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CURVES IN HOMOGENEOUS SPACES

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1. Introduction. Let $\bar{G}$ be a Lie group with connected Lie subgroup $\bar{H}$, and let $M(t)$, $N(t)$ be real analytic curves in $\mathscr{G}$, the Lie algebra of $\bar{G}$, with $M(0) = N(0) = 0 \in \mathscr{G}$. The main result in this paper is a Lie algebraic condition which is necessary and sufficient for

$$\exp M(t) \cdot \bar{H} = \exp N(t) \cdot \bar{H} \quad \text{for all } t \in \bar{R}.$$

If $\bar{H}$ is closed in $\bar{G}$, then this implies that curves $\exp M(t) \cdot \bar{H}$ and $\exp N(t) \cdot \bar{H}$ in the homogeneous space $\bar{G}/\bar{H}$ are identical.

This problem has been studied for $M(t) = tX$ and $N(t) = tY$, where $X, Y \in \mathscr{G}$. In [**1**] Goto shows that $\exp tX \cdot \bar{H} = \exp tY \cdot \bar{H}$ for all $t \in \bar{R}$ if and only if $X - Y \in \mathscr{H}$, the Lie algebra of $\bar{H}$, and $\mathrm{ad}_X{}^k Y \in \mathscr{H}$ for $k = 1, 2, \ldots,$ where $\mathrm{ad}_X Y = [X, Y]$. In this case the algebraic criterion involves Lie brackets of $d/dt\ M(t)|_{t=0}$ and $d/dt\ N(t)|_{t=0}$. Our main result, Theorem 1, considers the case where $M(t)$, $N(t)$ are arbitrary real analytic functions, and the algebraic criterion involves Lie brackets of higher order derivatives of $M(t)$ and $N(t)$.

Goto also proves that if $X, Y \in \mathscr{G}$ then $\exp \bar{R}X \cdot \bar{H} = \exp \bar{R}Y \cdot \bar{H}$ if and only if $\exp X \cdot \bar{H} = \exp \alpha t Y \cdot \bar{H}$ for some nonzero constant α. If $\bar{H}$ is closed in $\bar{G}$ this results in necessary and sufficient conditions for the orbits of $\bar{H}$ under the one-parameter groups $\exp \bar{R}X$ and $\exp \bar{R}\ Y$ to agree in $\bar{G}/.\bar{H}$ This result is generalized in Theorem 3 to include the orbits of connected subgroups of $\bar{G}$ with one-dimensional orbits in $\bar{G}/\bar{H}$.

This paper is organized as follows: in Section 2 we introduce notations and present some basic results from Lie theory which are used in later sections. In Section 3 we prove the main results, Theorem 1 and Theorem 3, and present an example.

2. Notation and preliminary results. Let G be a Lie group with Lie algebra of right-invariant vector fields, $\mathscr{G}$, and let $\bar{H}$ be a connected Lie subgroup of $\bar{G}$ with corresponding Lie algebra $\mathscr{H}$. Let e denote the identity element in $\bar{G}$. If $P(t)$ is a smooth curve in $\mathscr{G}$ we set $P^k(t) = d^k/dt^k\, P(t)$. We identify the tangent space to $\mathscr{G}$ at a point with $\mathscr{G}$ and consider $P^k(t) \in \mathscr{G}$.

We denote by $\bar{R}$ the additive group of real numbers. Let $X, Y \in \mathscr{G}$. We define $\mathrm{ad}_X{}^n Y$ inductively as follows: $\mathrm{ad}_X{}^0 Y = Y$, $\mathrm{ad}_X{}^k Y = [X, \mathrm{ad}_X{}^{k-1} Y]$.

Let $x = \exp X$. The mapping $A_x : g \to xgx^{-1}$ of $\bar{G} \to \bar{G}$ has differential

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$dA_x = \mathrm{Ad}(x) : \mathscr{G} \to \mathscr{G}$. The *Campbell-Baker-Hausdorff formula* for right-invariant vector fields asserts that

$$\mathrm{Ad}(x)(Y) = Y - \mathrm{ad}_X Y + \frac{1}{2!}\mathrm{ad}_X^2 Y - \frac{1}{3!}\mathrm{ad}_X^3 Y + \dots$$

(c.f. p. 118 of [**2**]).

Suppose $x, y \in \bar{G}$. The mapping $R_x : y \to yx$ from $G \to G$ has differential dR_x and for each $X \in \mathscr{G}$, $dR_x X(y) = X(yx)$ as X is a right-invariant vector field on $\bar{G}$.

We will use the fact that the derivative of the exp mapping is described by the formula (c.f. [**2**]):

$$d\exp_X Y(e) = (dR_{\exp X})_e \circ \frac{1 - e^{\mathrm{ad}_X}}{-\mathrm{ad}_X} Y(e)$$

$$= Y(\exp X) + \frac{1}{2!}\mathrm{ad}_X Y(\exp X) + \frac{1}{3!}\mathrm{ad}_X^2 Y(\exp X) \dots$$

A Lie subalgebra $\mathscr{E}$ of $\mathscr{G}$ is said to satisfy the *chain condition* if for each nonzero ideal $\mathscr{I}$ in $\mathscr{E}$ there exists and ideal $\mathscr{I}'$ of $\mathscr{I}$ of codimension 1. One can define a *solvable* Lie algebra as one which satisfies the chain condition (c.f. [**2**]). If follows that $\mathscr{E}$ is solvable if and only if there exists a sequence

(*) $\mathscr{E} = \mathscr{E}_n \supset \mathscr{E}_{n-1} \supset \dots \supset \mathscr{E}_1 \supset \mathscr{E}_0 = \{0\}$

where $\mathscr{E}_k$ is an ideal in $\mathscr{E}_{k+1}$ of codimension 1 for $0 \leqq k \leqq n - 1$. We call (*) the *descending chain* for $\mathscr{E}$.

3. Curves and orbits of subgroups in homogeneous spaces. Let $\bar{G}$ be a Lie group with Lie algebra $\mathscr{G}$ and $\bar{H}$ a connected Lie subgroup with Lie algebra $\mathscr{H}$. The following theorem generalizes Proposition 6 of [**1**]. The techniques used differ from those of [**1**] and if $M(t)$ is replaced by tX and $N(t)$ by tY in the proof of Theorem 1, a quick proof is obtained for Proposition 6 of [**1**].

THEOREM 1. *Let $M(t)$ and $N(t)$ be real analytic curves in $\mathscr{G}$ which pass through the zero vector field when $t = 0$. Consider the curve*

$$Q(t) = \sum_{l,k=0}^{\infty} \frac{(-1)^l}{(k+1)!l!}\mathrm{ad}^l_{N(t)}\mathrm{ad}^k_{M(t)} M^1(t) + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!}\mathrm{ad}^k_{N(t)} N^1(t)$$

in $\mathscr{G}$. Then $\exp M(t) \cdot \bar{H} = \exp N(t) \cdot \bar{H}$ *for all $t \in \bar{R}$ if and only if $Q^k(0) \in \mathscr{H}$ for $k = 0, 1, 2, \dots$.*

Proof. We begin by noting that $\exp M(t) \cdot \bar{H} = \exp N(t) \cdot \bar{H}$ for all $t \in \bar{R}$ if and only if $\exp(-N(t)) \exp M(t) \in \bar{H}$ for all $t \in R$. Set $C(t) = \exp(-N(t)) \exp M(t)$. If we identify the tangent space of $\bar{G}$ at $C(t)$ with $\mathscr{G}$, then $C(t) \in \bar{H}$ for all $t \in \bar{R}$ if and only if $C^1(t) \in \mathscr{H}$ for all $t \in \bar{R}$. Since $C^1(t)$ is a real analytic curve in the vector space $\mathscr{G}$, it suffices to show that $C^1(t) \in \mathscr{H}$ for t in some open neighborhood of 0 in $\bar{R}$. If we can show that $C^1(t) = Q(t)$ defined above then we are done, as the real analytic function $Q(t) \in \mathscr{H}$ for t in some neighborhood of 0 if and only if the Taylor coefficients $d^kQ/dt^k(t)|_{t=0} \in \mathscr{H}$ for $k = 0, 1, 2, \ldots$.

The derivative of the curve $C(t)$ can be expressed as

$$C^1(t) = \frac{\partial}{\partial t} \exp(-N(t)) \exp M(s)\Big|_{s=t} + \frac{\partial}{\partial s} \exp(-N(t)) \exp M(s)\Big|_{s=t}.$$

Set $a(t, s) = \exp(-N(t)) \exp M(s)$. Then

$$\frac{\partial}{\partial t} \exp(-N(t)) \exp M(s) = \frac{\partial}{\partial t} R_{\exp M(s)} \exp(-N(t))$$

$$= (dR_{\exp M(s)})_{a(t,s)} \circ (d \exp)_{-N(t)}(-N^1(t))(e)$$

$$= (dR_{\exp M(s)})_{a(t,s)} \circ (dR_{\exp -N(t)})_e \circ \frac{1 - e^{ad_{-N(t)}}}{-\mathrm{ad}_{-N(t)}}(-N^1(t))(e)$$

$$= + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \mathrm{ad}^k{}_{N(t)} N^1(t)(a(t, s)).$$

To evaluate the second term in the expression for $C^1(t)$ we proceed as follows: first we note that

$$\exp(-N(t)) \exp M(s) = \exp(-N(t)) \exp M(s) \exp N(t) \exp -N(t)$$
$$= R_{\exp -N(t)} \circ A_{\exp -N(t)} (\exp M(s))$$

where $A_x(y) = xyx^{-1}$ for $x, y \in \bar{G}$. Then

$$\partial/\partial s \exp(-N(t)) \exp M(s) = (dR_{\exp -N(t)})_{p_2} \circ (\mathrm{Ad}\ (\exp -N(t))_{p_1}$$
$$\circ (d \exp)_{M(s)} \circ M^1(s)(e)$$

where p_1, p_2 are the appropriate points. Applying the formulas from Section 2 for Ad and d exp, we obtain the expression

$$\frac{\partial}{\partial s} \exp(-N(t)) \exp M(s) = \sum_{k,l=0}^{\infty} \frac{(-1)^l}{l!(k+1)!} \mathrm{ad}^l{}_{N(t)} \mathrm{ad}^k{}_{M(s)} M^1(s)(a(t, s)).$$

If we set $s = t$ and identify the tangent space of $\bar{G}$ at $C(t)$ with $\mathscr{G}$ if follows that $C^1(t) = Q(t)$ and the proof is complete.

COROLLARY. *Let $X, Y \in \mathscr{G}$. Then* $\exp tX \cdot \bar{H} = \exp tY \cdot \bar{H}$ *if and only if* $Y - X \in \mathscr{H}$ *and* $\mathrm{ad}_X{}^k Y \in \mathscr{H}$ *for* $k = 1, 2, \ldots$.

Proof. Set $M(t) = tY$ and $N(t) = tX$. Then Theorem 1 applies. Here

$$Q(t) = \sum_{l=0}^{\infty} \frac{(-1)^l t^l}{l!} \mathrm{ad}_X{}^l Y - X,$$

$Q(0) = Y - X$, and $d^k/dt^k Q(t)|_{t=0} = (-1)^k \mathrm{ad}_X{}^k Y$. Theorem 1 asserts that $\exp tX \cdot \bar{H} = \exp tY \cdot \bar{H}$ if and only if $Y - X \in \mathscr{H}$ and $(-1)^k \mathrm{ad}_X{}^k Y \in \mathscr{H}$ for $k = 1, 2, \ldots$. This completes the proof.

Remark. The techniques employed in this proof can be used to obtain necessary and sufficient conditions in terms of Lie algebras for

$$\exp M_1(t) \exp M_2(t) \ldots \exp M_m(t)\bar{H} = \exp N_1(t) \ldots \exp N_n(t)\bar{H}$$

for all $t \in R$.

Suppose that $\bar{E}$ and $\bar{F}$ are connected Lie subgroups of $\bar{G}$, and $\bar{H}$ is a closed connected subgroup of $\bar{G}$. In the case where $\dim \bar{E} = \dim \bar{F} = 1$, Goto has found necessary and sufficient conditions, in terms of Lie algebras, for the equality $\bar{E} \cdot \bar{H} = \bar{F} \cdot \bar{H}$ (c.f. [**1**]). That is, the orbits of $e\bar{H}$ in the homogeneous space $\bar{G}/\bar{H}$ under $\bar{E}$ and $\bar{F}$ are identical. If we allow $\dim \bar{E}$ and $\dim \bar{F}$ to be arbitrary, clearly a necessary condition for the orbits to be the same is that the dimensions n and m of the submanifolds $\bar{E} \cdot \bar{H}$ and $\bar{F} \cdot \bar{H}$ of $\bar{G}/\bar{H}$ be the same. If $\mathscr{E}, \mathscr{F}, \mathscr{H}$ are the Lie algebras corresponding to $\bar{E}$, $\bar{F}$ and $\bar{H}$ respectively, then this condition becomes (c.f. [**2**]),

$$\dim \mathscr{E} - \dim \mathscr{E} \cap \mathscr{H} = \dim \mathscr{F} - \dim \mathscr{F} \cap \mathscr{H} = n.$$

Suppose that the dimension of the manifold $\bar{E} \cdot \bar{H}$ is one. That is, $\dim \mathscr{E} - \dim \mathscr{E} \cap \mathscr{H} = 1$. The following lemma shows that there exists an element X in $\mathscr{E}$ such that $(\exp \bar{R}X) \cdot \bar{H} = \bar{E} \cdot \bar{H}$.

LEMMA 2. *Let $\bar{E}$ and $\bar{H}$ be a connected Lie subgroups of the connected Lie group $\bar{G}$ with* $\dim \mathscr{E} - \dim \mathscr{E} \cap \mathscr{H} = 1$. *Then there exists a vector field X in $\mathscr{E}$ such that* $(\exp \bar{R}X) \cdot \bar{H} = \bar{E} \cdot \bar{H}$.

Proof. Let $\mathscr{S} = \mathscr{E} \cap \mathscr{H}$. Then $\mathscr{S}$ is a Lie subalgebra of $\mathscr{E}$ of codimension 1. It suffices to show that there exists X in $\mathscr{E}$ such that $\bar{E} = (\exp \bar{R}X) \cdot \bar{S}$, where $\bar{S}$ is the connected Lie subgroup of $\bar{E}$ with Lie algebra $\mathscr{S}$. Since $\bar{S} \subset \bar{H}$ this implies that $\bar{E} \cdot \bar{H} = (\exp \bar{R}X) \cdot \bar{S} \cdot \bar{H} = (\exp \bar{R}X) \cdot \bar{H}$.

Using Levi's Theorem (c.f. [**7**]) we can write $\mathscr{E}$ as the vector space direct sum $\mathscr{E} = \mathscr{R} \oplus \mathscr{A}$ where $\mathscr{R}$ is the radical of $\mathscr{E}$ and $\mathscr{A}$ a semi-simple Lie subalgebra of $\mathscr{E}$. Either $\mathscr{R} \subset \mathscr{S}$ or $\mathscr{R} \not\subset \mathscr{S}$. We will consider these two cases separately.

Suppose that $\mathscr{R} \subset \mathscr{S}$. Let R and A denote the connected Lie subgroups of E with Lie algebras $\mathscr{R}$ and $\mathscr{A}$ respectively. Since $\mathscr{R}$ is an ideal in $\mathscr{E}$ we know that R is a normal subgroup of E so that $\bar{E} = \bar{A} \cdot \bar{R}$. Thus $\bar{E} = \bar{A} \cdot \bar{S}$, since $R \subset S$. We can further simplify things by decomposing $\mathscr{A}$ as the Lie algebra direct sum of simple Lie algebras. Thus $\mathscr{A} = \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \ldots \oplus \mathscr{A}_k$ where $\mathscr{A}_i$ is simple for $i = 1, 2, \ldots, k$ (c.f. [**7**] or [**2**]), and $\bar{A} = \bar{A}_1 \otimes \ldots \otimes \bar{A}_k$ is the corresponding decomposition for $\bar{A}$. At least one of the $\bar{A}_i$'s is not in $\bar{S}$ and by relabelling we can have $\bar{A}_1 \not\subset \bar{S}$.

Claim: $\bar{E} = \bar{A}_1 \cdot \bar{S}$: Since $\mathscr{S}$ has codimension 1 in $\mathscr{E}$ we can choose a basis $\{X_1, X_2, \ldots, X_n\}$ for $\mathscr{A}$ with $X_1 \in \mathscr{A}_1$ and $X_2, \ldots, X_n \in \mathscr{S} \cap \mathscr{A}$. For $i = 2, \ldots, n$ we have $X_i = Y_1 + Y_2 + \ldots + Y_k$ where $Y_j \in \mathscr{A}_j$ for $j = 1, 2, \ldots, k$. For $a, b \in R$ we have $\exp aX_i \exp bX_1 = (\exp AX_i \exp bX_1$

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exp $-aX_i$) exp aX_i and using the Campbell-Baker-Hausdorff formula and the above decomposition of X_i the expression in brackets is contained in exp $\mathscr{A}_1$. Thus exp aX_i exp $bX_1 \in \bar{A}_1 \cdot \bar{S}$ and since $\bar{A}$ is the group generated by {exp $\bar{R}X_1$, exp $\bar{R}X_2, \ldots,$ exp $\bar{R}X_n$} we have shown that $\bar{A} \subset \bar{A}_1 \cdot \bar{S}$ and hence $\bar{E} = \bar{A} \cdot \bar{S} = A_1 \cdot \bar{S}$.

We now observe that $\mathscr{A}_1 \cap \mathscr{S}$ is a codimension one Lie subalgebra of the simple real Lie algebra $\mathscr{A}_1$. Let $\mathscr{B} = \mathscr{A}_1 \cap \mathscr{S}$. Then $\mathscr{A}_1 \supset \mathscr{B}$ with codimension = 1. Theorem 1 of [**6**] asserts that one and only one of the following occurs: (i) $\mathscr{B}$ is an ideal; (ii) $\mathscr{B}$ contains an ideal $\mathscr{I}$ of $\mathscr{A}_1$ such that $\mathscr{A}_1/\mathscr{I}$ is isomorphic to the real two dimensional nonabelian Lie algebra; (iii) $\mathscr{B}$ contains an ideal $\mathscr{I}$ of $\mathscr{A}$ such that $\mathscr{A}_1/\mathscr{I}$ is isomorphic to sl(2, R). Since $\mathscr{A}_1$ is a simple Lie algebra, case (i) is eliminated. In case (ii) we must have $\mathscr{I} = \{0\}$, hence $\mathscr{A}_1/\mathscr{I} \approx \mathscr{A}_1$ is the solvable 2 dimensional Lie algebra, a contradiction. We are left with case (iii) with $\mathscr{I} = \{0\}$ and $\mathscr{A}_1$ isomorphic to sl(2, R). It is easy to show that $\mathscr{A}_1$ has a basis $\{L, M, N\}$ with $M, N \in \mathscr{B}$, and $[M, N] = N$, $[L, M] = L$, $[N, L] = M$. By direct computation in sl(2, R) one can verify that $\bar{A}_1 =$ (exp $\bar{R}X$) $\cdot$ (exp $\bar{R}M$) $\cdot$ (exp $\bar{R}N$) where $X = L + M - 2N$. Since exp $\bar{R}M \subset \bar{S}$ and exp $\bar{R}N \subset \bar{S}$ we have shown that

$$\bar{E} = \bar{A}_1\bar{S} = (\exp \bar{R}X) \cdot \bar{S}.$$

This completes the proof in the case where $\mathscr{R} \subset \mathscr{S}$.

Suppose that $\mathscr{R} \not\subset \mathscr{S}$. Choose a basis $\{Y_0, Y_1, \ldots, Y_n\}$ for $\mathscr{E}$ where $Y_0 \in \mathscr{R}$ and $Y_1, \ldots, Y_n \in \mathscr{S}$. Since $\mathscr{R}$ is an ideal in $\mathscr{S}$ it follows that for each $a, b \in \bar{R}$

$$\exp aY_i \exp bY_0 \in \bar{R} \cdot (\exp aY_i).$$

Thus $\bar{E} = \bar{R} \cdot \bar{S}$. Let

$$\mathscr{R} = \mathscr{B}_n \supset \mathscr{B}_{n-1} \supset \ldots \supset \mathscr{B}_1 \supset \mathscr{B}_0 = \{0\}$$

be the descending chain for the solvable Lie algebra $\mathscr{R}$. Let k be the smallest positive integer for which $\mathscr{B}_k \not\subset \mathscr{S}$. Then there exists a basis $\{V_1, \ldots, V_n\}$ for $\mathscr{R}$ with $\{V_1, \ldots, V_i\}$ a basis for $\mathscr{B}_i$ for $i = 1, \ldots, n$, $V_k \notin \mathscr{S}$, and $V_l \in \mathscr{S}$ for $l \neq k$. Let B_i denote the connected Lie subgroup of $\bar{R}$ with Lie algebra $\mathscr{B}_i$. Then

$$\bar{B}_i = \exp \bar{R}V_1 \cdot \exp \bar{R}V_2 \ldots \exp \bar{R}V_i.$$

for $i = 1, \ldots, n$. In particular $\bar{R} = \bar{B}_n = \bar{B}_{k-1} \cdot \exp \bar{R}V_k \cdot \exp \bar{R}V_{k+1} \ldots \exp \bar{R}V_n$. Since $\bar{B}_{k-1}$ is a normal subgroup of $\bar{B}_k = \bar{B}_{k-1} \cdot \exp \bar{R}V_k$,

$$\bar{B}_{k-1} \cdot \exp \bar{R}V_k = \exp \bar{R}V_k \cdot \bar{B}_{k-1}$$

and $\bar{R} = \exp \bar{R}V_k \cdot \bar{B}_{k-1} \cdot \exp \bar{R}V_{k+1} \ldots \exp \bar{R}V_n$. By construction $\bar{B}_{k-1}$, exp $\bar{R}V_{k+1}, \ldots,$ exp $\bar{R}V_n \subset S$, hence $\bar{E} = \bar{R}\bar{S} = (\exp \bar{R}V_k) \cdot \bar{S}$. Setting $X = V_k$ completes the proof.

This result motivates the following definition:

Definition. An element X in $\mathscr{E}$ is called an *$\mathscr{H}$-free vector* if

$$\bar{E} \cdot \bar{H} = (\exp \bar{R}X) \cdot \bar{H}$$

Lemma 2 asserts that if $\dim \mathscr{E} - \dim \mathscr{E} \cap \mathscr{H} \leqq 1$ then there exists an $\mathscr{H}$-free vector in $\mathscr{E}$.

The following result is a necessary and sufficient condition for $\bar{E} \cdot \bar{H} = \bar{F} \cdot \bar{H}$, in terms of Lie algebras, where the orbits are submanifolds of $\bar{G}/\bar{H}$ of dimension zero or one.

THEOREM 3. *Suppose that $\bar{G}$ is a Lie group with Lie algebra $\mathscr{G}$, $\bar{E}$, $\bar{F}$ are connected Lie subgroups with Lie algebras $\mathscr{E}$ and $\mathscr{F}$ respectively. Let $\bar{H}$ be a closed connected Lie subgroup of $\bar{G}$ with Lie algebra $\mathscr{H}$ and* $\dim \mathscr{E} - \dim \mathscr{E} \cap \mathscr{H} = \dim \mathscr{F} - \dim \mathscr{F} \cap \mathscr{H} \leqq 1$. *Then there exists $\mathscr{H}$-free vectors $X \in \mathscr{E}$ and $Y \in \mathscr{F}$ and*

$$\bar{E} \cdot \bar{H} = \bar{F} \cdot \bar{H}$$

if and only if $X - \alpha Y \in \mathscr{H}$ for some real $\alpha \neq 0$ and $\mathrm{ad}_X{}^k Y \in \mathscr{H}$ *for* $k = 1, 2, \ldots$.

Proof. The existence of the $\mathscr{H}$-free vectors X and Y is guarenteed by Lemma 2. Thus $\bar{E} \cdot \bar{H} = (\exp \bar{R}X) \cdot \bar{H}$ and $\bar{F} \cdot \bar{H} = (\exp \bar{R}Y) \cdot \bar{H}$. In [**1**] Goto shows that $(\exp \bar{R}X) \cdot \bar{H} = (\exp \bar{R}Y) \cdot \bar{H}$ if and only if there exists a real $\alpha \neq 0$ such that $\exp tX \cdot \bar{H} = \exp t\alpha Y \cdot \bar{H}$ for all t in $\bar{R}$. It follows from the corollary to Theorem 1 that $\bar{E} \cdot \bar{H} = \bar{F} \cdot \bar{H}$ if and only if $X - \alpha Y \in \mathscr{H}$ and $\mathrm{ad}_X{}^k Y \in \mathscr{H}$ for $k = 1, 2, \ldots$. This completes the proof.

Example. Let $\bar{G}$ be a Lie group with Lie algebra $\mathscr{G}$ and let $\bar{H}$ be a connected Lie subgroup with Lie algebra $\mathscr{H}$. Suppose $X, Y, Z \in \mathscr{G}$ and $[Y, Z] = 0$. Then Theorem 1 supplies necessary and sufficient conditions for $\exp tX \cdot \bar{H} = \exp (tY + t^2 Z) \cdot \bar{H}$ for all $t \in R$. We can set $N(t) = tX$ and $M(t) = tY + t^2 Z$. Then

$$Q(t) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} t^l \mathrm{ad}_X{}^l (Y + 2 + tZ) - X$$

and

$$Q(0) = Y - X$$
$$Q^k(0) = (-1)^k \mathrm{ad}_X{}^k Y + (-1)^{k-1} 2k\, \mathrm{ad}_X{}^{k-1} Z \quad \text{for } k \geqq 1.$$

Thus $\exp tX \cdot \bar{H} = \exp (tY + t^2 Z) \cdot \bar{H}$ if and only if $X - Y \in \mathscr{H}$ and $\mathrm{ad}_X{}^k Y - 2k\, \mathrm{ad}_X{}^{k-1} Z \in \mathscr{H}$ for $k = 1, 2, \ldots$.

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