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On the Representations of a Number as the Sum of Three Cubes and a Fourth or Fifth Power*

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Abstract. Let $R_k(n)$ denote the number of representations of a natural number n as the sum of three cubes and a kth power. In this paper, we show that $R_3(n) \ll n^{5/9+\varepsilon}$, and that $R_4(n) \ll n^{47/90+\varepsilon}$, where $\varepsilon > 0$ is arbitrary. This extends work of Hooley concerning sums of four cubes, to the case of sums of mixed powers. To achieve these bounds, we use a variant of the Selberg sieve method introduced by Hooley to study sums of two kth powers, and we also use various exponential sum estimates.

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1. Introduction

In [5], Hooley showed that the number of representations of a natural number n as the sum of four cubes is $O(n^{(11/18)+\varepsilon})$, where $\varepsilon > 0$ is arbitrary. In forthcoming papers [12, 13], the author extends this to show that if $k \geqslant 3$ is an odd integer, then the number of representations of n as the sum of four kth powers is $O(n^{11/(6k)+\varepsilon})$. In this paper, we show that similar results can be obtained for sums of mixed powers, as stated in the following theorem.

THEOREM 1.1. *Let $R_k(n)$ denote the number of solutions to*

$$X^3 + Y^3 + Z^3 + W^k = n, \tag{1.1}$$

where X, Y, Z, W are nonnegative integers. Then $R_4(n) \ll n^{(5/9)+\varepsilon}$, and $R_5(n) \ll n^{(47/90)+\varepsilon}$.

By a counting argument, we note that the expected order of magnitude of $R_k(n)$ is $n^{1/k}$, and that up to a constant, $R_k(n)$ achieves this size infinitely often. To consider the strength of Theorem 1.1, one can obtain $R(n) \ll n^{(1/3)+(1/k)+\varepsilon}$ by using standard

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estimates for the divisor function. Thus, we have achieved a savings over the trivial estimate of $1/36$ in the exponent when $k = 4$, and of $1/90$ when $k = 5$. Our methods do not surpass the trivial estimate when $k \geqslant 6$, because the trivial estimate is not enough larger than $1/3$ in these cases.

It is known from Brüdern [1] that almost all positive integers can be written as a sum of three cubes and a fourth power, where these represented numbers n satisfy $R_4(n) \gg n^{17/144}$. Also, Ming-Gao Lu [9] has shown that almost all positive integers can be written as a sum of three cubes and a fifth power, where these represented numbers satisfy $R_5(n) \gg n^{(43/515)-\varepsilon}$. In [2], Brüdern shows that if $\tilde{R}_k(n)$ is the number of representations of n as the sum of six cubes and two fourth powers, then $\tilde{R}_k(n) \gg n^{3/2}$ for all large n, which is the expected order of magnitude. It might be expected that more recent developments in the circle method due to Vaughan and Wooley would enable us to achieve the correct lower bound for $R_k(n)$ when $k = 4$, 5, or possibly even 6.

The methods used to prove Theorem 1.1 are based on those developed by Hooley in [5]. The central idea is that since $X^3 + Y^3$ has a linear factor, we can transform (1.1) into

$$2r(r^2 + 3s^2) = n - Z^3 - W^k, \tag{1.2}$$

and then apply the Selberg sieve method to exploit the term $3s^2$ which arises. The error terms which arise can be treated by exponential sums. These sums are more difficult to treat than those appearing in [5], because of the inhomogeneity of the equations under consideration. To bound the sums, we use methods developed by Hooley [6, 7] which are based upon Deligne's resolution of the Riemann hypothesis for L-functions of algebraic varieties over finite fields.

We note at this point that our methods allow us to achieve $R_2(n) \ll n^{3/4+\varepsilon}$, which is better than the previously mentioned trivial estimate of $n^{5/6}$. However, if we transform (1.1) into (1.2), where $k = 2$, and now fix r and Z, this gives us $n^{2/3}$ choices for r and Z. When r and Z are fixed, then (1.2) is a binary quadratic form in W and s, so that there are at most n^ε solutions for W and s, which shows that $R_2(n) \ll n^{2/3+\varepsilon}$, which is better than the bound our methods achieve.

For convenient reference, we will make a few comments about the notation used in this paper. We note that ε denotes a sufficiently small positive real number, where the value of ε is free to change as needed throughout. We use $\ll$ and $\gg$ to denote Vinogradov's familiar notation, where the constants depend at most on ε. As usual, the greatest common divisor of $u_1, \ldots, u_j$ is denoted by $(u_1, \ldots, u_j)$; $p^\alpha \| x$ means that $p^\alpha \mid x$ but $p^{\alpha+1} \nmid x$; the divisor function of n will be denoted by $\tau(n)$; $\omega(u)$ denotes the number of prime factors of u; $\sigma_j(u)$ denotes the sum of the jth powers of the divisors of u; $\|\varsigma\|$ denotes the distance of ς from the nearest integer; $[x]$ denotes the greatest integer not exceeding x; $\lceil x \rceil$ denotes the smallest integer greater than or equal to x; $\mathbb{F}_q$ denotes the finite field with q elements; we denote $e(x) = \mathrm{e}^{2\pi i x}$. The Legendre symbol will be written as $(a|p)$ or (a/p).

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2. Initial Transformations and Introduction of the Sieve

We now begin the proof of Theorem 1.1 by first transforming (1.1) into a form suitable for application of a sieve method. We will assume throughout that $k = 4$ or 5. Let $R'_k(n)$ denote the number of solutions to (1.1) for which at least two of the variables X, Y, Z are nonzero. Then

$$R_k(n) \ll R'_k(n) + n^{1/3}. \tag{2.1}$$

For any representation of n arising in $R'_k(n)$, we can choose two of the variables X, Y, Z to have the same parity, and such that at least one of them is nonzero. Therefore

$$R'_k(n) \ll \sum_{\substack{X^3+Y^3+Z^3+W^k=n \\ X>0, X \geqslant Y; X \equiv Y \pmod 2}} 1. \tag{2.2}$$

By substituting

$$X = r + s, \qquad Y = r - s, \tag{2.3}$$

where r is a positive integer and s is a nonnegative integer, we see that

$$R'_k(n) \ll \nu(n, k), \tag{2.4}$$

where $\nu(n, k)$ is the number of solutions in r, s, Z ,W of

$$2r(r^2 + 3s^2) = n - Z^3 - W^k \tag{2.5}$$

such that r is a positive integer, and s, Z, W are nonnegative integers.

Let $\Xi(n, k, r)$ denote the number of solutions to (2.5) in s, Z, W for a fixed value of r. Then

$$\nu(n, k) = \sum_{r \leqslant N} \Xi(n, k, r), \tag{2.6}$$

where

$$N = N(n) = (n/2)^{1/3} < n^{1/3}. \tag{2.7}$$

We can now introduce a sieve to take advantage of the term $3s^2$ which appears in (2.5). To do this, we will replace that term by a member of a larger set which includes all numbers of the form $3s^2$, and which will be a set surviving a sieving process. Namely, let $\mathfrak{S} = \mathfrak{S}(n, r)$ be the set of all integers (positive or negative) that are

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not quadratic non-residues, modulo p, for all primes p such that

$$p \nmid r, \tag{2.8}$$

$$p \nmid n, \quad \left(\frac{3}{p}\right) = 1, \quad p > D_1, \tag{2.9}$$

where D_1 is a suitable sufficiently large absolute constant exceeding 5.

We can now use Selberg's upper bound sieve method as described in Chapter One of [4] to obtain an upper bound for the characteristic function of $\mathfrak{S}$. Let d denote a square-free number (possibly 1) consisting entirely of prime factors p satisfying (2.8) and (2.9), and let $\mathfrak{H}(d)$ denote the set of all integers (possibly negative) that are quadratic non-residues modulo each prime divisor of d (where $\mathfrak{H}(1)$ is the set of all integers). We now introduce real numbers $\lambda_d = \lambda_{d,n,k,r}$ which satisfy the conditions that $\lambda_1 = 1$ and $\lambda_d = 0$ for $d > \xi = n^\beta$, where β will be determined later to satisfy $0 < \beta < 1/3$, and where it is understood that ξ and β will depend on k. Then considering

$$\left(\sum_{\substack{d \\ u \in \mathfrak{H}(d)}} \lambda_d\right)^2 = \sum_{\substack{d \\ u \in \mathfrak{H}(d)}} \rho_d \tag{2.10}$$

as a function of u, we see that this function is non-negative and is equal to 1 when u is three times a square, and that this will be an upper bound for the characteristic function of $\mathfrak{S}$. It is convenient to note that we can express ρ_d as

$$\rho_d = \sum_{[d_1, d_2] = d} \lambda_{d_1} \lambda_{d_2}, \tag{2.11}$$

so that $\rho_d = 0$ for $d > \xi^2 = n^{2\beta}$.

Combining this upper bound for $\mathfrak{S}$ with the definition of $\Xi(n, k, r)$, we obtain

$$\Xi(n, k, r) \leqslant \sum_{Z, W, l} \sum_{\substack{d \\ L \in \mathfrak{H}(d)}} \rho_d, \tag{2.12}$$

where the first summation is over $0 \leqslant Z \leqslant n^{1/3}$, $0 \leqslant W \leqslant n^{1/k}$, and over l satisfying $2r(r^2 + l) = m - Z^3 - W^k$. Let $\Phi(n, k, r, d)$ denote the number of solutions in L_d, Z, W of the conditions

$$2r(r^2 + L_d) = n - Z^3 - W^k; \quad Z \leqslant n^{1/3}, \ W \leqslant n^{1/k}; \quad L_d \in \mathfrak{H}(d), \tag{2.13}$$

and let

$$\Theta(n, k, r) = \sum_{d \leqslant \xi^2} \rho_d \Phi(n, k, r, d). \tag{2.14}$$

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Then

$$\Xi(n,k,r) \leqslant \Theta(n,k,r), \tag{2.15}$$

and by (2.6), we see that

$$v(n,k) \leqslant \sum_{r \leqslant N} \Theta(n,k,r). \tag{2.16}$$

In order to estimate $\Theta(n,k,r)$, we need to transform $\Phi(n,k,r,d)$. Let l_d throughout refer to an integer belonging to any given complete set of incongruent representatives of $\mathfrak{H}(d)$, modulo d. Then the number of solutions to (2.13) is the same as the number of solutions in l_d, Z, W of

$$2r(r^2 + l_d) \equiv n - Z^3 - W^k \pmod{2rd}, \tag{2.17}$$

such that $Z \leqslant n^{1/3}$ and $W \leqslant n^{1/k}$. Let $\Upsilon(n,k,r,d)$ denote the number of solutions in l_d, Z, W of (2.17) with $Z, W < 2rd$. Then we can rewrite $\Phi(n,k,r,d)$ as

$$\Phi(n,k,r,d) = \frac{([n^{1/3}]+1)([n^{1/k}]+1)}{4r^2d^2}\Upsilon(n,k,r,d) + \Phi_2(n,k,r,d), \tag{2.18}$$

where $\Phi_2(n,k,r,d)$ is defined by this relation. Since $(r,d) = 1$, then $\Upsilon(n,k,r,d)$ is the number of simultaneous solutions of the congruences

$$n - Z^3 - W^k \equiv 0 \pmod{2r}, \tag{2.19}$$

$$n - Z^3 - W^k \equiv 2r(r^2 + l_d) \pmod{d}, \tag{2.20}$$

for which $Z, W < 2rd$. Therefore we can write

$$\Upsilon(n,k,r,d) = \psi(n,k,r)\gamma(n,k,r,d), \tag{2.21}$$

where $\psi(n,k,r)$ is the number of incongruent solutions in Z and W, modulo $2r$, of (2.19), and where $\gamma(n,k,r,d)$ is the number of incongruent solutions in Z, W, l_d, modulo d, of (2.20). If we define

$$\Phi_1(n,k,r,d) = \frac{\gamma(n,k,r,d)}{d^2}, \tag{2.22}$$

then (2.18) can be written as

$$\Phi(n,k,r,d) = \frac{([n^{1/3}]+1)([n^{1/k}]+1)\psi(n,k,r)}{4r^2}\Phi_1(n,k,r,d) + \Phi_2(n,k,r,d). \tag{2.23}$$

Thus, if we let

$$\Theta_i(n,k,r) = \sum_{d \leqslant \xi^2} \rho_d \Phi_i(n,k,r,d) \tag{2.24}$$

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for $i = 1, 2$, then (2.14) yields

$$\Theta(n,k,r) = \frac{([n^{1/3}]+1)([n^{1/k}]+1)\psi(n,k,r)}{4r^2}\Theta_1(n,k,r) + \Theta_2(n,k,r). \tag{2.25}$$

To conclude our preliminary work, let

$$v_1(n,k) = \frac{([n^{1/3}]+1)([n^{1/k}]+1)}{4}\sum_{r\leqslant N}\frac{\psi(n,k,r)\Theta_1(n,k,r)}{r^2}, \tag{2.26}$$

and let

$$v_2(n,k) = \sum_{r\leqslant N}\Theta_2(n,k,r), \tag{2.27}$$

so that by (2.16) and (2.25), we obtain

$$v(n,k) \leqslant v_1(n,k) + v_2(n,k). \tag{2.28}$$

3. Lemmata on Congruences

We next develop some lemmata which will be useful in estimating $v_1(n,k)$ and $v_2(n,k)$, and which are the analogues of Lemmata 1 and 2 from Hooley's work [5].

LEMMA 3.1. *Let $g(u; v)$ be the multiplicative function of u defined on prime powers by*

$$g(p^\alpha; v) = \begin{cases} p^{\alpha-2}, & \text{if } \alpha \geqslant 3, \text{ and } p^3 \mid v, \\ 1, & \text{otherwise.} \end{cases} \tag{3.1}$$

Then we have

$$\psi(n,k,r) \ll (2k+1)^{(r)} r\, g(r; n).$$

Proof. If p^α is any prime power, let $\psi_1(n,k,p^\alpha)$ denote the number of solutions in Z and W, modulo p^α, of

$$n - Z^3 - W^k \equiv 0 \pmod{p^\alpha}. \tag{3.2}$$

On noting that

$$\psi(n,k,r) \leqslant 4\prod_{p^\alpha\|r}\psi_1(n,k,p^\alpha), \tag{3.3}$$

it suffices to show that for any prime power p^α, we have

$$\psi_1(n,k,p^\alpha) \leqslant (2k+1)\, g(p^\alpha; n). \tag{3.4}$$

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We let p^α be any prime power, and examine the possible values of α. If $\alpha = 1$, then for each given value of W, there are at most three values of Z, modulo p, satisfying (3.2), so that

$$\psi_1(n, k, p^\alpha) \leqslant 3p. \tag{3.5}$$

When $\alpha \geqslant 2$, we first examine solutions of (3.2) which do not satisfy the condition

$$Z \equiv W \equiv 0 \pmod{p}. \tag{3.6}$$

These solutions satisfy either $n - Z^3 \not\equiv 0$, mod p, or $m - W^k \not\equiv 0$, mod p. In the first case, for each such Z, there are at most k values of W satisfying (3.2), and in the second case, for each appropriate W, there are at most three values of Z satisfying (3.2). Consequently there are at most $2kp^\alpha$ solutions of (3.2) which do not satisfy (3.6).

It remains to consider the solutions of (3.2) satisfying (3.6) when $\alpha \geqslant 2$. When $\alpha = 2$, there are at most p^2 such solutions, modulo p^α. When $\alpha \geqslant 3$, there will only be solutions satisfying (3.2) and (3.6) if $p^3 \mid n$, in which case there will be at most $p^{2\alpha-2}$ such solutions modulo p^α.

So combining the conclusions of the previous two paragraphs, we see that if $\alpha = 2$, we have

$$\psi_1(n, k, p^\alpha) \leqslant (2k+1)p^\alpha, \tag{3.7}$$

and if $\alpha \geqslant 3$, then

$$\psi_1(n, k, p^\alpha) \leqslant \begin{cases} 2kp^\alpha, & \text{if } p^3 \nmid n \\ (2k+1)p^{2\alpha-2} & \text{if } p^3 \mid n. \end{cases} \tag{3.8}$$

Upon combining (3.5), (3.7), and (3.8), we see that (3.4) holds, which proves the lemma. □

Another result we will require in our analysis of $v_1(n, k, r)$ is expressed in the following lemma.

LEMMA 3.2. *Let $T(n, k, r, p)$ denote the number of solutions in a, Z, W modulo p, of the congruence*

$$2r(r^2 + a^2) \equiv n - Z^3 - W^k \pmod{p}. \tag{3.9}$$

If $p \nmid 2r$, then we have

$$T(n, k, r, p) = p^2 + O(p^{3/2}).$$

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Proof. Let

$$b_j(u) = \sum_{x=1}^{p} e(ux^j/p), \qquad c(u) = \sum_{x=1}^{p} e(uq(x)/p), \tag{3.10}$$

where $q(x) = 2r(r^2 + x^2)$. It is known that if $(u, p) = 1$, then $|b_j(u)| \leqslant (j-1)p^{1/2}$; see, for instance, Lemma 4.3 in [11]. Then we have

$$T(n, k, r, p) = \frac{1}{p} \sum_{u=1}^{p} b_3(u) b_k(u) c(u) e(-un/p). \tag{3.11}$$

By applying the triangle and Cauchy-Schwarz inequalities, we obtain

$$|T(n, k, r, p) - p^2| \leqslant \left(\max_{1 \leqslant v \leqslant p-1} |b_3(v)| \right) \frac{1}{p} \sum_{u=1}^{p-1} |b_k(u) c(u)| \tag{3.12}$$

$$\leqslant 2p^{1/2} \left(\frac{1}{p} \sum_{u=1}^{p} |b_k(u)|^2 \right)^{1/2} \left(\frac{1}{p} \sum_{u=1}^{p} |c(u)|^2 \right)^{1/2}. \tag{3.13}$$

By considering the number of solutions to the underlying congruences $u^k \equiv v^k \pmod p$ and $q(u) \equiv q(v) \pmod p$ of the sums in (3.12), it follows from orthogonality that

$$|T(n, k, r, p) - p^2| \leqslant (2p^{1/2})(kp)^{1/2}(2p)^{1/2},$$

which gives the desired result. □

4. Estimation of $v_1(n, k)$ by the Selberg Sieve

In order to achieve a bound for $v_1(n, k)$, we will employ Selberg's sieve method to bound $\Theta_1(n, k, r)$, where the condition for each prime p which we are sieving out is the property of being a quadratic non-residue, modulo p.

Let p denote a prime satisfying (2.8) and (2.9). Recalling (2.20) and that if $d = p$, then l_p must lie in $\mathfrak{H}(p)$, we see that

$$\gamma(n, k, r, p) = p^2 - \frac{1}{2} T(n, k, r, p) - \frac{1}{2} \psi_1(n - 2r^3, k, p),$$

because each of the p^2 choices of Z and W allows only one possible value, modulo p, for l_p, so the latter two terms will subtract off those values for which l_p is not in $\mathfrak{H}(p)$. (Here, $T(n, k, r, p)$ is defined as in Lemma 3.2, and $\psi_1(n - 2r^3, k, p)$ as defined in Lemma 3.1 compensates for the solutions of (3.9) for which $p \mid a$.) Combining this result with (3.5) and Lemma 3.2 gives

$$\gamma(n, k, r, p) = \frac{1}{2} p^2 + O(p^{3/2}), \tag{4.1}$$

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and from (2.22), this gives

$$\Phi_1(n, k, r, p) = \frac{1}{2} + O(p^{-1/2}). \tag{4.2}$$

Since by (2.9), one has $p > D_1$, then if D_1 is chosen sufficiently large, we have

$$0 < \Phi_1(n, k, r, p) < 1. \tag{4.3}$$

In order to utilize Selberg's sieve method, let

$$f(d) = f(n, k, r, d) = \frac{1}{\Phi_1(n, k, r, d)}, \tag{4.4}$$

and following Hooley's treatment in [4], let

$$f_1(d) = \sum_{k|d} \mu(k) f(d/k) = \prod_{p|d} \big(f(p) - 1\big), \tag{4.5}$$

where we recall that d is square-free. Note that by (2.22), $f(d)$ is multiplicative, and that since $f(p) > 1$, then $f_1(d) > 0$. From (2.24), (2.11), and (4.4), we see that

$$\Theta_1(n, k, r) = \sum_{d_1, d_2 \leqslant \xi} \frac{\lambda_{d_1} \lambda_{d_2}}{f([d_1, d_2])}. \tag{4.6}$$

Since the sum on the right hand side of (4.6) is the sum which appears in the main term of Selberg's method, then from [4], we have that $\Theta_1(n, k, r)$ has a minimum value of $1/V(\xi)$ subject to the constraints on λ_d, where

$$V(\xi) = V_{n,k,r}(\xi) = \sum_{d \leqslant \xi} \frac{\mu^2(d)}{f_1(d)}, \tag{4.7}$$

and where the λ_d which give this minimum are given by

$$\lambda_d = \frac{\mu(d)}{V(\xi)} \sum_{d_3|d} \frac{\mu^2(d_3)}{f_1(d_3)} \sum_{\substack{d_4 \leqslant \xi/d \\ (d_4, d)=1}} \frac{\mu^2(d_4)}{f_1(d_4)}. \tag{4.8}$$

When D_1 is sufficiently large, it follows from (4.2), (4.4), and (4.5) that when $p > D_1$,

$$f_1(p) = 1 + O(p^{-1/2}) < 2 \quad \text{and} \quad f_1(p) > 1/2. \tag{4.9}$$

Then by (4.7) and recalling that primes dividing d must satisfy (2.8) and (2.9), we obtain

$$V(\xi) \geqslant 1 + \frac{1}{2} \sum_{D_1 < p \leqslant \xi} 1, \tag{4.10}$$

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where the summation is over primes p satisfying $p \nmid rn$ and $(3|p) = 1$. Therefore,

$$V(\xi) > \frac{D_2 \xi}{\log \xi}, \tag{4.11}$$

for some constant D_2, since $r \leqslant N < n$ by (2.7), and since $\xi = n^{\beta}$, where $0 < \beta < 1/3$. This gives

$$\Theta_1(n, k, r) \ll n^{-\beta+\varepsilon}. \tag{4.12}$$

So along with (2.26), this gives

$$v_1(n, k) \ll n^{(1/3)+(1/k)-\beta+\varepsilon} \sum_{r \leqslant N} \frac{\psi(n, k, r)}{r^2}. \tag{4.13}$$

To estimate the sum in (4.13), we see from Lemma 3.1 that

$$\sum_{r \leqslant N} \frac{\psi(n, k, r)}{r^2} \ll \sum_{r \leqslant N} r^{-1+\varepsilon} g(r; n). \tag{4.14}$$

When $p \mid n$,

$$\sum_{j=0}^{[\log n]} \frac{g(p^j; n)}{p^j} \leqslant 1 + \frac{1}{p} + \frac{\log n}{p^2},$$

so that since $g(u; n)$ is multiplicative in u, we obtain

$$\sum_{r \leqslant N} r^{-1+\varepsilon} g(r; n) \leqslant \prod_{\substack{p \leqslant n \\ p \nmid n}} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p|n} \left(1 + \frac{1}{p} + \frac{\log n}{p^2}\right). \tag{4.15}$$

The final product in (4.15) satisfies

$$\prod_{p|n} \left(1 + \frac{1}{p} + \frac{\log n}{p^2}\right) \ll 2^{\omega(n)} (\log n)^{(\log n)^{1/2}} \prod_{\substack{p > (\log n)^{1/2} \\ p|n}} \left(1 - \frac{1}{p}\right)^{-1}. \tag{4.16}$$

From (4.14), (4.15), and (4.16), we have

$$\sum_{r \leqslant n} \frac{\psi(n, k, r)}{r^2} \ll n^{\varepsilon} \prod_{p \leqslant n} \left(1 - \frac{1}{p}\right)^{-1} \ll n^{\varepsilon}. \tag{4.17}$$

Finally, by using this estimate for the sum in (4.13), we see that

$$v_1(n, k) \ll n^{(1/3)+(1/k)-\beta+\varepsilon}. \tag{4.18}$$

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5. Expression for $v_2(n, k)$ in Terms of Exponential Sums

We now need to estimate $v_2(n, k)$. To do this, we will express $\Phi(n, k, r, d)$ using exponential sums, and then appeal to (2.18) to obtain an expression for $\Phi_2(n, k, r, d)$. Let

$$\mathcal{N}(n, k, r, d, b, c) = \sum_{\substack{0 \leqslant Z \leqslant n^{1/3} \\ 0 \leqslant W \leqslant n^{1/k} \\ Z \equiv b,\ W \equiv c \pmod{2rd}}} 1.$$

Then from (2.13) and (2.17), we have

$$\Phi(n, k, r, d) = \sum_{l_d} \sum_{0 \leqslant b, c < 2rd} \mathcal{N}(n, k, r, d, b, c), \tag{5.1}$$

where the inner sum is over b and c such that

$$2r(r^2 + l_d) \equiv n - b^3 - c^k \pmod{2rd}. \tag{5.2}$$

By orthogonality, one has

$$\begin{aligned} 4r^2d^2\mathcal{N}(n, k, r, d, b, c) &= \sum_{\substack{0 \leqslant Z \leqslant n^{1/3} \\ 0 \leqslant W \leqslant n^{1/k}}} \sum_{0 \leqslant u, v < 2rd} e((u(b - Z) + v(c - W))/2rd) \\ &= \tilde{n}_3\tilde{n}_k + \tilde{n}_k \sum_{0<u<2rd} \theta_{u,3} e(ub/2rd) + \tilde{n}_3 \sum_{0<v<2rd} \theta_{v,k} e(vc/2rd) + \\ &\quad + \sum_{0<u,v<2rd} \theta_{u,3}\theta_{v,k} e((ub + vc)/2rd), \end{aligned} \tag{5.3}$$

where $\tilde{n}_j = [n^{1/j}] + 1$, and

$$\theta_{w,j} = \theta_{w,j,2rd,n} = \sum_{0 \leqslant V \leqslant n^{1/j}} e(-wV/2rd) \ll \frac{1}{\|w/2rd\|}. \tag{5.4}$$

Therefore, by substituting (5.3) back into (5.1), and comparing this with (2.18), we see that

$$\begin{aligned} \Phi_2(n, k, r, d) = &\frac{[n^{1/k}] + 1}{4r^2d^2} \sum_{0<u<2rd} \theta_{u,3} S(n, k, r, d; u, 0) + \\ &+ \frac{[n^1/3] + 1}{4r^2d^2} \sum_{0<v<2rd} \theta_{v,k} S(n, k, r, d; 0, v) + \\ &+ \frac{1}{4r^2d^2} \sum_{0<u,v<2rd} \theta_{u,3}\theta_{v,k} S(n, k, r, d; u, v), \end{aligned} \tag{5.5}$$

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where

$$S(n,k,r,d;u,v)=\sum_{l_d}\sum_{0\leqslant b,c<2rd} e((ub+vc)/2rd), \tag{5.6}$$

with the inner sum of (5.6) being over b and c satisfying (5.2).

Let $S^*(n,k,r,d;u,v;l_d)$ be the inner sum of (5.6), so that

$$S(n,k,r,d;u,v)=\sum_{l_d} S^*(n,k,r,d;u,v;l_d). \tag{5.7}$$

On noting from (2.7), (2.24), and (2.27) that

$$rd\leqslant N\xi^2<n^{(1/3)+2\beta}\leqslant n,$$

then it follows from (5.4) and (5.5) that

$$\begin{aligned}\Phi_2(n,k,r,d) \ll \frac{n^{1/k}}{rd}\sum_{0<|u|<n}\frac{|S(n,k,r,d;u,0)|}{|u|}+\\ +\frac{n^{1/3}}{rd}\sum_{0<|v|<n}\frac{|S(n,k,r,d;0,v)|}{|v|}+\\ +\sum_{0<|u|,|v|<n}\frac{|S(n,k,r,d;u,v)|}{|u||v|}.\end{aligned} \tag{5.8}$$

Next, from (2.27), (2.24), and (5.8), we obtain

$$\begin{aligned}v_2(n,k) \ll n^{1/k}\sum_{0<|u|<n}\frac{1}{|u|}\sum_{r\leqslant N}\frac{1}{r}\sum_{d\leqslant\xi^2}\frac{|\rho_{d,r}||S(n,k,r,d;u,0)|}{d}+\\ +n^{1/3}\sum_{0<|v|<n}\frac{1}{|v|}\sum_{r\leqslant N}\frac{1}{r}\sum_{d\leqslant\xi^2}\frac{|\rho_{d,r}||S(n,k,r,d;0,v)|}{d}+\\ +\sum_{0<|u|,|v|<n}\frac{1}{|u||v|}\sum_{r\leqslant N}\sum_{d\leqslant\xi^2}|\rho_{d,r}||S(n,k,r,d;u,v)|.\end{aligned} \tag{5.9}$$

We now need to bound the size of ρ_d which occurs in our expression for $v_2(n,k)$, so we now consider (4.8), which gave the optimal values for the λ_d. In examining the sums given in (4.8), if we recall (4.9) and note that d_3 and d_4 are square-free, we see that

$$\sum_{d_3|d}\frac{\mu^2(d_3)}{f_1(d_3)}\leqslant\tau(d)2^{\omega(d)}\ll d^{\varepsilon}\ll(\xi^2)^{\varepsilon}\ll n^{\varepsilon}, \tag{5.10}$$

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and that

$$\sum_{\substack{d_4 \leqslant \xi/d \\ (d_4,d)=1}} \frac{\mu^2(d_4)}{f_1(d_4)} \leqslant \sum_{d_4 \leqslant \xi/d} \frac{\mu^2(d_4)}{f_1(d_4)} \ll \sum_{d_4 \leqslant \xi/d} 2^{\omega(d_4)} \ll \frac{n^\varepsilon \xi}{d}. \tag{5.11}$$

By using (4.11) and the latter two estimates in (4.8), we get

$$\lambda_d \ll \frac{n^\varepsilon}{d}, \tag{5.12}$$

so that by (2.11), we have

$$\rho_d \ll n^\varepsilon \sum_{[d_1,d_2]=d} \frac{1}{d_1 d_2} \ll \frac{n^\varepsilon}{d} \sum_{[d_1,d_2]=d} 1 \ll \frac{n^\varepsilon \tau(d)^2}{d} \ll \frac{n^\varepsilon}{d}.$$

By substituting this bound for ρ_d into (5.9), we obtain

$$v_2(n,k) \ll n^{(1/k)+\varepsilon} v_3(n,k) + n^{(1/3)+\varepsilon} v_4(n,k) + n^\varepsilon v_5(n,k), \tag{5.13}$$

where

$$v_3(n,k) = \sum_{0<|u|<n} \frac{1}{|u|} \sum_{r \leqslant N} \frac{1}{r} \sum_{d \leqslant \xi^2} \frac{|S(n,k,r,d;u,0)|}{d^2}, \tag{5.14}$$

$$v_4(n,k) = \sum_{0<|v|<n} \frac{1}{|v|} \sum_{r \leqslant N} \frac{1}{r} \sum_{d \leqslant \xi^2} \frac{|S(n,k,r,d;0,v)|}{d^2}, \tag{5.15}$$

and

$$v_5(n,k) = \sum_{0<|u|,|v|<n} \frac{1}{|u||v|} \sum_{r \leqslant N} \sum_{d \leqslant \xi^2} \frac{|S(n,k,r,d;u,v)|}{d}. \tag{5.16}$$

In order to estimate $v_3(n,k)$, $v_4(n,k)$, and $v_5(n,k)$ we will first examine some results about exponential sums.

6. Some Results on Exponential Sums

We now develop some results about exponential sums which will assist us in estimating $S(n,k,r,d;u,v)$. The first lemma will allow us to exhibit a multiplicative property of an exponential sum under suitable conditions.

LEMMA 6.1. *Let* $\Psi(m;x,y)$ *indicate a condition on a positive integer* m *and integers* x *and* y *satisfying the following two properties:*

(1) *If* $x' \equiv x \pmod m$ *and* $y' \equiv y \pmod m$, *then* $\Psi(m;x,y)$ *is equivalent to* $\Psi(m;x',y')$.

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(2) *If* m_1 *and* m_2 *are coprime, then* $\Psi(m_1m_2; x, y)$ *holds if and only if* $\Psi(m_1; x, y)$ *and* $\Psi(m_2; x, y)$ *both hold.*

Let the exponential sum $P(m; u, v)$ *be given by*

$$P(m; u, v) = \sum_{\substack{\Psi(k;x,y) \\ 0 \leqslant x,y<m}} e((ux + vy)/m). \tag{6.1}$$

Then if $(m_1, m_2) = 1$*, we have*

$$P(k_1k_2; u, v) = P(m_1; \bar{m}_2u, \bar{m}_2v)P(m_2; \bar{m}_1u, \bar{m}_1v), \tag{6.2}$$

where $\bar{m}_1$ *and* $\bar{m}_2$ *are defined by the congruences*

$$m_1\bar{m}_1 \equiv 1 \pmod{m_2}, \qquad m_2\bar{m}_2 \equiv 1 \pmod{m_1}.$$

Proof. The proof relies on the Chinese Remainder Theorem, and Hooley gives a sketch of the proof following Lemma 3 in [5]. □

This leads to the following useful corollary.

LEMMA 6.2. *Let* $P(m; u,v)$ *be defined as in Lemma 6.1. If* u *and* v *are given and* $(m_1, m_2) = 1$*, then there exist integers* u_1,v_1,u_2,v_2 *such that*

$$P(m_1m_2; u, v) = P(m_1; u_1, v_1)P(m_2; u_2, v_2),$$

with

$$(m_1m_2, u, v) = (m_1, u_1, v_1)(m_2, u_2, v_2).$$

We will also need a bound on exponential sums which comes from a result of Chalk and Smith [3], which they proved using algebraic geometry.

LEMMA 6.3. *Let*

$$Q_k(m; u, v; \mu) = \sum_{\substack{x^3+y^k \equiv \mu \pmod{m} \\ 0 \leqslant x,y<m}} e\Big(\frac{ux + vy}{m}\Big). \tag{6.3}$$

If $p \nmid (u, v)$*, then*

$$\big|Q_k(p; u, v; \mu)\big| \leqslant (k^2 + 2k - 3)p^{1/2} + k^2. \tag{6.4}$$

Proof. From Theorem 2 in [3], it suffices to show that for every t in $\mathbb{F}_p$, we have that $ux + vy - t$ does not divide $x^3 + y^k - \mu$ in $\mathbb{F}_p[x, y]$. It is easy to check that if $p \nmid (u, v)$ and $k = 4$ or 5, then $ux + vy - t$ can not divide $x^3 + y^k - \mu$. □

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The estimate given in the preceding lemma allows us to bound an exponential sum arising from $S^*(n, k, r, d; u, v; l_d)$. In order to obtain a bound for $Q_k(m; u, v; \mu)$ when m is a prime power, we require the following results.

LEMMA 6.4. *Let*

$$W(m; a_1, a_2) = \sum_{0 \leqslant x < m} e\left(\frac{a_1 x^j + a_2 x}{m}\right). \tag{6.5}$$

Then for $\alpha \geqslant 3$,

$$W(p^\alpha; a_1, a_2) \ll p^{\alpha/2}(p^\alpha, a_1, a_2)^{1/4}(p^\alpha, a_2)^{1/4}. \tag{6.6}$$

Proof. For $j = 1, 2$, let $a_j = p^{\gamma_j}\tilde{a}_j$, where $p \nmid \tilde{a}_j$. (If either $a_j = 0$, we can replace it by p^α without affecting (6.6).) We note that the result is trivial when $p^\alpha \mid (a_1, a_2)$, so that we can suppose that at least one of the γ_j is less than α.

Suppose that $\gamma_1 > \gamma_2$. Then

$$W(p^\alpha; a_1, a_2) = p^{\gamma_2} W(p^{\alpha-\gamma_2}; p^{\gamma_1-\gamma_2}\tilde{a}_1, \tilde{a}_2).$$

However, $W(p^\alpha; b_1, b_2) = 0$ if $p \mid b_1$ but $p \nmid b_2$, so that in this case, $W(p^\alpha, a_1, a_2) = 0$.

Therefore, we can now restrict our attention to the case where $\gamma_1 \leqslant \gamma_2$. Then

$$W(p^\alpha; a_1, a_2) = p^{\gamma_1} W(p^{\alpha-\gamma_1}; \tilde{a}_1, p^{\gamma_2-\gamma_1}\tilde{a}_2). \tag{6.7}$$

From the proof of Lemma 7 in [8], we have that if $p \nmid b_1$, then

$$W(p^\alpha; b_1, b_2) \ll p^{\alpha/2}(p^\alpha, a_2)^{1/4}.$$

Thus, we obtain from (6.7) that

$$\begin{aligned} W(p^\alpha; a_1, a_2) &\ll p^{\gamma_1+(\alpha-\gamma_1)/2}(p^{\gamma_2-\gamma_1}, p^{\alpha-\gamma_1})^{1/4} \\ &= p^{(\alpha/2)+(\gamma_1/4)}(p^{\gamma_2}, p^\alpha)^{1/4}. \end{aligned} \tag{6.8}$$

By recalling that $\gamma_1 \leqslant \gamma_2$, we see that (6.6) follows from (6.8). □

LEMMA 6.5. *Let*

$$E_k(p^\alpha; v) = \sum_{\substack{1 \leqslant y_1, y_2 \leqslant p^\alpha \\ y_1^k \equiv y_2^k \pmod{p^\alpha}}} e\left(\frac{v(y_1 - y_2)}{p^\alpha}\right). \tag{6.9}$$

Then

$$E_k(p^\alpha; v) \ll p^\alpha(v, p^\alpha)^{(k-2)/(k-1)}. \tag{6.10}$$

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Proof. The cases $\alpha = 1$ and 2 are trivial, so we can suppose $\alpha \geqslant 3$. We begin by noting that the contribution to $E_k(p^\alpha; v)$ from y_1, y_2 for which $p^{\lceil \alpha/k \rceil} \mid (y_1, y_2)$ is

$$\sum_{\tilde{y}_1, \tilde{y}_2 = 1}^{p^{\alpha - \lceil \alpha/k \rceil}} e\left(\frac{v(\tilde{y}_1 - \tilde{y}_2)}{p^{\alpha - \lceil \alpha k \rceil}}\right) = \left| \sum_{\tilde{y}_1 = 1}^{p^{\alpha - \lceil \alpha/k \rceil}} e\left(\frac{v\tilde{y}_1}{p^{\alpha - \lceil \alpha/k \rceil}}\right) \right|^2 = \begin{cases} p^{2(\alpha - \lceil \alpha/k \rceil)}, & \text{if } p^{\alpha - \lceil \alpha/k \rceil} \mid v \\ 0, & \text{otherwise.} \end{cases} \tag{6.11}$$

Therefore, this contribution does not exceed our bound in (6.10).

We can now restrict our attention to the contribution from y_1, y_2 for which $y_1 = p^\gamma \tilde{y}_1$ and $y_2 = p^\gamma \tilde{y}_2$, where $0 \leqslant \gamma < \lceil \alpha/k \rceil$ and $p \nmid \tilde{y}_1 \tilde{y}_2$. Let $\mathcal{U}_k(m)$ denote the set of reduced residues ω modulo m for which $\omega^k \equiv 1 \pmod{m}$. Then for a given value of γ in the above range, the contribution to $E_k(p^\alpha; v)$ from the associated values of y_1, y_2 is

$$\sum_{\substack{1 \leqslant \tilde{y}_1, \tilde{y}_2 \leqslant p^{\alpha - \gamma} \\ \tilde{y}_1^k \equiv \tilde{y}_2^k \pmod{p^{\alpha - \gamma k}} \\ p \nmid \tilde{y}_1 \tilde{y}_2}} e\left(\frac{v(\tilde{y}_1 - \tilde{y}_2)}{p^{\alpha - \gamma}}\right). \tag{6.12}$$

Note that $\tilde{y}_1^k \equiv \tilde{y}_2^k \pmod{p^{\alpha - \gamma k}}$ if and only if there exists $\omega \in \mathcal{U}_k(p^{\alpha - \gamma k})$ such that $\tilde{y}_1 \equiv \omega \tilde{y}_2 \pmod{p^{\alpha - \gamma k}}$. Therefore, if we replace $\tilde{y}_1$ by $\omega \tilde{y}_2 + c p^{\alpha - \gamma k}$, we can rewrite (6.12) as

$$\sum_{\omega \in \mathcal{U}_k(p^{\alpha - \gamma k})} \sum_{\substack{\tilde{y}_2 = 1 \\ p \nmid \tilde{y}_2}}^{p^{\alpha - \gamma}} e\left(\frac{v(\omega - 1)\tilde{y}_2}{p^{\alpha - \gamma}}\right) \sum_{c=1}^{p^{\gamma(k-1)}} e\left(\frac{vc}{p^{\gamma(k-1)}}\right). \tag{6.13}$$

Since the inner sum is 0 unless $p^{\gamma(k-1)} \mid v$, and since $\mathcal{U}_k(p^{\alpha - \gamma k})$ has at most k elements, the contribution from a particular value of γ is at most $kp^{\alpha + \gamma(k-2)}$ if $p^{\gamma(k-1)} \mid v$, and is 0 otherwise. Let $p^a \| v$, where we adopt the convention that $a = \infty$ if $v = 0$. Then a particular γ satisfying $0 \leqslant \gamma \leqslant \lceil \alpha/k \rceil - 1$ will only contribute to $E_k(p^\alpha; v)$ when $\gamma \leqslant \frac{a}{k-1}$. By summing the contribution from the values of γ with $0 \leqslant \gamma \leqslant \lceil \alpha/k \rceil - 1$, and noticing that all other values of γ are handled by (6.11), we see that (6.10) holds. □

The preceding two lemmata allow us to obtain a bound for $Q_k(p^\alpha; u, v; \mu)$.

LEMMA 6.6. *Let* $Q_k(p^\alpha; u, v; \mu)$ *be defined as in* (6.3), *let* $\nabla(m; a)$ *be the multiplicative function of m defined on prime powers by*

$$\nabla(p^\alpha; a) = \begin{cases} 1, & \text{if } \alpha = 1 \text{ and } p \nmid a, \\ p^{\alpha/2} & \text{otherwise,} \end{cases} \tag{6.14}$$

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and let $\mathcal{H}(m; w_1, w_2; a)$ *be the multiplicative function of m satisfying*

$$\mathcal{H}(p^\alpha; w_1, w_2; a) = \begin{cases} (p^\alpha, w_1)^{1/4}(p^\alpha, w_2)^{\frac{k-2}{2(k-1)}}, & \text{if } \alpha \geqslant 3 \text{ and } p \mid a, \\ 1, & \text{otherwise.} \end{cases} \tag{6.15}$$

Then we have

$$Q_k(m; u, v; \mu) \ll m^{1/2+\varepsilon} \nabla(m; (u, v)) \mathcal{H}(m; u, v; \mu). \tag{6.16}$$

Proof. By repeated application of Lemma 6.2, it suffices to prove that for any prime power p^α,

$$Q_k(p^\alpha; u, v; \mu) \leqslant A p^{\alpha/2} \nabla(p^\alpha; (u, v)) \mathcal{H}(p^\alpha; u, v; \mu), \tag{6.17}$$

where A is a positive constant, since $A^{\omega(m)} \ll m^\varepsilon$. So let p^α be a given prime power. If $\alpha = 1$, then (6.17) follows from Lemma 6.3. If $\alpha = 2$, then (6.17) follows from (3.7). If $\alpha \geqslant 3$ and $p \nmid \mu$, then (6.17) follows from (3.8).

It remains to consider the cases where $\alpha \geqslant 3$ and $p \mid \mu$. By applying the Cauchy-Schwarz inequality, we obtain the bound

$$\begin{aligned} |Q_k(p^\alpha; u, v; \mu)| &= \left| \frac{1}{p^\alpha} \sum_{t=1}^{p^\alpha} \sum_{x=1}^{p^\alpha} e\left(\frac{tx^3 + ux}{p^\alpha}\right) \sum_{y=1}^{p^\alpha} e\left(\frac{ty^k + vy - t\mu}{p^\alpha}\right) \right| \\ &\leqslant \left(\frac{1}{p^\alpha} \sum_{t=1}^{p^\alpha} |W(p^\alpha; t, u)|^2 \right)^{\frac{1}{2}} \left(\frac{1}{p^\alpha} \sum_{t=1}^{p^\alpha} \left| \sum_{y=1}^{p^\alpha} e\left(\frac{ty^k + vy}{p^\alpha}\right) \right|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{6.18}$$

Moreover, by Lemma 6.4, the first term of (6.18) is bounded above by

$$\left(\frac{1}{p^\alpha} \sum_{t=1}^{p^\alpha} p^\alpha (p^\alpha, t)^{1/2} (p^\alpha, u)^{1/2} \right)^{1/2} \ll p^{\alpha/2} (p^\alpha, u)^{1/4}. \tag{6.19}$$

We can rewrite the second term of (6.18) as $E_k(p^\alpha; v)^{1/2}$, so that (6.17) follows from using the bounds from (6.19) and Lemma 6.5 in (6.18). This completes the proof of the lemma. □

We have now laid the framework to obtain an expression for $S(n, k, r, d; u, v)$ that will be used to estimate $v_3(n, k)$, $v_4(n, k)$, and $v_5(n, k)$. Let

$$U^*(n, k, r, d; a_1, a_2; l_d) = \sum_{0 \leqslant b, c < d} e((a_1 b + a_2 c)/d), \tag{6.20}$$

where the sum is over b, c for which

$$2r(r^2 + l_d) \equiv n - b^3 - c^k \pmod{d}. \tag{6.21}$$

If $S^*(n, k, r, d; u, v; l_d)$ is as defined in (5.7), then by Lemma 6.1, and by recalling that

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$(2r, d) = 1$,

$$S^*(n, k, r, d; u, v; l_d) = Q_k(2r; \bar{d}u, \bar{d}v; n)U^*(n, k, r, d; \overline{2r}u, \overline{2r}v; l_d), \tag{6.22}$$

where $d\bar{d} \equiv 1 \pmod{2r}$, and $2r\overline{2r} \equiv 1 \pmod{d}$. Let

$$U(n, k, r, d; a_1, a_2) = \sum_{l_d} \sum_{0 \leqslant b,c<d} e((a_1 b + a_2 c)/d), \tag{6.23}$$

where the sum is over l_d, b, c satisfying (6.21). Then by (6.22) and (5.7),

$$S(n, k, r, d; u, v) = Q_k(2r; \bar{d}u, \bar{d}v; n)U(n, k, r, d; \overline{2r}u, \overline{2r}v). \tag{6.24}$$

Recalling that $(2r, \bar{d}) = 1$, we can use Lemma 6.6 to obtain

$$S(n, k, r, d; u, v) \ll r^{1/2+\varepsilon}\nabla(r; (u, v))\mathcal{H}(r; u, v; n)|U(n, k, r, d; \overline{2r}u, \overline{2r}v)|. \tag{6.25}$$

7. Estimation of $v_3(n, k)$ and $v_4(n, k)$

In order to bound $v_3(n, k)$ and $v_4(n, k)$, it is sufficient to bound $U(n, k, r, d; \overline{2r}u, \overline{2r}v)$ trivially in (6.25), so that

$$S(n, k, r, d; u, v) \ll r^{1/2+\varepsilon}d^2\nabla(r; (u, v))\mathcal{H}(r; u, v; n). \tag{7.1}$$

We will need the following result.

LEMMA 7.1. *Let $F_1(m; a)$ and $F_2(m; a)$ be the multiplicative functions of m defined on prime powers by*

$$F_1(p^\alpha; a) = \begin{cases} p^{\alpha/4}, & \text{if } p \mid a, \\ 1, & \text{otherwise}, \end{cases} \tag{7.2}$$

and by

$$F_2(p^\alpha; a) = \begin{cases} p^{\alpha(k-2)/2(k-1)}, & \text{if } p \mid a, \\ 1, & \text{otherwise}. \end{cases} \tag{7.3}$$

If $j = 1$ or 2 and $a \leqslant n$, then

$$\sum_{0<t\leqslant n} F_j(t; n)t^{-1} \ll \sum_{0<t\leqslant n} F_j(t; n)\nabla(t; a)t^{-1} \ll n^\varepsilon. \tag{7.4}$$

Proof. Since $F_1(t; a) \leqslant F_2(t; a)$ and $\nabla(t; a) \geqslant 1$, it suffices to show that the final inequality in (7.4) holds when $j = 2$, so let $j = 2$. By multiplicativity in t, and since

$$\frac{k-2}{2(k-1)} + \frac{1}{2} - 1 = \frac{-1}{2(k-1)},$$

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we have that the left side of (7.4) is bounded by

$$\prod_{\substack{p \nmid na \\ p \leqslant n}} \left(1 + \frac{2}{p} + \sum_{m \geqslant 3} p^{-m/2}\right) \prod_{\substack{p \mid na \\ p \leqslant n}} \left(1 + \sum_{m \geqslant 1} p^{-m/2(k-1)}\right)$$
$$\ll (na)^{\varepsilon} \prod_{p \leqslant n} \left(1 - \frac{1}{p}\right)^{3} \ll n^{\varepsilon},$$

where we have used that $C^{\omega(na)} \ll (na)^{\varepsilon}$ for any constant C. This completes the proof of the lemma. □

By using the bound for S from (7.1) in (5.14), we obtain

$$\begin{aligned} v_3(n,k) &\ll \sum_{0<|u|<n} \frac{1}{|u|} \sum_{r \leqslant n^{1/3}} r^{-1/2+\varepsilon} \sum_{d \leqslant \xi^2} \nabla(r;u)\mathcal{H}(r;u,0;n) \\ &\ll n^{(1/6)+2\beta+\varepsilon} \sum_{0<u<n} \frac{F_1(u;n)}{u} \sum_{r \leqslant n^{1/3}} \frac{\nabla(r;u)F_2(r;n)}{r} \\ &\ll n^{(1/6)+2\beta+\varepsilon}, \end{aligned} \tag{7.5}$$

since Lemma 7.1 provides upper bounds for the final two sums in (7.5).

In a similar manner, we obtain from (5.15) that

$$\begin{aligned} v_4(n,k) &\ll \sum_{0<|v|<n} \frac{1}{|v|} \sum_{r \leqslant n^{1/3}} r^{-1/2+\varepsilon} \sum_{d \leqslant \xi^2} \nabla(r;v)\mathcal{H}(r;0,v;n) \\ &\ll n^{(1/6)+2\beta+\varepsilon} \sum_{0<v<n} \frac{F_2(v;n)}{v} \sum_{r \leqslant n^{1/3}} \frac{\nabla(r;v)F_1(r;n)}{r} \\ &\ll n^{(1/6)+2\beta+\varepsilon}. \end{aligned} \tag{7.6}$$

8. Estimation of $U(n,k,r,d;u,v)$

In order to bound $v_5(n,k)$ we will need a more precise estimate for $U(n,k,r,d;u,v)$ than that used in (7.1). In order to achieve this, we will use estimates based on the work of Deligne. The methods we use are based on those developed by Hooley in [6, 7]. We will require the following lemma concerning absolute irreducibility, which is Theorem III.1 B of Schmidt [10].

LEMMA 8.1. *Let*

$$\eta(x,y) = \varphi_0 y^m + \varphi_1(x) y^{m-1} + \cdots + \varphi_m(x),$$

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where φ_0 *is a non-zero constant, be a polynomial with coefficients in a field K. Let*

$$\delta(\eta) = \max_{1 \leqslant i \leqslant m} \frac{1}{i} \deg(\varphi_i),$$

and suppose that $\delta(\eta) = j/m$ *with* $(j, m) = 1$. *Then* $\eta(x, y)$ *is absolutely irreducible.*

The next lemma will provide a bound for $U(n, k, r, d; u, v)$.

LEMMA 8.2. *If d is square-free number whose prime factors satisfy* (2.8) *and* (2.9), *then*

$$U(n, k, r, d; u, v) \ll d\,(d, u). \tag{8.1}$$

Proof. We point out that it should be possible to obtain the bound $d\,(d, u, v)$, but this bound will prove sufficient for our purposes. By repeated application of Lemma 6.2, it suffices to show that

$$U(n, k, r, p; u, v) \ll p\,(p, u) \tag{8.2}$$

for any prime p satisfying (2.8) and (2.9). Since (8.2) holds trivially when $(p, u) = p$, we can suppose that $p \nmid u$. Let

$$G(n, k, r, p; u, v) = \sum_{0 \leqslant x, y, z < p} e\left(\frac{ux + vy}{p}\right), \tag{8.3}$$

where the sum is over x, y, z satisfying

$$2r(r^2 + z^2) \equiv n - x^3 - y^k \pmod{p}. \tag{8.4}$$

Then

$$U(n, k, r, p; u, v) = -\frac{1}{2} G(n, k, r, p; u, v) + \sum_{0 \leqslant x, y < p} e\left(\frac{ux + vy}{p}\right)$$
$$+ \sum_{\substack{0 \leqslant x, y < p \\ 2r^3 \equiv n - x^3 - y^k \pmod{p}}} e\left(\frac{ux + vy}{p}\right), \tag{8.5}$$

since l_p must be a quadratic nonresidue modulo p. The second term on the right side of (8.5) is zero because $p \nmid u$, and the final sum in (8.5) is $O(p)$, so that

$$U(n, k, r, p; u, v) \ll G(n, k, r, p; u, v) + O(p). \tag{8.6}$$

Therefore (8.2) follows from showing that

$$G(n, k, r, p; u, v) \ll p. \tag{8.7}$$

Let

$$h(x, y, z) = x^3 + y^k + 2r(r^2 + z^2) - n, \quad \text{and} \quad w(x, y) = ux + vy, \tag{8.8}$$

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where for convenience, we omit the dependence on n, k, r, u, and v. Next, let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$. We point out that (2.9) gives that $p > 3$, and that $2r \neq 0$ when considered as an element of $\mathbb{F}_p$. Also, since u is an integer and $p \nmid u$, then $u \neq 0$ in $\mathbb{F}_p$. In [7], Hooley uses Deligne's resolution of the Riemann hypothesis for algebraic varieties over finite fields to develop conditions under which (8.7) will hold. Theorem 5 of [7] shows that (8.7) is true provided that the following two conditions are satisfied:

(1) If $t \in \overline{\mathbb{F}_p}$, then the curve defined by $h(x, y, z) = 0$ and $w(x, y) = t$ is absolutely irreducible over $\overline{\mathbb{F}_p(t)}$, the algebraic closure of the function field $\mathbb{F}(t)$, for all but at most D_3 values of t, where D_3 is some absolute constant.
(2) For all $t \in \overline{\mathbb{F}_p}$ and for any natural number α, the number of $(x, y, z) \in \mathbb{F}_{p^\alpha}^3$ satisfying $h(x, y, z) = 0$ and $w(x, y) = t$ is $O(p^\alpha)$.

Note that since $p \nmid u$, then $w(x, y) = t$ means that $x = u^{-1}(t - vy)$. From this expression for x, we find that $h(x, y, z)$ becomes

$$h_t(y, z) = u^{-3}(t - vy)^3 + y^k + 2r(r^2 + z^2) - n. \tag{8.9}$$

Then the number of solutions to $h_t(y, z) = 0$ is $O(p^\alpha)$, because for each choice of $y \in \mathbb{F}_{p^\alpha}$, there are at most two values of z satisfying the equation, so that Condition (2) holds.

Showing that condition 1 holds will require proving that $h_t(y, z)$ is absolutely irreducible except for at most D_3 values of t. If $k = 5$, then by Lemma 8.1, we have that $h_t(y, z)$ is absolutely irreducible for all values of $t \in \overline{\mathbb{F}_p}$, because $(2, k) = 1$ in this case. When $k = 4$, we will have to resort to direct attempts at factoring $h_t(y, z)$, which will show that $h_t(y, z)$ is absolutely irreducible over $\overline{\mathbb{F}_p(t)}$ unless t is one of the solutions of certain polynomials.

So suppose that $k = 4$, and that

$$h_t(y, z) = g_1(y, z)g_2(y, z) \tag{8.10}$$

is a non-trivial factorization of $h_t(y, z)$, where $g_1, g_2 \in \overline{\mathbb{F}_p(t)}[y, z]$. The degree of g_1 and g_2 with respect to z can not be two, or else the other polynomial would be constant. Therefore, on multiplying g_1 and g_2 by suitable elements of $\overline{\mathbb{F}_p}$, we can suppose without loss of generality that

$$g_j(y, z) = \tilde{g}_j(y) + \gamma z, \quad j = 1, 2, \tag{8.11}$$

where $\gamma^2 = 2r$. On equating the coefficients involving z in (8.9) and (8.10), we find that

$$\tilde{g}_1(y) = -\tilde{g}_2(y). \tag{8.12}$$

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Let $\tilde{g}_1(y) = c_2y^2 + c_1y + c_0$. From (8.9), (8.10), (8.11), and (8.12), we see that

$$-\tilde{g}_1(y)^2 = y^4 + u^{-3}(t - vy)^3 + 2r^3 - n.$$

By equating the coefficients of y in this last equation, we see that

$$c_2^2 = -1, \tag{8.13}$$

$$2c_1c_2 = u^{-3}v^3, \tag{8.14}$$

$$c_1^2 + 2c_0c_2 = -3u^{-3}v^2t, \tag{8.15}$$

$$2c_0c_1 = 3u^{-3}vt^2, \tag{8.16}$$

$$c_0^2 = n - 2r^3 - u^{-3}t^3. \tag{8.17}$$

Suppose first of all that $v = 0$ in $\mathbb{F}_p$, and that $t \neq 0$. Note from (8.13) that $c_2 \neq 0$. This means that $c_1 = 0$ by (8.14), and then that $c_0 = 0$ from (8.15). Then from (8.17), we see that

$$t^3 = u^3(n - 2r^3). \tag{8.18}$$

Next suppose that $v \neq 0$ in $\mathbb{F}_p$, that $t \neq 0$, and that (8.18) does not hold. By (8.17), our final supposition shows that $c_0 \neq 0$. Using (8.13), (8.14), and (8.16) to solve for $2c_1$, we see that

$$-u^{-3}v^3c_2 = 3c_0^{-1}u^{-3}vt^2. \tag{8.19}$$

By squaring both sides of (8.19), and then using (8.17) to substitute for c_0^2, we obtain

$$(u^{-3}t^3 + 2r^3 - n)v^4 = 9t^4. \tag{8.20}$$

So if (8.10) is a nontrivial factorization of $h_t(y, z)$, then t is either zero or satisfies (8.18) or (8.20). Since the number of solutions t in $\overline{\mathbb{F}_p}$ of (8.18) or (8.20) are bounded by 3 and 4 respectively, then there are at most eight values of t in $\overline{\mathbb{F}_p}$ for which $h_t(y, z)$ fails to be absolutely irreducible. This proves that condition 1 holds, which by our previous discussion proves (8.7). The lemma now follows from (8.6). □

9. Estimation of $v_5(n, k)$

We can now use the bound for U from the previous section to obtain a bound for $v_5(n, k)$. From (5.16), (6.25), Lemma 8.2, and recalling that $(\overline{2r}, d) = 1$ in (6.25), we have that

$$\begin{aligned} v_5(n,k) &\ll \sum_{0<|u|,|v|<n} \frac{1}{|u||v|} \sum_{r \leqslant N} \sum_{d \leqslant \xi^2} r^{1/2+\varepsilon} \nabla(r; (u,v)) \mathcal{H}(r; u, v; n)(d, u) \\ &\ll n^{(1/2)+\varepsilon} \sum_{0<u,v<n} \frac{F_1(u;n)}{uv} \sum_{r \leqslant n^{1/3}} \frac{\nabla(r; (u,v)) F_2(r;n)}{r} \sum_{d \leqslant \xi^2} (d, u). \end{aligned} \tag{9.1}$$

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The final sum in (9.1) satisfies

$$\sum_{d \leqslant \xi^2} (d, u) \leqslant \sum_{\delta | u} \delta \sum_{\substack{d \leqslant \xi^2 \\ \delta | d}} 1 \leqslant \xi^2 \tau(u) \ll n^{2\beta+\varepsilon}. \tag{9.2}$$

By Lemma 7.1, the sums over u and r in (9.1) are $O(n^{\varepsilon})$, so that from (9.1) and (9.2), we have

$$v_5(n, k) \ll n^{(1/2)+2\beta+\varepsilon}. \tag{9.3}$$

10. Completion of the Proof

We now complete the proof of Theorem 1.1. By (5.13), (7.5), (7.6), and (9.3), we have

$$v_2(n, k) \ll n^{(1/2)+2\beta+\varepsilon}. \tag{10.1}$$

By (2.1) and (2.4), we have

$$R_k(n) \ll v(n, k) + n^{1/3} \tag{10.2}$$

From (2.28), (4.18), and (10.1), we see that choosing β to be $\frac{1}{3k} - \frac{1}{18}$ will minimize our bound for $v(n, k)$, so that (10.2) will yield Theorem 1.1.

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