdot\end{bmatrix}\begin{bmatrix}\cdot&1&\cdot\\ 1&\cdot&\cdot\\ \cdot&\cdot&1\end{bmatrix}\begin{bmatrix}\cdot&1&\cdot\\ \cdot&\cdot&1\\ 1&\cdot&\cdot\end{bmatrix}\begin{bmatrix}\cdot&\cdot&1\\ 1&\cdot&\cdot\\ \cdot&1&\cdot\end{bmatrix}\begin{bmatrix}\cdot&\cdot&1\\ \cdot&1&\cdot\\ 1&\cdot&\cdot\end{bmatrix} \quad (1)$$

so treated yield

$$\begin{bmatrix}1&\cdot&\cdot\\ \cdot&1&\cdot\\ \cdot&\cdot&1\end{bmatrix}\begin{bmatrix}1&\cdot&\cdot\\ \cdot&\cdot&1\\ \cdot&1&\cdot\end{bmatrix}\begin{bmatrix}1&\cdot&\cdot\\ 1&-1&-1\\ \cdot&\cdot&1\end{bmatrix}\begin{bmatrix}1&\cdot&\cdot\\ \cdot&\cdot&1\\ 1&-1&-1\end{bmatrix}\begin{bmatrix}1&\cdot&\cdot\\ 1&-1&-1\\ \cdot&1&\cdot\end{bmatrix}\begin{bmatrix}1&\cdot&\cdot\\ \cdot&1&\cdot\\ 1&-1&-1\end{bmatrix} \quad (2)$$

where the semi-isolated leading unit elements constitute the scalar representation and in the last two rows and columns we see the familiar [2, 1] rational representation of the symmetric group of order 3!, namely

$$\begin{bmatrix}1&\cdot\\ \cdot&1\end{bmatrix}\begin{bmatrix}\cdot&1\\ 1&\cdot\end{bmatrix}\begin{bmatrix}-1&-1\\ \cdot&1\end{bmatrix}\begin{bmatrix}\cdot&1\\ -1&-1\end{bmatrix}\begin{bmatrix}-1&-1\\ 1&\cdot\end{bmatrix}\begin{bmatrix}1&\cdot\\ -1&-1\end{bmatrix}. \quad (3)$$

[1] *Combinatory Analysis*, 1916, vol. i, 153.

In the general case of order $n!$ this reduction gives at once, and in a rational canonical form, the irreducible $[n-1, 1]$ representation, and is perhaps the easiest way of obtaining it.

Compounds of the $[n-1, 1]$ *representation*

3. We proceed to form the compounds of the matrices of the $[n-1, 1]$ representation. As is well known, the characteristic polynomials of the permutation matrices are

$$(1-x)^n,\ (1-x^2)(1-x)^{n-2},\ \ldots.,\ 1-x^n, \tag{1}$$

corresponding respectively to conjugate classes or cycle-types $[1^n]$, $[2, 1^{n-2}]$, ..., $[n]$ of the permutations. Removing from each the factor $1-x$ belonging to the scalar representation, we have the characteristic polynomials of the matrices of the $[n-1, 1]$ representation in their conjugate classes,

$$(1-x)^{n-1},\ (1-x^2)(1-x)^{n-3},\ \ldots.,\ 1+x+x^2+\ldots.+x^{n-1}. \tag{2}$$

Now the latent roots of $A^{(k)}$ are the k-ary products of the latent roots of A, and so the successive traces of $A^{(0)}$, $A^{(1)}$, ..., $A^{(n)}$ are the elementary symmetric functions of the latent roots, that is to say, are the coefficients of $1, -x, x^2, -x^3, \ldots.$ in the characteristic polynomials of the present case, for these are evidently self-reciprocal polynomials. Hence the table M of characters of the compounds in question may be constructed as follows: let the polynomials (2) be expanded in powers of $-x$, and let the respective sets of coefficients of $(-x)^k$ in each, for $k = 0, 1, 2, \ldots., n-1$ be entered as successive columns of a matrix of n rows and $P(n)$ columns. This is the matrix M. For example when $n = 4$ we have

	$(1-x)^3$	$(1-x^2)(1-x)$	$(1-x^2)(1+x)$	$1-x^3$	$1+x+x^2+x^3$
1	1	1	1	1	1
$-x$	3	1	-1	.	-1
x^2	3	-1	-1	.	1
$-x^3$	1	-1	1	1	-1

(3)

and the 4-by-5 table of coefficients is the matrix M.

Identification of the matrix of characters

4. We shall now prove that M is simply $\hat{G}$. For in the first place, the classical relations of Frobenius[1] for group characters may

[1] See for example D. E. Littlewood, *The Theory of Group Characters*, 1940, 63-67.

be expressed thus,

$$\{s_{(\lambda)}\} = G'\{h_{(\lambda)'}\}, \tag{1}$$

where G' is the transposed matrix of group characters. Here we mean by $\{s_{(\lambda)}\}$ the column vector having for elements the ordered products of sums of powers, for example

$$\{s_1^4 \quad s_2 s_1^2 \quad s_2^2 \quad s_3 s_1 \quad s_4\} \tag{2}$$

in the case $n = 4$. The partition of n shown by the suffixes of the factors is denoted by (λ), the conjugate partition by $(\lambda)'$, while $\{h_{(\lambda)'}\}$ denotes the corresponding column vector having for elements the bialternant symmetric functions of Jacobi, for example

$$h_{(21^2)} = \begin{vmatrix} h_1 & h_2 & h_4 \\ h_0 & h_1 & h_3 \\ . & h_0 & h_2 \end{vmatrix}. \tag{3}$$

By way of illustration the first of the relations of Frobenius for $n = 4$ is

$$s_1^4 = h_4 + 3\,h_{(31)} + 2\,h_{(2^2)} + 3\,h_{(21^2)} + h_{(1^4)}. \tag{4}$$

Now in symmetric polynomials any identity that is homogeneous and of degree n in n variables continues to hold for any greater number of variables. Let us then define symmetric functions depending as follows on t, an arbitrary parameter: $s_r = 1 - t^r$, $r = 1, 2, 3, \ldots\ldots$ This sufficiently defines the generating functions of the fundamental symmetric functions, and it is these generating functions, not the arguments, that are of interest. From the relations existing[1] between generating functions we find at once that the complete homogeneous symmetric functions h are generated in this case by

$$(1 - tx)\,(1 - x)^{-1} = 1 + (1 - t)\,x + (1 - t)\,x^2 + \ldots\ldots, \tag{5}$$

so that $h_0 = 1$, $h_r = 1 - t$, $r = 1, 2, 3, \ldots\ldots$

It is now possible to deduce the values of the bialternants corresponding to unicursal partitions. The distinctive feature of these[2] is that the first subdiagonal contains elements $h_0 (= 1)$ exclusively. So, illustrating once again by $n = 4$, we have in the present case such results as

$$h_{(21^2)} = \begin{vmatrix} 1-t & 1-t & 1-t \\ 1 & 1-t & 1-t \\ & 1 & 1-t \end{vmatrix} = t^2\,(1 - t), \tag{6}$$

[1] MacMahon, *Combinatory Analysis*, vol. i, 3·7.

[2] *Ibid.*, vol. i, 200.

https://doi.org/10.1017/S0013091500008610 Published online by Cambridge University Press

and indeed in general, by the operations $\text{row}_1 - \text{row}_2$, $\text{row}_2 - \text{row}_3$, $\text{row}_3 - \text{row}_4$ and so on we see that if the unicursal partition is of r parts, then the value of $h_{(\lambda)'}$ is $(-t)^{r-1}(1-t)$. We see equally that any bialternant corresponding to a non-unicursal partition must acquire, when so reduced, zero elements above the diagonal and at least one zero element in the diagonal, and so must vanish identically in t.

Transcribing therefore the relations of Frobenius to this case, we have the general result, which the following will serve to illustrate,

$$\{s_1^4 \quad s_2 s_1^2 \quad s_2^2 \quad s_3 s_1 \quad s_4\} = \hat{G}' \{h_4 \quad h_{(31)} \quad h_{(21^2)} \quad h_{(1^4)}\}, \tag{7}$$

that is,

$$\{(1-t)^4 \quad (1-t^2)(1-t)^2 \quad (1-t^2)^2 \quad (1-t^3)(1-t) \quad 1-t^4\}$$
$$= (1-t)\,\hat{G}'\{1 \quad -t \quad t^2 \quad -t^3 \quad t^4\}. \tag{8}$$

Comparing this with the table of §3 (3), and bearing in mind that t is arbitrary and that the elements of $\{1 \quad -t \quad t^2 \quad -t^3\}$ and of the general vector of similar kind are linearly independent, we conclude that $M = \hat{G}$.

Reduction of compound permutation matrices

5. We can now see what matrix of characters would have replaced $\hat{G}$, had we taken compounds of the original unreduced permutation matrices. For in that case each polynomial generating a column of the matrix would have contained the additional factor $1 - x$; the first column in our illustration being generated by $(1-x)^4$ instead of $(1-x)^3$. The resulting matrix would therefore be such that the *first differences* of its rows would give $\hat{G}$. It follows that the compounds of permutation matrices, for $k = 0, 1, 2, \ldots, n$, are reducible respectively to the following simple direct sums of irreducible representations,

$$[n],\ [n] \dotplus [n-1, 1],\ [n-1, 1] \dotplus [n-2, 1^2],\ \ldots,\ [2, 1^{n-2}] \dotplus [1^n]. \tag{1}$$

This is a much simpler resolution than the corresponding resolution of the successive induced matrices of permutation matrices, though it leaves untouched the question of finding the characters corresponding to non-unicursal partitions. On the other hand, the compounds of the $[n-1, 1]$ representation give directly, not merely the characters, but the special matrix representations themselves, and in suitable rational canonical form.

Compound matrices of direct sums

6. The result of § 5 (1) might have been deduced from a theorem on compound matrices which, with its extensions and the analogous theorems for induced matrices, is of independent value. In its simplest case it concerns $(B \dot{+} C)^{(k)}$, where B and C are submatrices in direct sum, that is, aligned in isolation from each other down the diagonal of a partitioned matrix. The term "direct sum" usually refers to submatrices square in shape, of order $m \times m$; here we shall extend it to the case of rectangular submatrices. The theorem is that, in a sense about to be described,

$$(B \dot{+} C)^{(k)} = B^{(k)} \dot{+} B^{(k-1)} \times C^{(1)} \dot{+} B^{(k-2)} \times C^{(2)} \dot{+} \ldots . \dot{+} C^{(k)}, \quad (1)$$

where $L \times M$ denotes the direct product matrix of L and M. This useful theorem has been given before by the author,[1] but implicitly only and in respect not of identity but of collineatory equivalence. It has also been given explicitly, but still in respect of equivalence, by D. E. Littlewood.[2] An explicit enunciation and proof have been given by W. Ledermann.[3] We shall establish it as an identity for the k^{th} compound of

$$A = \begin{bmatrix} B & . \\ . & C \end{bmatrix}, \quad (2)$$

where B and C may be rectangular. By Laplacian expansion any minor of order k in this matrix, not identically zero in the elements b_{ij}, c_{ij}, must be the product of a non-vanishing subminor of order p in B by the complementary subminor of order $k - p$ in C. But the matrix having such products as elements, all duly arranged in lexical order of rows and columns, is the direct product $B^{(p)} \times C^{(k-p)}$. If therefore we adopt a compound lexical order (Aitken, *op. cit.*, 367) which first exhausts the suffixes of elements in B, then of those in B and C together, and if we order all minors accordingly, we arrive at the matrix on the right of (1). It is to be noted that with this convention of order (1) is an identity, and not simply a collineatory equivalence. An example is

$$\begin{bmatrix} a_1 & b_1 & c_1 & . \\ a_2 & b_2 & c_2 & . \\ . & . & . & d_3 \end{bmatrix}^{(2)} = \begin{bmatrix} |a_1 b_2| & |a_1 c_2| & |b_1 c_2| & . & . & . \\ . & . & . & a_1 d_3 & b_1 d_3 & c_1 d_3 \\ . & . & . & a_2 d_3 & b_2 d_3 & c_2 d_3 \end{bmatrix}, \quad (3)$$

[1] *Proc. London Math. Soc.* (2), **38** (1935), 367, 370.

[2] *Proc. London Math. Soc.* (2), **40** (1936), 375, or *Theory of Group Characters*, 198.

[3] *Proc. Roy. Soc., Edin.*, **56** (1936), 77-78.

that is,

$$\begin{bmatrix} B & . \\ . & C \end{bmatrix}^{(2)} = \begin{bmatrix} B^{(2)} & . \\ . & B \times C \end{bmatrix}, \qquad (4)$$

since here $C^{(2)}$ does not exist.

The theorem (1) is the analogue, in respect of direct sum, direct product and the forming of compound matrices, of Vandermonde's identity of combinatory algebra, namely

$$(m + n)_{(k)} = m_{(k)} + m_{(k-1)}\, n + m_{(k-2)}\, n_{(2)} + \ldots . + n_{(k)}, \qquad (5)$$

where $m_{(k)} = m\,(m-1)\,(m-2)\ldots .(m-k+1)/k!$. Indeed if B and C are unit matrices of orders $m \times m$ and $n \times n$ respectively, Vandermonde's theorem expresses the equality of the traces on both sides of (1).

It is almost evident that theorems such as (1) referring to completely isolated submatrices have counterparts referring to semi-isolated submatrices; and semi-isolation is sufficient for reducibility. In such cases we envisage, instead of the direct sum, a succession of submatrices semi-isolated on the one side, either above or below the diagonal, but the same side for all. In our application to compound permutation matrices we use the semi-isolated case corresponding to

$$(1 \dot{+} B)^{(k)} = B^{(k-1)} \dot{+} B^{(k)}, \qquad (6)$$

which gives directly the result of § 5 (1) on the reducibility of compound permutation matrices, when once the theorem of § 3 has been established.

Further useful generalizations are worthy of record. Let us arrange minors of order k according to a compound lexical order that exhausts first the suffixes of elements in B, then of those in B and C, then of those in B, C and D, and so on. Then, analogous to the Vandermondian multinomial expansion having powers replaced by factorial polynomials, there is an expansion for the k^{th} compound of

$$\begin{bmatrix} B & . & . & .. & . \\ . & C & . & .. & . \\ . & . & D & .. & . \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ . & . & . & .. & H \end{bmatrix}, \qquad (7)$$

where $B, C, \ldots, H$ are in general rectangular but isolated. We have merely to interpret combinatory numbers by compound matrices, sum by direct sum, in the general sense here employed, product by direct product. There is also the semi-isolated analogue.

https://doi.org/10.1017/S0013091500008610 Published online by Cambridge University Press

Induced matrices of direct sums

7. There are equally valuable identities for induced matrices of direct sums, and it will be well to mention them for reference. The identity for the binary case has been given by Ledermann (*op. cit.*). Let $m_{[k]}$ denote the combinatory number which, when m is a positive integer, enumerates the combinations of m things k at a time when all possible repetitions are permitted. Then there is the corresponding Vandermondian identity

$$(m+n)_{[k]} = m_{[k]} + m_{[k-1]}n + m_{[k-2]}n_{[2]} + \dots + n_{[k]}, \qquad (1)$$

easily established by combinatorial reasoning. If it is recalled that the elements of $A^{[k]}$, the k^{th} induced matrix of A, are the minor *permanents* of order k extracted from A and arranged in lexical order, and that in minor permanents repetition of rows and columns is allowed (always with the convention that for each set of p repeated columns, not rows, the permanent is to be divided by $p!$), we have by the same reasoning as before, and with the same agreement as to order, the identity

$$[B \dotplus C]^{[k]} = B^{[k]} \dotplus B^{[k-1]} \times C^{[1]} \dotplus B^{[k-2]} \times C^{[2]} \dotplus \dots \dotplus C^{[k]}, \qquad (2)$$

in the extended sense of the direct sum. There is a corresponding analogue for the Vandermondian expansion of $(m+n+p+\dots+r)_{[k]}$, and there are also the counterparts of these theorems for the case of semi-isolated submatrices.

The corresponding expansions for direct products of submatrices in direct sum or semi-isolation are well known and easy to establish; they are analogous in all respects to the multiplication of linear multinomial functions, and if the convention of compound lexical order be observed, are true identities.

MATHEMATICAL INSTITUTE,
THE UNIVERSITY, EDINBURGH, 1.