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THE DISTRIBUTION OF PRIME NUMBERS IN SHORT INTERVALS

VALERIIA STARICHKOVA

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In this thesis, we focus on the problem of primes in short intervals. We will explore the main ingredients in the works by Ingham [9], Heath-Brown and Iwaniec [6], and Baker and Harman [1], such as zero-density estimates, their weighted variation and sieve methods. The thesis structure is primarily based on the work [1] and, in particular, on the proof that there exists a prime number in the interval $[x - x^{0.54}, x]$ for all x large enough. Each chapter is connected with a tool from this paper and we expand on each topic.

We will say that the prime number theorem holds in $[x - y, x]$, $y = y(x)$, if the number of primes in such an interval is asymptotic to $y(x)/\log x$ as x tends to infinity. In Chapter 2, we cover the classic zero-density approach for deriving results on primes in short intervals due to Hoheisel [7] and Ingham [9]. We prove a generalised version of Ingham's theorem, which connects zero-density estimates and zero-free regions for the Riemann zeta function ζ with primes in short intervals; our version of the theorem explicitly shows the dependence of the interval length on the combination of the zero-free regions and the zero-density estimates. Namely, [9, Theorem 1] states that:

- zero-free regions for $\zeta(\sigma + iT)$ of the form $\sigma > 1 - \eta(T)$ with $\log\log T/\log T = o(\eta(T))$ and
- zero-density estimates of the form $N(\sigma, T) \ll T^{b(1-\sigma)} \log^B T$, which hold for constants $b, B > 0$ uniformly in $\frac{1}{2} \le \sigma \le 1$,

imply that the prime number theorem holds in $[x - y, x]$, $y = x^\theta$, if $1 - 1/b < \theta < 1$.

We slightly generalise Ingham's theorem by considering the intervals of length $y = x^\theta g(x), g(x) = x^{o(1)}$. This allows us to extend the range for θ down to $1 - 1/b$

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inclusive if the combination of the zero-density estimates and zero-free regions is *good enough*. For example, as a corollary, if $\zeta(\sigma + iT)$ is nonzero in a vertical strip $\sigma \le 1 - \eta$ for a positive number η and, as before, $N(\sigma, T) \ll T^{b(1-\sigma)} \log^B T$, then the prime number theorem holds in $[x - y, x], y = x^\theta g(x)$ with $g(x)$ being a finite power of $\log x$. Even if it is unlikely that we can reach such powerful zero-free regions in the near future, it seems interesting to explore the role of the zero-free regions in the classic zero-density approach, which is done in Chapter 2.

The bottleneck of the classic zero-density approach is improving upon the constant b from the upper bound for $N(\sigma, T)$, which is a very hard task. In 2024, Guth and Maynard [3] were the first ones in more than 50 years to improve upon Huxley's result [8]. They pushed Huxley's constant $b = 12/5$ down to $30/13$, and thus proved the prime number theorem in $[x - x^\theta, x]$ for $17/30 < \theta < 1$.

In Chapter 3, we introduce the weighted zero-density approach due to Heath-Brown and Iwaniec [6], which allows one to avoid the bottlenecks of the classic approach and provide a lower bound for primes in short intervals. We generalise and simplify some estimates by Heath-Brown and Iwaniec, and explain how to switch from their form of the weighted zero-density estimates to the slightly different form by Baker and Harman. In addition, we slightly improve upon the weighted zero-density estimates used by Baker and Harman.

In Chapter 4, we introduce the linear sieve, Harman's sieve and Harman's comparison principle, which connects the estimates for *rough* numbers in short and long intervals; rough numbers are integers without small prime factors. In addition, we prove the asymptotic for rough numbers in the sets where the prime number theorem holds (we describe this below).

Let us recall the prime number theorem with the sharpest error term:

$$\pi(x) = \mathrm{Li}(x) + O\biggl(x \exp\Bigl(-A\frac{\log^{3/5} x}{(\log\log x)^{1/5}}\Bigr)\biggr),$$

with $A = 0.2098$ (see [2]). It implies that the prime number theorem holds in the interval $\mathcal{B} = [x - Y, x], Y = Y(x)$ as long as $x \exp(-A \log^{3/5} x/(\log\log x)^{1/5}) = o(Y/\log x)$ and $Y \le x$. This can be generalised to the asymptotic for rough numbers in the same interval: let $S(\mathcal{B}, z)$ denote the number of integers in $\mathcal{B}$, prime factors of which are larger or equal to z. Then $S(\mathcal{B}, z)$ is asymptotic to $(uY/\log x)\omega(u)$, where $u = \log x/\log z$ and $\omega(u)$ is the Buchstab function (see [10, Theorem 7.11] and [4, A.2]).

The classic zero-density approach, as mentioned above, implies that the prime number theorem holds in even shorter intervals $[x - y, x]$ with $x^{17/30} \le y \le x$. Hence, we provide the asymptotic for rough numbers in such intervals by modifying the proof of [10, Theorem 7.11]. We note that our result is stated for $x^{7/12} \le y \le x$ since [3] was out of reach when the thesis was submitted, so we relied on the prime number theorem in short intervals proven by Heath-Brown [5].

Chapter 5 contains technical results connected to estimates for weighted sums of the form

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$$\sum_{\substack{x-y\leq mnr\leq x\\ m\sim M\\ n\sim N}} a_m b_n, \tag{1}$$

where m, n are integers and r is prime. The notations $m \sim M$ and $n \sim N$ mean $M \leq m \leq 2M$ and $N \leq n \leq 2N$, respectively; in other words, m and n are restricted to dyadic intervals. We can derive the estimates for the sums in (1) from the similar sums of the form $\sum a_m b_n \Lambda(r)$, where $\Lambda(r)$ is the von Mangoldt function. The latter sums can be estimated by the means of the weighted zero-density approach from Chapter 3. Next, we provide the estimates for the sums (1) with the prime parameter r restricted to a dyadic region. The results from Chapter 5 are crucial for understanding the argument from [1].

In Chapter 6, we explain the argument from [1] for $\theta = 0.54$ and formulate a general form of the computational problem which is resolved in a particular case by Baker and Harman.

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VALERIIA STARICHKOVA,
School of Physical, Environmental and Mathematical Sciences,
UNSW Canberra, Canberra ACT 2610, Australia
e-mail: v.starichkova@unsw.edu.au

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