Canad. Math. Bull. Vol. 19 (2), 1976

LOCAL COMPACTNESS IN SET VALUED FUNCTION SPACES

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1. Recently Hunsaker and Naimpally [2] have proved: The pointwise closure of an equicontinuous family of point compact relations from a compact T_2-space to a locally compact uniform space is locally compact in the topology of uniform convergence. This is a generalization of the same result of Fuller [1] for single valued continuous functions.

For a range space which is locally compact normal and uniform theorem B below is an improvement on the result of Hunsaker and Naimpally quoted above [see Remark 3 at the end of this paper].

For general topological spaces the notion of equicontinuity has been generalized to "even-continuity" [see [5]] and "regularity" [4] which are equivalent under reasonable conditions [4]. Their natural generalizations [3], however, for the set valued function spaces are quite distinct and together yield, as shown in [3], an Ascoli type theorem for the set of all set valued functions with point compact images, continuous with respect to the finite topology [6] on the hyperspace of the range space, and having the "compact-open" topology. It seems natural to ask if these conditions would also give a result analogous to that of Fuller quoted above. The purpose of this paper is to prove that they do.

We need a few notations and definitions before stating the main results. For any space Y let 2^Y denote the set of all non-empty closed subsets of Y and $C(Y)=\{A\in 2^Y : A$ is compact$\}$. For any set $U\subset Y$ let $L(U)=\{A\in 2^Y : A\cap U\neq\varnothing\}$ and $M(U)=\{A\in 2^Y : A\subset U\}$. Then the topology generated by all sets $L(U)[M(U)]$ as a sub-base [base], where U is any open set in Y, will be denoted by $\tau[\kappa]$ and are the so called lower semi finite [upper semi finite] topologies. The smallest topology containing both τ and κ is the so called finite topology [6] and will be denoted by ν. A set valued function $f:X\to Y$ assigns to each $x\in X$ a closed and non-empty subset $f(x)$ of Y. Thus f defines a single valued function $\hat{f}:X\to 2^Y$ and conversely. For any topology t on 2^Y, f is t-continuous if $\hat{f}$ is continuous with respect to t. In particular f is said to be continuous [l.s.c.; u.s.c.] if f is ν–[τ–; κ–] continuous.

Let $F=F(X, Y)$ denote the set of all set valued functions from X into Y,

Received by the editors May 8, 1974.

https://doi.org/10.4153/CMB-1976-029-2 Published online by Cambridge University Press

$C(X, Y)=\{f\in F: f(x)$ is compact for each $x\in X\}$ and $S(X, Y)$ be the set of all continuous functions in $C(X, Y)$. For any subset G of F and $A\subset X$, let $G(A)=\bigcup\{f(x): f\in G$ and $x\in A\}$, and $G(x)=G(\{x\})$. We say that G is *regular at* $x\in X$ if given any open set U in Y and $H\subset G$ such that $\overline{H(x)}\subset U$, there exists an open set V containing x such that $H(V)\subset U$. G is *regular* if it is regular at each $x\in X$. We say that G is *evenly continuous* at $x\in X$ if given any $y\in Y$ and closed neighbourhood U of y there exist open sets V and W containing x and y respectively such that if for any $g\in G$, $g(x)\cap W\neq\emptyset$, then $g^{-1}[U]=\{z\in Y: g(z)\cap U\neq\emptyset\}\supset V$. The point open topology $P_\kappa[P_\tau]$ on F is the topology generated by all sets of the type $M(x, U)=\{f\in F: f(x)\subset U\}$, $[L(x, U)=\{f\in F: f(x)\cap U\neq\emptyset\}]$, where $x\in X$ and $U\subset Y$ is open. The topology P_ν is the smallest topology containing P_τ and P_κ. Finally, for any $f\in F$, $\mathscr{G}(f)=\bigcup\{x\times f(x): x\in X\}$ is called the graph of f. Note that if $G\subseteq C(X, Y)$ is regular and evenly continuous then $G\subset S(X, Y)$ provided Y is regular.

The main results of this paper are:

THEOREM A. *Let X be a compact T_2-space and Y be a locally compact T_2-space. If $F\subset C(X, Y)$ is regular and evenly continuous, then $\bar{F}$, the closure of F in $(C(X, Y), P_\nu)$, is locally compact.*

THEOREM B. *Let X be a compact T_2-space and Y be a locally compact, normal T_2-space. If $F\subset C(X, Y)$ is regular, then $\bar{F}$, the closure of F in $(C(X, Y), P_\nu)$, is locally compact.*

2. REMARK 1. Let X be compact and Y be regular.

If $f\in S(X, Y)$, then $\mathscr{G}(f)\in C(X\times Y)$: since f is continuous clearly the set valued function $f': X\to X\times Y$ defined by $f'(x)=\{(x, y): y\in f(x)\}$ for each $x\in X$ is also continuous, that is, $\hat{f}': X\to C(X\times Y)$ is ν-continuous. Hence $\hat{f}'(X)=\{\{x\}\times f(x)\in C(X\times Y): x\in X\}$ is a compact subset of $(C(X\times Y), \nu)$. Thus by theorem (2.5.2) [6, p. 157], $\mathscr{G}(f)=\{(x, y): x\in X, y\in f(x)\}=\bigcup\{\hat{f}'(x): x\in X\}$ is a compact subset of $X\times Y$.

THEOREM 1. *Let Y be a regular space. Let $G\subset C(X, Y)$ be regular and evenly continuous and $\{f_\alpha: \alpha\in D\}$ be a net in G converging to $f\in C(X, Y)$ with respect to P_ν. Then f is continuous.*

Proof. We shall show that (a) f is u.s.c. and (b) f is l.s.c.

f is u.s.c.: Let $x\in X$ and $U\supset f(x)$ be open in Y. Then there is an open set V in Y containing $f(x)$, such that, $\bar{V}\subset U$. Since $\{f_\alpha\}$ converges to f with respect to P_κ there exists an $\alpha_0\in D$ such that for all $\alpha\geq\alpha_o$, $\alpha\in D$, $f_\alpha(x)\subset V$. Hence $\overline{G(x)}\subset U$, where $G=\{f_\alpha: \alpha\in D$ and $\alpha\geq\alpha_0\}$. By regularity there exists an open set W containing x such that $G(W)\subset U$. We claim that $f(W)\subset U$: Suppose not. Then there exists a $z\in W$ such that $f(z)\not\subset U$. Let 0 be an open set in Y such that $0\cap f(z)\neq\emptyset$ and $0\cap U=\emptyset$, for one may assume without loss of

https://doi.org/10.4153/CMB-1976-029-2 Published online by Cambridge University Press

generality, because Y is regular, that U is a closed neighbourhood of $f(x)$. But then for all $\alpha \geq \alpha_0$, $f_\alpha(z) \cap 0 = \emptyset$ contradicting that $\{f_\alpha\}$ converges to f with respect to P_τ. This establishes the above claim and proves that f is u.s.c.

f is l.s.c.: Let $x \in X$, U_0 be an open set in Y and $f(x) \cap U_0 \neq \emptyset$. Let $U \subset U_0$ be a closed neighbourhood of $y \in U_0 \cup f(x)$. Then by even continuity of F there exist open sets V containing y and W containing x such that $f \in F$ and $f(x) \cap V \neq \emptyset$ then $W \subset F^{-1}(U)$. Since $\{f_\alpha\}$ converges to f with respect to P_τ and $f(x) \cap (U \cap V) \neq \emptyset$, there exists an $\alpha_0 \in D$, such that, for all $\alpha \in D$ and $\alpha \geq \alpha_0$, $f_\alpha(x) \cap V \neq \emptyset$. Hence for any $z \in W$, $f_\alpha(z) \cap U \neq \emptyset$ for all $a \in D$ and $\alpha \geq \alpha_0$. We claim that for any $z \in W$, $f(z) \cap U \neq \emptyset$: Suppose not, then there exists in Y an open set $0 \supset f(z)$ such that $0 \cap U = \emptyset$, but then $f_\alpha(z) \not\subset 0$ for any $\alpha \in D$ and $\alpha \geq \alpha_0$ contradicting the fact that $\{f_\alpha\}$ converges to f with respect to P_κ. The above claim thus proves that f is l.s.c.

THEOREM 2. *Let $F \subset C(X, Y)$ be regular and evenly continuous, X be compact T_2 and Y be a regular space. If $\{f_\alpha\}$ is a net in F converging to $g \in F$ with respect to P_ν, then $\mathcal{G}(f_\alpha)$ converges to $\mathcal{G}(g)$ in $X \times Y$ with respect to ν.*

Proof. We shall show that $\mathcal{G}(f_\alpha)$ converges to $\mathcal{G}(g)$, (a) with respect to τ, and (b) with respect to κ. Note that by theorem 1, g is continuous and hence $\mathcal{G}(g) \in C(X \times Y)$ by Remark 1 and the fact that ν on $C(X \times Y)$ is T_2.

Proof of (a): Let $U \times V$ be a basic open set in $X \times Y$ and $\mathcal{G}(g) \cap U \times V \neq \emptyset$. Then there is an $x \in X$ such that $x \times g(x) \cap U \times V \neq \emptyset$. Since $\{f_\alpha\}$ converges to g with respect to P_τ and $g(x) \cap V \neq \emptyset$ there exists an α_0 such that $\alpha \geq \alpha_0$ implies $f_\alpha(x) \cap V \neq \emptyset$. Hence, $x \times f_\alpha(x) \cap U \times V \neq \emptyset$ for $\alpha \geq \alpha_0$ and $\mathcal{G}(f_\alpha)$ converges to $\mathcal{G}(g)$ with respect to τ.

Proof of (b): Let W be an open set in $X \times Y$ containing $\mathcal{G}(g)$. Since $x \times g(x)$ is compact there are open sets $U(x)$ and $V(x)$ containing x and $g(x)$, such that, $\overline{U(x)} \times \overline{V(x)} \subset W$. Since $\{f_\alpha\}$ converges to g with respect to P_κ there exists an $\alpha(x)$, such that, $\alpha \geq \alpha(x)$ implies $f_a(x) \subset V(x)$. Since $g(x)$ is compact we may assume without loss of generality that $\overline{F_x(x)} \subset V(x)$ where $F_x = \{f_a : \alpha \geq \alpha(x)\}$. By regularity of F there exists an open set $0(x)$ containing x such that $F_x(0(x)) \subset V(x)$ and $0(x) \subset U(x)$. Hence for any $z \in 0(x)$ and $f \in F_x$, $z \times f(z) \subset W$. Now, since X is compact there exists a finite open cover $\{0(x_i) : 1 \leq i \leq n\}$ of X and if $\alpha_0 \geq \alpha(x_i)$, $i = 1, \ldots, n$, then for any $\alpha \geq \alpha_0$, $x \times f_\alpha(x) \subset W$ for each $x \in X$. That is, $\mathcal{G}(f_\alpha) \subset W$ for all $\alpha \geq \alpha_0$ and the proof is complete.

THEOREM 3. *Let Y be a regular space and $F \subset C(X, Y)$ be evenly continuous and regular. Let $\{f_\alpha\}$ be a net in F and $\mathcal{G}(f_\alpha)$ converge to $A \in C(X \times Y)$ with respect to ν. Then $A = \mathcal{G}(g)$ for some $g \in S(X, Y)$, and $\{f_\alpha\}$ converges to g with respect to P_ν.*

Proof. Let $g(x) = \{y : (x, y) \in A\}$. Then clearly $g(x)$ is compact for each $x \in X$ and to show that $g \in C(X, Y)$ it is enough to note that $g(x) \neq \emptyset$ for each x.

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Indeed, if this were not so then for some $x \in X$, $x \times Y \cap A = \emptyset$, that is, $A \subset (X - \{x\}) \times Y = W$. But from the given convergence there exists an α_0 such that for $\alpha \geq \alpha_0$, $\mathcal{G}(f_\alpha) \subset W$ implying that $f_\alpha(x) = \emptyset$ for all $\alpha \geq \alpha_0$ which is false. Clearly $\mathcal{G}(g) = A$.

To prove that $\{f_\alpha\}$ converges to g with respect to P_ν, we prove (a) convergence with respect to $P_{\kappa'}$ (b) convergence with respect to P_τ.

(a) Let $x \in X$ and U be an open set containing $g(x)$. Then $X \times U \cup (X - \{x\}) \times Y = W$ is an open set containing $\mathcal{G}(g) = A$, and since $\{\mathcal{G}(f_\alpha)\}$ converges to $\mathcal{G}(g)$ with respect to κ there exists an α_0 such that $\mathcal{G}(f_\alpha) \subset W$ for all $\alpha \geq \alpha_0$. Hence $x \times f_\alpha(x) \subset X \times U$, that is, $f_\alpha(x) \subset U$, and this proves (a).

(b) Let $x \in X$, U be open in Y and $g(x) \cap U \neq \emptyset$. Suppose that for a cofinal set E of values of α, $f_\alpha(x) \cap U = \emptyset$. Let $H = \{f_\alpha : \alpha \in E\}$ and $V \subset U$ be a closed neighbourhood of some point $z \in g(x) \cap U$. Then $\overline{H(x)} \subset Y - V = W$, and by regularity of F there exists an open set 0 containing x such that $H(0) \subset W$. Now $\mathcal{G}(g) \cap 0 \times V \neq \emptyset$, but for all $\alpha \in E$, $\mathcal{G}(f_\alpha) \cap 0 \times V = \emptyset$ contradicting that $\{\mathcal{G}(f_\alpha)\}$ converges to $\mathcal{G}(g)$ with respect to τ. Hence $f_\alpha(x) \cap U \neq \emptyset$ for all $\alpha \geq \alpha_0$ for some α_0, and this proves (b).

Finally it follows from theorem 1 that $g \in S(X, Y)$.

Proof of theorem A. Since F is regular and evenly continuous by theorem 1, $\bar{F} \subset S(X, Y)$. By remark 1, the mapping $f \to \mathcal{G}(f)$ associates to each element f of $\bar{F}$ an element $\mathcal{G}(f)$ of $C(X \times Y)$ and is clearly 1–1. Theorems 2 and 3 imply that the above correspondence is a homeomorphism of $(\bar{F}, P_\nu)$ into $(C(X, Y), \nu)$ and furthermore that $\{G(f) : f \in \bar{F}\}$ is a closed subset of $C(X \times Y)$. Hence $(C(X, Y), \nu)$ being locally compact [6, prop (4.4.1), p. 162] so is (F, P_ν). This proves theorem A.

We need the following result to prove Theorem B.

THEOREM 4. *Let Y be a normal T_2-space and $F \subset F(X, Y)$ be regular. Then $\bar{F}$, the closure of F in $(F(X, Y), P_\nu)$, is also regular.*

Proof. Let $x \in X$ be arbitrary. We shall show that $\bar{F}$ is regular at x. So let U be an open set in Y, $H \subset \bar{F}$ and $\overline{H(x)} \subset U$. Since Y is normal there is an open set $W \supset \overline{H(x)}$ and $\bar{W} \subset U$. Let $G = \{f \in F : f(x) \subset W\}$. Then $H_1 = H \cap F \subset G$, $\overline{G(x)} \subset U$, and by regularity of F there exists an open set V containing x such that $G(V) \subseteq U$. We claim that if $g \in H$ then $g(V) \subset U$. Suppose not. Then $g \in H - H_1$ and there exists a net $\{f_\alpha, \alpha \in D, \geq\}$ in F converging to g with respect to P_ν. Since $g(x) \subset W$ and convergence of the net with respect to P_ν implies convergence with respect to P_κ there exist an $\alpha_0 \in D$ such that if $\alpha \in D$ and $\alpha \geq \alpha_0$, then $f_\alpha(x) \subset W$, that is, $f_\alpha \in G$. But if $g(V) \not\subset U$ then there exists a $z \in V$ such that $g(z) - U \neq \emptyset$, and hence an open set 0, such that, $0 \cap g(z) \neq \emptyset$ but $0 \cap U = \emptyset$ [For using normality of Y we may get another open set W' such that $W \subseteq W'$ and $\bar{W}' \subseteq U$ and work with W' if necessary]. But this leads to a

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contradiction, since the net $\{f_\alpha\}$ converges to g also with respect to p_τ implying that there exists an $\alpha_1 \in D$ such that for all $\alpha \in D$ and $\alpha \geq \alpha_1$, $f_\alpha(z) \cap 0 \neq \emptyset$. This completes the proof.

Proof of theorem B. Let $f \in \bar{F}$. We shall exhibit a compact neighbourhood in P_ν of f in $\bar{F}$. By theorem 4, $\bar{F}$ is regular, so f is u.s.c. By [7, Prop. (2.3), p. 34], $f(X)$ is compact. Let U be an open subset of Y containing $f(X)$ and having compact closure. We shall show that $N = \bar{F} \cap C(X, \bar{U})$ is the required neighbourhood of f. Now $C(X, \bar{U}) = F(X, \bar{U})$, and $(F(X, \bar{U}), P_\nu)$ is homeomorphic to $\prod\{Z_x : x \in X\}$ with the product topology, where Z_x is a copy of $2^{\bar{U}}$ with topology ν for each $x \in X$. Since (Z_x, ν) is compact [6, th (4.2), p. 161], $(C(X, \bar{U}), P_\nu)$ is compact. Since N is a closed subset it is compact. Thus to complete the proof it is enough to show that N is a P_ν-neighbourhood of f.

For each $x \in X$ there exists an open neighbourhood $V(x)$ containing $f(x)$, with $\overline{V(x)} \subset U$. Let $H_x = \{h \in C(X, Y) : h(x) \subset V(x)\}$. Then H_x is an open neighbourhood of f in P_ν and $\overline{H_x(x)} \subset U$. Since $\bar{F}$ is regular, there exists an open set $W(x)$ containing x such that $(H_x \cap \bar{F})(W(x)) \subset U$. Let $W(x_1), \ldots, W(x_n)$ cover X. Then $H = H_{x_1} \cap \cdots \cap H_{x_n}$ is open in P_ν, $f \in H$ and $(H \cap \bar{F})(X) \subset U$. Therefore N which contains $H \cap \bar{F}$ is a neighbourhood of f in P_ν.

Remark 2. The author is thankful to the referee for correcting earlier proof of this theorem.

Remark 3. For this remark see the definition of equicontinuity of Hausaker and Naimpally [2]. It is easy to see that if Y is a compact uniform space then equicontinuity implies regularity. In fact $G \subset F(X, Y)$* is equicontinuous if and only if it is regular and evenly continuous. To see that regularity is decidedly weaker than equicontinuity it is enough to consider a sequence of continuous functions converging pointwise to an u.s.c. function which is not l.s.c. then the sequence is regular but not evenly continuous and hence not equicontinuous.

Example. Let N be the set of all positive integers and $A = \{0\} \cup \{1/n : n \in N\}$.

We define a topology in terms of neighbourhoods of points in $Y = N \times N \cup \{0\} \times A$ as follows: For $y \in N \times N$ any subset of Y containing y is a neighbourhood of y. If $y = (0, 1/n)$ then any set containing y and a cofinite subset of $n \times N$ is a neighbourhood of y. If $y = (0, 0)$ then any set containing a set of the type

$$\{(0,0)\} \cup \{(m, n) : m \in N_1, n \in F_m\} \cup \{(0, 1/n) : n \geq k \quad \text{for some } k\}$$

is a neighbourhood, where N_1 is a cofinite subset of N and $F_m = N$ except for a finite number of elements $N_2 \subseteq N_1$, for each of which F_m is cofinite in N. Y with the topology generated by these neighbourhoods is compact T_2. T_2 is

* [It is enough to assume that $\overline{G(x)}$ is compact for each $x \in X$].

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clear. To see compactness consider an open covering $\mathcal{U}$ of Y. First pick an open set $U \in \mathcal{U}$ containing $(0, 0)$. This leaves a finite number of "rows" $m \times N$, $m \in N - N_1$, together with a finite set of elements of Y not covered by U. Next pick one element from $\mathcal{U}$ containing $(0, 1/m)$ for each $m \in N - N_1$. Then the finite set thus obtained from $\mathcal{U}$ leaves only a finite set in Y still not covered.

Let X be the space obtained by giving A the relative topology from the usual topology of the reals.

We now define the functions $f_n : X \to Y$, $n = 0, 1, 2, \ldots$ as follows:

For $n \neq 0$, $f_n(1/m) = 1/m$ if $m \leq n$, and for $m > n$, that is $m = n + k$,

$$f_n(1/m) = \{(i, j+n) : j = 1, \ldots, k;\ i \leq j\}$$

and

$$f_n(0) = \{(0, 0)\} \cup \{(i, j+n) : j \in N,\ i \leq j\} \cup \{(0, 1), \ldots, (0, 1/n)\}.$$

For $n = 0$, $f_0(1/m) = (0, 1/m)$, and $f_0(0) = \{0\} \times A$.

It is then easy to check that each f_n, $n \neq 0$, is continuous, and f_0 is u.s.c. but not l.s.c. Also for each $x \in X$,

$$\limsup_{n \in N} f_n(x) = \liminf_{n \in N} f_n(x) = f_0(x)$$

Since Y is compact T_2 this implies that $\{f_n\}$ converges to f_0 in p_ν. This is thus the type of example mentioned in Remark 3.

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