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ON SAFETY AND RIVALS IN POWER-STRUCTURES

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It is shown that every minimal power-structure other than the singleton power-structure has at least two contenders; that is, any contender has at least one rival. The safe core of a power-structure is empty if and only if the power-structure is minimal.

'Nessun dorma'—Turandot.

The theory of coups d'état in power-structures, introduced by the author in [**1**], represents an organisation as a finite lattice $(X, \leqslant)$, and a *power-structure* within the organisation as a convex $\vee$-subsemilattice B of X. The *boss* of the power-structure is $\vee B$, the lattice supremum of the subset B. A *coup* occurs in B when a subset T of B containing $\vee B$ is removed from B so as to leave a subset $B \backslash T$ which again is a power-structure (the *surviving power-structure* of the coup), the cardinality of T being minimal for all possible choices of such subsets. (The removal of T from B is metaphorical only; the sets X, B and the lattice structure are not to be thought of as changing at some point of time.) The set T is called the *topple set* of the coup, and completely characterises the coup; its cardinality $|T|$ is called the *stability* of the power-structure B, written $t(B)$. The member $\vee(B \backslash T)$ is the boss of the surviving power-structure. A given power-structure may admit more than one coup. Any boss c of a surviving power-structure is called a *contender* of B; the coup is said to *promote* c. Any member d of B who is covered by the boss of B (that is, $d < \vee B$, and $d < x < \vee B$ for no $x \in B$, hence for no $x \in X$) is called a *deputy* of B.

Introduce the notation $(\leqslant a)$ for the set $\{x \in X : x \leqslant a\}$. It was shown in [**1**] that every contender c is necessarily a deputy; and that the topple set of the coup promoting c is unique, $= B \backslash (\leqslant c)$. Thus the stability of B is

$$(1) \qquad t(B) = |B \backslash (\leqslant c)| = \min\{|B \backslash (\leqslant d)| : d \text{ is a deputy of } B\}.$$

A *minimal power-structure* is a power-structure B which is minimal among all power-structures having the same boss and stability as B. That is, if $\mathcal{P}$ denotes the set of all power-structures of the organisation X, then B in $\mathcal{P}$ is called minimal if for all $C \in \mathcal{P}$, $C \subset B$ and $\vee C = \vee B$ imply $t(C) < t(B)$. ($\subset$ denotes strict inclusion.) In this note we prove the

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THEOREM OF THE RIVAL. *Every minimal power-structure other than a singleton power-structure has at least two contenders.*

This theorem was stated without proof in [**1**]. To quote the interpretation given there: 'A deputy d in an organisation, having in mind to promote himself to top man by means of a coup, must first asses the power-structures as he sees them, or perhaps assess what subsets of the organisation must be deemed to be power-structures if a coup successful to him is to be found. There may be advantages, springing say from matters of secrecy or from national security, in finding the minimal such power-structures allowing a successful coup in his favour. The quoted result shows that any minimal power-structure must give to at least one other deputy the status of a rival to d.' Any power-structure having d as its only deputy has of course d as its only contender, the only available successor to its current boss—presumably a desirable situation for d. There certainly exist power-structures with many deputies only one of whom is a contender; the theorem tells us that such power-structures are not minimal.

The proof uses the following result from [**1**] (Lemma 13, Theorem 14):

LEMMA. *Let* $B, C \in \mathcal{P}$ *and* $\vee B = \vee C$. *Then* B *covers* C *in* $(\mathcal{P}, \subseteq)$ *if and only if* $B = C \cup \{z\}$ *for some minimal member* z *of* B; *and in that case*

$$t(B) = \text{ either } t(C) + 1 \text{ or } t(C),$$

according as there does or does not exist a topple set of B *containing* z.

PROOF OF THE THEOREM: Let M be a minimal power-structure in an organisation X, with $|M| \geqslant 2$. If $|M| = 2$ then M has the form $\{a, d\}$ with $a > d$, and $K = \{a\}$ is a power-structure having the same boss a and stability 1 as M, contradicting the minimality of M. So in fact $|M| \geqslant 3$.

M must have at least one contender; suppose it has exactly one, c say.

Assume that M has no deputy other than c. Then M has the form

$$M = \{a, c\} \cup E, \qquad \text{where } a > c > e \text{ for all } e \in E;$$

the only possible topple set is $\{a\}$, and $t(M) = 1$. Since E is not empty, M has a minimal member other than c; let y be one such, and write

$$K = M \backslash \{y\}.$$

K must be a power-structure with the same boss a as M, since $y \neq a$ and K is order-convex and an increasing subset of M. Since $K \subset M$ and M covers K in $\mathcal{P}$, the lemma gives $t(M) = t(K)$. This contradicts the minimality of M. Therefore M

has at least one deputy distinct from c. Let z be a minimal member of M which is $\leqslant c$ (there exists at least one). Suppose $z = c$. Then M must be of the form

(2) $$M = \{a, c, d_1, d_2, \ldots, d_n\} \qquad \text{for some } n \geqslant 1,$$

where $a = \vee M$, and $c, d_1, d_2, \ldots, d_n$ are distinct deputies and therefore each minimal. For if some deputy, say d_1, were not minimal, then

$$|M \backslash (\leqslant d_1)| \leqslant |M| - 2 < |M \backslash (\leqslant c)|,$$

showing that the promotion of c would involve removal of a set $M \backslash \{c\}$ of greater cardinality than would the promotion of d_1, contradicting the assumption that c is a contender. But the form (2) implies that all the deputies $d_1, d_2, \ldots, d_n$ are contenders, contradicting the assumption that M has only the contender c.

Thus, $z < c$. Put

$$J = M \backslash \{z\}.$$

Let d be a deputy of M other than c. Since d is not a contender,

$$|M \backslash (\leqslant d)| > t(M).$$

We cannot have $z = d$. If $z < d$ then $M \backslash (\leqslant d) = J \backslash (\leqslant d)$, so

$$|J \backslash (\leqslant d)| > t(M);$$

if $z \not< d$ then

$$|J \backslash (\leqslant d)| = |M \backslash (\leqslant d)| - 1 \geqslant t(M);$$

so in either case,

(3) $$|J \backslash (\leqslant d)| \geqslant t(M).$$

Now $t(J) = \min_f \{|J \backslash (\leqslant f)| \colon f \text{ is a deputy of } J\}$; the deputies of J are precisely the deputies of M, including c, so

$$t(J) = \min_d \{|J \backslash (\leqslant c)|, \quad |J \backslash (\leqslant d)| \colon d \text{ is a deputy of } M,\ d \neq c\};$$

and

(4) $$|J \backslash (\leqslant c)| = |M \backslash (\leqslant c)| = t(M).$$

By (3) and (4),

$$t(J) = t(M).$$

But since $J \subset M$ and J is a power-structure of X and $\vee J = a = \vee M$, the minimality of M gives $t(J) < t(M)$. This contradiction proves the theorem.

If a power-structure B is not minimal, an induction argument shows that there must exist a power-structure C covered by B for which $\vee C = \vee B$ and $t(C) = t(B)$. By the lemma, $C = B \backslash \{z\}$ for some member z which is minimal in B and belongs to no topple set of B. The set of members which belong to no topple set is the set

$$S = \{x \in B : x \leqslant c \text{ for every contender } c \text{ of } B\},$$

called the *safe core* of B since it is precisely these members who survive every possible coup in B. Thus any non-minimal power-structure has a nonempty safe core. The argument is reversible, so we have a characterisation of minimal power-structures:

THEOREM. *A power-structure is minimal if and only if its safe core is empty.*

Thus no one is safe in a minimal power-structure.

COROLLARY. *Every power-structure which is a root system with two or more contenders is minimal. No chain is minimal.*

If B is a nonminimal power-structure with safe core S, then it is possible to obtain from it a minimal power-structure by deletion of part or all of S, that is, there exists a subset S' of S such that $B' := B \backslash S'$ is a minimal power-structure, with $t(B') = t(B)$.

REFERENCES

[1] J.B. Miller, 'Introduction to a theory of coups', *Algebra Universalis* **9** (1979), 346–370.

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