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$C(X)$ AS A DUAL SPACE

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It is known [1] that for compact Hausdorff X, $C(X)$ is the dual of a Banach space if and only if X is hyperstonian, that is the closure of an open set in X is again open and the carriers of normal measures in $C(X)^*$ have dense union in X. With the desiratum of proving that $C(X)$ is always the dual of some sort of space we broaden the concept of Banach space as follows. A Banach space may be comfortably regarded as a pair (E, B) where E is a topological linear space and B is a subset of E; the requisite property is that the Minkowski functional of B be a complete norm whose topology coincides with that of E. For an arbitrary such pair, we may imitate the definition of the dual of a Banach space, and define $(E, B)^*$ by providing the vector-space of continuous linear functionals on E with the "norm"

$$||\psi|| = \sup\{|\psi(b)|: b \in B\}.$$

Say that (E, B) is a Λ-*space* (where Λ denotes the real or complex scalar field) if $(E, B)^*$ is a Banach space. Our main result is obtained with the help of the adjoint functor theorem (stated below) of category theory.

MAIN THEOREM. *Let X be an arbitrary topological space. Then there exists a Λ-space (E, B), with E topologically isomorphic to a product of copies of Λ, such that the* sup-*normed Banach space $C_0(X)$ of bounded continuous Λ-valued functions is linearly isometric to $(E, B)^*$.*

In developing the proof we point out how an adjoint functor arises naturally to surmount the original obstruction, and how the concept of "Λ-space" is itself suggested by the adjoint.

We are grateful to S. Swaminathan for making us aware of [1] and to the referee for helpful criticism.

In raising the question "is $C(X)$ a dual space?" two fundamental constructions come into play:

1. If F, F' are Banach spaces, the vector space $\mathscr{L}(F, F')$ of continuous linear maps $F \to F'$ is a Banach space in the norm

$$||\psi|| = \sup\{||\psi(x)||: ||x|| \leqq 1\}.$$

2. If X is a compact Hausdorff space and F is a Banach space, the vector space $C(X, F)$ of continuous maps $X \to F$ is a Banach space in the norm

$$||f|| = \sup\{||f(x)||: x \in X\}.$$

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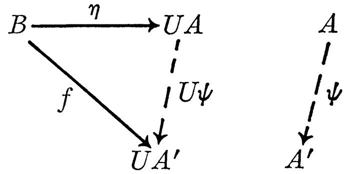
The original question "is X represented by an F such that $C(X, \Lambda) \cong \mathscr{L}(F, \Lambda)$" and the similarity of the norm formulas, beg comparison with a central definition of category theory:

Definition 1. Let $\mathscr{A}$, $\mathscr{B}$ be categories, let $U: \mathscr{A} \rightarrow \mathscr{B}$ be a functor and let B be an object in $\mathscr{B}$. A *free* $\mathscr{A}$*-object over* B *with respect to* U is a pair (A, η) with A an object in $\mathscr{A}$ and $\eta: B \rightarrow UA$ a morphism in $\mathscr{B}$ possessing the universal property that



for all similar pairs (A', f), there exists unique $\mathscr{A}$-morphism $\psi: A \rightarrow A'$ with $U\psi \cdot \eta = f$. For intuition, think of $\mathscr{A}$ as a category of "$\mathscr{B}$-objects with additional structure", U as the "underlying $\mathscr{B}$-object" functor, B as "an object of free generators", η as "inclusion of the generators", and the universal property as "unique extension by an $\mathscr{A}$-morphism of an arbitrary $\mathscr{B}$-morphism on the generators". *U has a left adjoint* if there exists a free (A, η) over B for every $\mathscr{B}$-object B.

Suppose, in particular, that Ban denotes the category of Banach spaces and norm-decreasing linear maps, that Top is the category of topological spaces and continuous maps and that U: Ban $\rightarrow$ Top is the unit disc functor. Consider a compact Hausdorff space X over which there exists free (F, η) with respect to U. Then "composing with η" is a linear map

$$-\cdot\eta: \mathscr{L}(F, F') \rightarrow C(X, F')$$

which (by the universal property) establishes a bijection of the unit balls, and is hence a linear isometry. In particular, $C(X) \cong F^*$.

Unhappily, the existence of free (F, η) over X does not characterize the hyperstonian spaces. Indeed, if (F, η) exists, the continuous map

$$X \xrightarrow{\eta} F \longrightarrow F^{**} \xrightarrow{(-\cdot\eta)^{-1*}} C(X)^*$$

is routinely checked to be the evaluation map sending $x \in X$ to its evaluation functional $f \mapsto f(x)$. Since this mapping is also injective (X is completely regular), X is metrizable. But not all hyperstonian spaces are metrizable; for example the β-compactification of an infinite discrete space is hyperstonian, but not metrizable.

Our immediate goal is to supplant the unit disc functor U: Ban $\rightarrow$ Top with another top-valued functor with respect to which free objects always exist. We pause, then, to consider some basic definitions and theorems which deal with this problem.

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Definition 2. Recall that $\mathscr{A}$ is *complete* [**3**, p. 44, 2.9, p. 47, 17.3, p. 27] if every set-indexed family $(A_\alpha: \alpha \in I)$ has a *product* $P_\beta: \prod A_\alpha \to A_\beta$ (not excluding the case $I = \emptyset$ wherein $\prod A_\alpha$ is a *terminal object* [**3**, p. 24, p. 14]) and if every pair $f, g: A_1 \rightrightarrows A_2$ of $\mathscr{A}$-morphisms has an *equalizer* $i: A \to A_1$ [**3**, p. 8].

Ban is complete. $\prod F_\alpha$ is the Banach space of all tuples (x_α) with $\sup\{||x_\alpha||: \alpha \in I\} < \infty$ with this supremum as the norm; $P_\beta(x_\alpha) = x_\beta$. The equalizer of f, g is the isometric inclusion of the closed subspace $\mathrm{Ker}(f - g)$ on which f and g agree.

Top is complete. $\prod X_\alpha$ is the usual Tychonoff product, and the equalizer of f, g is the subset on which f and g agree with the subspace topology.

The category Tls of topological linear spaces and continuous linear maps is complete. $\prod X_\alpha$ is the usual cartesian product vectorspace with the Tychonoff topology, and the equalizer of f, g is the linear subspace on which f, g agree provided with the subspace topology.

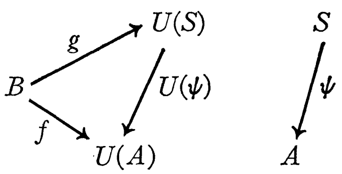
Let $\mathscr{A}$ be complete. A *complete subcategory* of $\mathscr{A}$ is a full subcategory $\mathscr{B}$ of $\mathscr{A}$ which is *closed under products* (B_α in $\mathscr{B}$ implies $\prod B_\alpha$, as computed in $\mathscr{A}$, is in $\mathscr{B}$) and *closed under equalizers* ($f, g: B_1 \rightrightarrows B_2$) in $\mathscr{B}$ and $i: A \to B_1$ an equalizer in $\mathscr{A}$ of f, g implies A is in $\mathscr{B}$). A complete subcategory is complete qua category.

THEOREM (Freyd adjoint functor theorem). *Let $\mathscr{A}$ be a complete category and let $U: \mathscr{A} \to \mathscr{B}$ be a functor. Then U has a left adjoint if and only if the following conditions hold:*

Ad 1. *Whenever* $\{A, P_\alpha: A \to A_\alpha\} = \prod A_\alpha$ *in* $\mathscr{A}$, $\{UA, U(P_\alpha): U(A) \to U(A_\alpha)\} = \prod U(A_\alpha)$ *in* $\mathscr{B}$.

Ad 2. *Whenever* $i: A \to A_1$ *is the equalizer of* $f, g: A_1 \rightrightarrows A_2$ *in* $\mathscr{A}$, $U(i)$ *is the equalizer of* $U(f), U(g)$ *in* $\mathscr{B}$.

Ad 3. *For each* B *in* $\mathscr{B}$ *there exists a set* $\mathscr{S}$ *of objects in* $\mathscr{A}$ *such that whenever*



A *is an* $\mathscr{A}$*-object and* $f: B \to U(A)$ *is a* $\mathscr{B}$*-morphism, there exist* $S \in \mathscr{S}$, $g: B \to U(S)$ *in* $\mathscr{B}$, *and* $\psi: S \to A$ *in* $\mathscr{A}$ *with* $U(\psi) \cdot g = f$.

For a proof see [**3**, 3.1, p. 124]. In Ad 3, we emphasize that $\mathscr{S}$ is a *set* as opposed to a proper class; more precisely, it must be legitimate to form $\prod \{S: S \in \mathscr{S}\}$ in $\mathscr{A}$. The class $\mathscr{S} = \{A:$ there exists $f: B \to U(A)\}$ has all desired properties except for "smallness".

It is possible to show that the unit disc functor Ban $\to$ top satisfies Ad 2 and Ad 3. However, Ad 1 fails for infinite products.

THEOREM 1. *Let X be a topological space. Then there exists a compact Hausdorff space βX such that $C_0(X)$ and $C(\beta X)$ are linearly isometric Banach spaces.*

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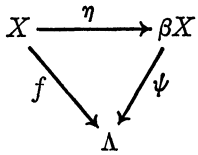
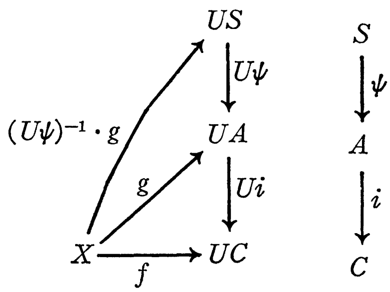
Proof. The argument is well-known when X is completely regular separated and βX is the β-compactification. Such βX is characterized by being free over X with respect to the inclusion functor from the category CT2 of compact Hausdorff spaces into completely regular separated spaces. The



universal property establishes a linear isomorphism $C_0(X) \to C(\beta X)$ because bounded subsets of Λ have compact closure. That the sup-norms are the same requires only that $\eta(X)$ be dense in βX and this is deducible purely from the universal property (the quickest proof following from the fact that βX is completely regular; also, c.f. the proof of Theorem 4(1) below). We have only to show that any topological space has a β-compactification, that is, that the inclusion functor U: CT2 $\to$ Top has a left adjoint. Ad 1 and Ad 2 are clear since CT2 is a complete subcategory. To prove Ad 3, let $X \in$ Top, set α to be the cardinal of the set of ultrafilters on the set X, and define $\mathscr{S}$ to be the set of all $S \in$ CT2 whose underlying set is a cardinal $\leqq \alpha$. Given $C \in$ CT2 and f: $X \to UC$ let A be the closure of $f(X)$ in C with inclusion



map i: $A \to C \in$ CT2. Then f factors through Ui by a unique continuous map g. For each element $x \in A$ there exists an ultrafilter $\mathscr{U}$ on $f(X)$ converging to x. As A is Hausdorff, the cardinal of A is dominated by α and there exists a homeomorphism ψ: $S \to A$ with $S \in \mathscr{S}$. That Ad 3 is satisfied is now clear, and the proof is complete.

Fix an arbitrary class, $\mathscr{F}$, of Banach spaces. Define $\mathscr{E}$ to be the full subcategory of Tls consisting of all $E \in$ TlS which are topologically isomorphic to a closed subspace of a product (in TlS) of elements of $\mathscr{F}$ (considered as topological linear spaces).

Thus, if $\mathscr{F}$ is all Banach spaces, $\mathscr{E}$ is the category of complete, separated locally convex spaces; if $\mathscr{F} = \{\Lambda\}$, $\mathscr{E}$ is the class of all E which are topologically isomorphic to a product (in TlS) of copies of Λ [**4**, p. 191, exercise 6].

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Let U: $\mathscr{E} \to$ Top be the underlying topological space functor.

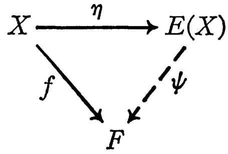
THEOREM 2. *U has a left adjoint.*

Proof. The terminal object, 0, of Tls belongs to $\mathscr{E}$. If i_α: $E_\alpha \to \prod_\beta F_{\alpha,\beta}$ is a closed embedding then

$$\prod i_\alpha : \prod E_\alpha \to \prod_{\alpha,\beta} F_{\alpha,\beta}$$

is again a closed embedding. Therefore $\mathscr{E}$ is closed under products. If $E \in \mathscr{E}$ and E' is a closed subspace of E then $E' \in \mathscr{E}$. In particular, $\mathscr{E}$ is closed under equalizers (since all spaces in $\mathscr{E}$ are Hausdorff). Ad 1 and Ad 2 are now clear. The proof of Ad 3 is entirely analogous to that in Theorem 1; define α to be the cardinal of the set of ultrafilters on the free linear span of the set X and consider the closure of the linear span of $f(X)$. The proof is complete.

For each topological space X let $(E(X), \eta)$ denote the free $\mathscr{E}$-object over X with respect to U. The universal property establishes a linear isomorphism



$-\cdot\eta$: $\mathscr{L}(E(X), F) \to C(X, F)$, for each $F \in \mathscr{F}$. When X is compact, the sup-norm on $C(X, F)$ transports to make $\mathscr{L}(E(X), F)$ into a Banach space $\mathscr{L}[E(X), F]$ in the norm

$$||\psi|| = \sup\{||\psi(b)||: b \in B\}$$

where $B = \eta(X)$. This motivates the definition of an $\mathscr{F}$*-space* as a pair (E, B) where $E \in$ Tls, $B \subset E$ are such that $\mathscr{L}(E, F)$ is a Banach space $\mathscr{L}[(E, B), F]$ in the norm

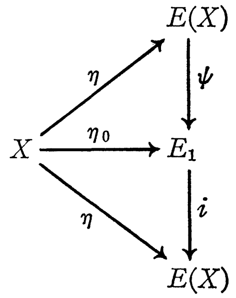
$$||\psi|| = \sup\{||\psi(b)||: b \in B\}$$

for all $F \in \mathscr{F}$. A Λ-*space* is a $\{\Lambda\}$-space. In view of Theorem 1 and the remarks preceding Theorem 2 we have proved the main theorem (stated at the beginning of the paper). We have also proved

THEOREM 3. *Let $\mathscr{F}$ be a class of Banach spaces and let X be a compact (not necessarily separated) topological space. Then there exists an $\mathscr{F}$-space (E, B) with E topologically isomorphic to a closed subspace of a topological linear space product of elements of $\mathscr{F}$, and with B compact such that the Banach spaces $C(X, F)$ and $\mathscr{L}[(E, B), F]$ are canonically linearly isometric for all $F \in \mathscr{F}$.*

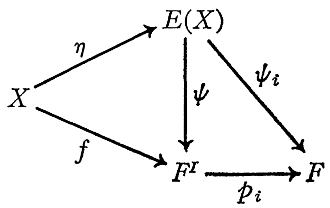
THEOREM 4. (1) *$E(X)$ is the closed linear span of $\eta(X)$.* (2) *X is completely regular separated if and only if η: $X \to E(X)$ is a homeomorphism into, providing some $F \in \mathscr{F}$ is non-zero.*

Proof. (1) While a Hahn-Banach argument works, there is a more basic reason. Let E_1 be the closed linear



span of $\eta(X)$. As $\mathscr{E}$ is closed hereditary, $E_1 \in \mathscr{E}$. Let i: $E_1 \to E(X)$ be the inclusion map. Then η factors through i by continuous η_0. By the universal property there exists ψ with $\psi\eta = \eta_0$. Since $i\psi \in \mathscr{E}$ and leaves η invariant, it follows that $i\psi = \text{id}$ and i is onto as desired.

(2) One way is clear. Conversely, let X be completely regular separated. There exists non-zero F in $\mathscr{F}$. Since the unit interval can be homeomorphicly embedded in F there exists a homeomorphism f of X into F^I for I a sufficiently large set. By the universal property, for each $i \in I$



there exists

$$E(X) \xrightarrow{\psi_i} F$$

in $\mathscr{E}$ with $\psi_i\eta = p_if$. There exists unique continuous (and linear) ψ with $p_i\psi = \psi_i$ for all i. $\psi\eta = f$ since the maps agree followed by each product projection. But whenever a composition of two continuous maps is a homeomorphism into, so is the first. The proof is complete.

The following theorem is roughly similar to some results of Edelstein [**2**].

THEOREM 5. *Let X be a completely regular separated space. Then there exists a set I and a homeomorphism η of X into the real topological linear space $\mathbf{R}^I$ with the following properties:*

(1) *Every continuous endomorphism f: $X \to X$ extends uniquely to a continuous linear endomorphism $\tilde{f}$: $\mathbf{R}^I \to \mathbf{R}^I$ such that $\tilde{f}\eta = \eta f$.*

(2) *Given two continuous endomorphisms f, g: $X \rightrightarrows X$, $(gf)^\sim = \tilde{g}\tilde{f}$. Thus every semigroup of mappings lifts to an isomorphic semigroup.*

(3) *If f: $X \to X$ is a homeomorphism onto, $\tilde{f}$ is a topological isomorphism onto.*

Proof. (2) and (3) are formal consequences of (1). To prove (1), set $\Lambda = \mathbf{R}$, let $\mathscr{E}$ correspond to $\mathscr{F} = \{\mathbf{R}\}$ and apply Theorem 2. $(E(X), \eta) = (\mathbf{R}^I, \eta)$ is the desired construction. The proof is complete.

References

1. W. G. Bade et al., *The space of all continuous functions on a compact Hausdorff space*, Notes for Mathematics 2906, Section 8 (University of California at Berkeley, 1957).
2. M. Edelstein, *On the representation of mappings of compact metrizable spaces as restrictions of linear transformations*, Can. J. Math. *22* (1970), 372–375.
3. B. Mitchell, *Theory of categories* (Academic Press, New York, 1965).
4. H. H. Schaeffer, *Topological vector spaces* (Macmillan, New York, 1966).

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