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MIXED INJECTIVE MODULES*

DERYA KESKIN TÜTÜNCÜ
Department of Mathematics, Hacettepe University, 06800 Beytepe, Ankara, Turkey
e-mail: keskin@hacettepe.edu.tr

SAAD H. MOHAMED
Department of Mathematics, Faculty of Education, Ain Shams University, Cairo, Egypt
e-mail: sshhmohamed@yahoo.ca

and NIL ORHAN ERTAŞ
Karabuk University, Department of Mathematics, 78050 Karabuk, Turkey
e-mail: orhannil@yahoo.com

Abstract. Since Azumaya introduced the notion of A-injectivity in 1974, several generalizations have been investigated by a number of authors. We introduce some more generalizations and discuss their connection to the previous ones.

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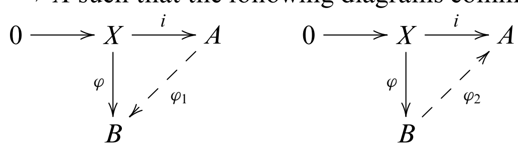
1. Introduction. All rings considered have unities, and all modules are unital right modules. The notations $N \leq^e M$ and $K \leq^{\oplus} M$ indicate that N is an *essential* submodule and K is a *direct summand* of M, respectively. A summand will always mean a direct summand. K is a *complementary summand* of L in M if $M = K \oplus L$. A *closed* submodule of M is one that has no proper essential extensions in M. A module M is *extending* if every closed submodule of M is a summand. The *graph* of a homomorphism $\varphi : X \longrightarrow Y$ is the submodule $\langle\varphi\rangle = \{x - \varphi(x) : x \in X\}$ of $X \oplus Y$. A homomorphism $\Psi : U \longrightarrow V$ is called *faithful* if $\Psi = 0$ only if $U = 0$. For modules A and B, $\varphi : A \geq X \longrightarrow B$ will denote a partial homomorphism $X \longrightarrow B$. B is said to be A-*injective* if for any $\varphi : A \geq X \longrightarrow B$, there exists a homomorphism $\varphi_1 : A \longrightarrow B$ that extends φ (see G. Azumaya, M-projective and M-injective modules, unpublished work, 1974, and [**1**]). Baba [**2**] generalized the notion of A-injectivity as follows:

B is *almost* A-*injective* if for any $\varphi : A \geq X \longrightarrow B$, there exists a homomorphism $\varphi_1 : A \longrightarrow B$ that extends φ (injectivity behaviour), or there exists a non-zero summand $A_2 \leq A$ and a homomorphism $\varphi_2 : B \longrightarrow A_2$ such that $\varphi_2\varphi = \pi_{A_2}|_X$, where π_{A_2} is the projection of A onto A_2 (which we refer to as opposite injectivity behaviour). We note that for an indecomposable module A, we have that B is almost A-injective if and only if for any $\varphi : A \geq X \longrightarrow B$, there exists a homomorphism $\varphi_1 : A \longrightarrow B$ or a homomorphism $\varphi_2 : B \longrightarrow A$ such that the following diagrams commute:



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* Dedicated to Professor Patrick F. Smith on his 65th birthday.

In the following we investigate cases where we have a mixture of the two behaviours. For $\varphi : A \geq X \longrightarrow B$, we associate a class, denoted by $[[\varphi : A \geq X \longrightarrow B]]$, consisting of all commutative diagrams

$$\begin{array}{ccccc} A_2 & \xleftarrow{\pi_{A_2}} & A & \xrightarrow{\pi_{A_1}} & A_1 \\ \Big\uparrow{\scriptstyle \varphi_2} & & \Big\uparrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \varphi_1} \\ & & X & & \\ & & \Big\downarrow{\scriptstyle \varphi} & & \\ B_2 & \xleftarrow[\pi_{B_2}]{} & B & \xrightarrow[\pi_{B_1}]{} & B_1 \end{array}$$

where $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, and π_{A_i} and π_{B_i}, $i = 1, 2$, are the natural projections. (The commutativity of the diagram is equivalent to: for $x = a_1 + a_2$ and $\varphi(x) = b_1 + b_2$, we have $\varphi_1(a_1) = b_1$ and $\varphi_2(b_2) = a_2$.)

B is said to be A-*ojective* if for any $\varphi : A \geq X \longrightarrow B$, there exists $D \in [[\varphi : A \geq X \longrightarrow B]]$, with φ_2 being a monomorphism [**5**, **8**]. As a generalization we say that B is A-*mixed injective* if for any $\varphi : A \geq X \longrightarrow B$, there exists $D \in [[\varphi : A \geq X \longrightarrow B]]$ with φ_2 faithful. [**4**, **7**] are the general references for notions of modules not defined in this work.

2. Mixed injectivity. In this section we study various types of generalizations of injectivity under one umbrella. First we note that

(1) $[[\varphi : A \geq X \longrightarrow B]]$ is not empty, as it always contains the trivial diagram

$$\begin{array}{ccccc} 0 & \xleftarrow{0} & A & \xrightarrow{1} & A \\ \Big\uparrow{\scriptstyle 0} & & \Big\uparrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle 0} \\ & & X & & \\ & & \Big\downarrow{\scriptstyle \varphi} & & \\ B & \xleftarrow[1]{} & B & \xrightarrow[0]{} & 0 \end{array}$$

By a non-trivial diagram, we mean one in which $A_2 \oplus B_1 \neq 0$. If such a diagram exists for each φ we say that B is A-*basic injective*.

(2) For $\varphi = 0$, we have the diagram

$$\begin{array}{ccccc} 0 & \xleftarrow{0} & A & \xrightarrow{1} & A \\ \Big\uparrow{\scriptstyle 0} & & \Big\uparrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle 0} \\ & & X & & \\ & & \Big\downarrow{\scriptstyle 0} & & \\ 0 & \xleftarrow[0]{} & B & \xrightarrow[1]{} & B \end{array}$$

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PROPOSITION 2.1. *For modules A and B,*

(1) *B is A-injective if and only if for any $\varphi : A \geq X \longrightarrow B$, there exists $D \in [[\varphi : A \geq X \longrightarrow B]]$ with $A_2 \oplus B_2 = 0$.*

(2) *B is A-ojective if and only if for any $\varphi : A \geq X \longrightarrow B$, there exists $D \in [[\varphi : A \geq X \longrightarrow B]]$ with $Ker\, \varphi_2 = 0$.*

(3) *B is A-mixed injective if and only if for any $\varphi : A \geq X \longrightarrow B$, there exists $D \in [[\varphi : A \geq X \longrightarrow B]]$ with φ_2 faithful.*

(4) *B is almost A-injective if and only if for any $\varphi : A \geq X \longrightarrow B$, there exists $D \in [[\varphi : A \geq X \longrightarrow B]]$ such that $A_2 = 0$ implies $B_2 = 0$.*

Proof. We only need to prove (4). Assume B is almost A-injective, and consider $\varphi : A \geq X \longrightarrow B$. The injectivity behaviour corresponds to the diagram

$$\begin{array}{ccccc} 0 & \xleftarrow{0} & A & \xrightarrow{1} & A \\ \Big\uparrow{\scriptstyle 0} & & \Big\uparrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \varphi_1} \\ & & X & & \\ & & \Big\downarrow{\scriptstyle \varphi} & & \\ 0 & \xleftarrow[0]{} & B & \xrightarrow[1]{} & B \end{array}$$

The opposite injectivity behaviour corresponds to the diagram

$$\begin{array}{ccccc} A_2 & \xleftarrow{\pi_{A_2}} & A & \xrightarrow{\pi_{A_1}} & A_1 \\ \Big\uparrow{\scriptstyle \varphi_2} & & \Big\uparrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle 0} \\ & & X & & \\ & & \Big\downarrow{\scriptstyle \varphi} & & \\ B & \xleftarrow[1]{} & B & \xrightarrow[0]{} & 0 \end{array}$$

with $A_2 \neq 0$.

Conversely, assume the condition. Given $\varphi : A \geq X \longrightarrow B$, we have a commutative diagram

$$\begin{array}{ccccc} A_2 & \xleftarrow{\pi_{A_2}} & A & \xrightarrow{\pi_{A_1}} & A_1 \\ \Big\uparrow{\scriptstyle \varphi_2} & & \Big\uparrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \varphi_1} \\ & & X & & \\ & & \Big\downarrow{\scriptstyle \varphi} & & \\ B_2 & \xleftarrow[\pi_{B_2}]{} & B & \xrightarrow[\pi_{B_1}]{} & B_1 \end{array}$$

We consider two cases.

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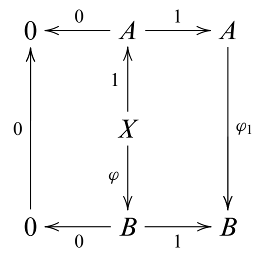
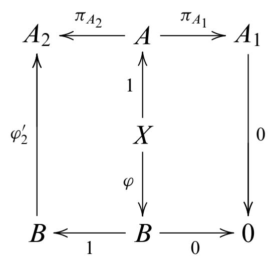
(1) $A_2 = 0$: The hypothesis implies $B_1 = B$ and we have the commutative diagram



which gives an injectivity behaviour.

(2) $A_2 \neq 0$: We may define $\varphi_2' : B \longrightarrow A_2$ as $\varphi_2' = \varphi_2$ on B_2 and $\varphi_2' = 0$ on B_1. Then the diagram reduces to



This is an opposite injectivity behaviour. □

The proof of the following lemma is straightforward.

LEMMA 2.2. *Let* $M = A \oplus B$*, where* $B \neq 0$, $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ *and* $\varphi_2 : B_2 \longrightarrow A_2$*. Consider the following conditions:*

(1) $A_2 \oplus B_2 = 0$.

(2) *Ker* $\varphi_2 = 0$.

(3) φ_2 *is faithful.*

(4) $A_2 = 0$ *implies* $B_2 = 0$.

(5) $A_2 \oplus B_1 \neq 0$.

Then (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(5).

As an immediate consequence of the above lemma and Proposition 2.1, we have the hierarchy

injectivity $\Rightarrow$ ojectivity $\Rightarrow$ mixed injectivity $\Rightarrow$ almost injectivity $\Rightarrow$ basic injectivity.

Now we give examples to separate these cases.

EXAMPLES 2.3. (1) Let $A = \mathbb{Z}_4$ and $B = \mathbb{Z}_6$. Then B is A-ojective, and is not A-injective.

(2) Let A be an injective module with exactly one non-zero proper submodule S. Let B be an indecomposable module that contains a simple submodule not isomorphic to S. Then B is A-mixed injective and is not A-ojective.

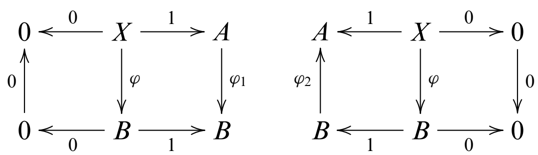
(3) Let A be an extending module whose socle is maximal and contains more than one homogeneous component. Let B be an indecomposable module such that A and B have no non-zero isomorphic submodules, and B is not A-ojective. Then B is almost A-injective and is not A-mixed injective (cf. Theorem 3.6).

(4) Let A be indecomposable. Let $B = B_1 \oplus B_2$ such that A and B have no non-zero isomorphic submodules, B_1 is A-injective and B is not A-injective. Then B is A-basic injective and is not almost A-injective.

However, for uniform modules, we have the following proposition.

PROPOSITION 2.4. *For an indecomposable module A and a uniform module B, the following are equivalent:*
(1) *B is A-basic injective.*
(2) *B is almost A-injective.*
(3) *B is A-mixed injective.*
(4) *B is A-ojective.*

Proof. We only need to prove (1)$\Rightarrow$(4). Given $\varphi : A \geq X \longrightarrow B$ without loss of generality, we may assume that $\varphi \neq 0$. The hypothesis gives only the following two diagrams:



In the second case, we have $\varphi_2\varphi = 1_X$. Hence, $\operatorname{Ker}\varphi_2 \cap \varphi(X) = 0$. However, $\varphi(X)$ is essential in B, and consequently $\operatorname{Ker}\varphi_2 = 0$. □

The above proposition yields the following generalization of [**8**, Theorem 13], which is also a generalization of [**3**, Lemma 8].

THEOREM 2.5. *Let $M = M_1 \oplus \cdots \oplus M_n$, where the M_i are uniform. Then M is extending and the decomposition is exchangeable if and only if M_i is M_j-basic injective for all $i \neq j$.*

Next we give characterizations for different types of injectivity analogous to that given in [**8**] for ojective modules. First we need some lemmas.

LEMMA 2.6. *Let $M = A \oplus B$ and $\varphi : A \geq X \longrightarrow B$. Then*
(1) *$X \oplus B = \langle\varphi\rangle \oplus B$.*
(2) *$Ker\,\varphi = \langle\varphi\rangle \cap A$.*
(3) *φ is a monomorphism if and only if $\langle\varphi\rangle \cap A = 0$.*
(4) *$\varphi = 0$ if and only if $\langle\varphi\rangle \leq A$.*

Proof. We prove only (2) and (4), the rest being obvious.
(2) $x \in \operatorname{Ker}\varphi \Rightarrow \varphi(x) = 0 \Rightarrow x - \varphi(x) = x \in \langle\varphi\rangle \cap X \leq A \cap \langle\varphi\rangle$;

$$\text{and}\quad a \in \langle\varphi\rangle \cap A \Rightarrow a = x - \varphi(x) \quad \text{for some} \quad x \in X$$
$$\Rightarrow x - a = \varphi(x) \in A \cap B = 0$$
$$\Rightarrow a = x \quad \text{and} \quad x \in \operatorname{Ker}\varphi.$$

(4) We have $\varphi = 0$ if and only if $X = \operatorname{Ker}\varphi = \langle\varphi\rangle \cap A$. Also $\varphi = 0$ if and only if $X = \langle\varphi\rangle$. Hence, $\varphi = 0$ if and only if $\langle\varphi\rangle = \langle\varphi\rangle \cap A$ if and only if $\langle\varphi\rangle \leq A$. □

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LEMMA 2.7. *Let* $N \leq A \oplus B$. *Then* $N \cap B = 0$ *if and only if there exists* $\varphi : A \geq X \longrightarrow B$ *such that* $N = \langle\varphi\rangle$. *Moreover,* $\varphi = 0$ *if and only if* $N \leq A$, *and* φ *is a monomorphism if and only if* $N \cap A = 0$.

Proof. ($\Rightarrow$): Define $X = A \cap (N \oplus B)$ and $\varphi : X \longrightarrow B$ as the restriction to X of the projection $N \oplus B \longrightarrow B$ along N. Given $n \in N$, let $n = a + b$ with $a \in A$ and $b \in B$. Hence, $a = n - b \in A \cap (N \oplus B) = X$. This gives $\varphi(a) = -b$; hence, $n = a - \varphi(a) \in \langle\varphi\rangle$. Now consider $x \in X$. Then $x = n + b$ with $n \in N$ and $b \in B$. Hence, $\varphi(x) = b$ and so $x - \varphi(x) = n \in N$. This proves that $N = \langle\varphi\rangle$.

($\Leftarrow$): Obvious.

The last statement follows from Lemma 2.6. □

Some arguments in the proof of the following theorem are similar to those given in [**8**, Theorem 7].

THEOREM 2.8. *B is A-basic injective if and only if for any submodule N of* $M = A \oplus B$ *with* $N \cap B = 0$, *we have* $M = N' \oplus A' \oplus B'$ *with* $A' \leq A, B' \leq B$ *and* $N \leq N' \neq M$. *Further, we have the following:*

(1) *B is A-injective if and only if* $M = N' \oplus B$.

(2) *B is A-ojective if and only if* $N' \cap B = 0$.

(3) *B is A-mixed injective if and only if* $N' \cap B$ *is not a non-zero complementary summand of* B' *in B.*

Proof. 'Only if': By Lemma 2.7, there is $\varphi : A \geq X \longrightarrow B$ such that $N = \langle\varphi\rangle$. The hypothesis yields a non-trivial diagram in $[[\varphi : A \geq X \longrightarrow B]]$. Then, by Lemma 2.6 (1),

$$\begin{aligned} M = A \oplus B &= A_1 \oplus B_1 \oplus A_2 \oplus B_2 \\ &= \langle\varphi_1\rangle \oplus B_1 \oplus A_2 \oplus \langle\varphi_2\rangle \\ &= \langle\varphi_1\rangle \oplus \langle\varphi_2\rangle \oplus A_2 \oplus B_1. \end{aligned}$$

We prove $\langle\varphi\rangle \leq \langle\varphi_1\rangle \oplus \langle\varphi_2\rangle$. Let $x = a_1 + a_2$ and $\varphi(x) = b_1 + b_2$. We get from the diagram $\varphi_1(a_1) = b_1$ and $\varphi_2(b_2) = a_2$. Hence,

$$x - \varphi(x) = a_1 - b_1 - (b_2 - a_2) = a_1 - \varphi_1(a_1) - (b_2 - \varphi_2(b_2)) \in \langle\varphi_1\rangle \oplus \langle\varphi_2\rangle.$$

Thus, $N = \langle\varphi\rangle \leq \langle\varphi_1\rangle \oplus \langle\varphi_2\rangle$. Define $N' = \langle\varphi_1\rangle \oplus \langle\varphi_2\rangle$, $A' = A_2$ and $B' = B_1$. Then $M = N' \oplus A' \oplus B'$ with $N \leq N' \neq M$.

'If': Consider $\varphi : A \geq X \longrightarrow B$. Clearly, $\langle\varphi\rangle \cap B = 0$. The hypothesis then yields a decomposition $M = N' \oplus A' \oplus B'$ with $A' \leq A, B' \leq B$ and $\langle\varphi\rangle \leq N' \neq M$. For simplicity, let $A' = A_2$ and $B' = B_1$. Then $M = N' \oplus A_2 \oplus B_1$. As $M \neq N'$, $A_2 \oplus B_1 \neq 0$. By the modular law, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where $A_1 = A \cap (N' \oplus B_1)$ and $B_2 = B \cap (N' \oplus A_2)$. Let η_1 denote the projection of M onto B_1 along $N' \oplus A_2$, and η_2 denote the projection of M onto A_2 along $N' \oplus B_1$. It is clear that $A_1 \oplus B_1 \leq \operatorname{Ker}\eta_2$ and $A_2 \oplus B_2 \leq \operatorname{Ker}\eta_1$. Also $\langle\varphi\rangle \leq N' \leq \operatorname{Ker}\eta_i$, $i = 1, 2$. Then for every $x \in X$, $\eta_i\varphi(x) = \eta_i(x)$ and $\eta_i\pi_{A_j} = 0 = \eta_i\pi_{B_j}$ for $j \neq i = 1, 2$. Define $\varphi_1 = \eta_1 \mid_{A_1}$ and $\varphi_2 = \eta_2 \mid_{B_2}$. Then $\pi_{B_1}\varphi(x) = \eta_1\pi_{B_1}\varphi(x) = \eta_1\varphi(x) = \eta_1(x) = \eta_1\pi_{A_1}(x) = \varphi_1\pi_{A_1}(x)$; $\varphi_2\pi_{B_2}\varphi(x) = \eta_2\pi_{B_2}\varphi(x) = \eta_2\varphi(x) = \eta_2(x) = \eta_2\pi_{A_2}(x) = \pi_{A_2}(x)$.

(1) Obvious.

(2) One can easily check that $\operatorname{Ker}\varphi_2 = N' \cap B$. Hence, φ_2 is a monomorphism if and only if $N' \cap B = 0$.

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(3) We have $B_2 = B \cap (N' \oplus A')$ and $\operatorname{Ker}\varphi_2 = N' \cap B$. If $N' \cap B \neq 0$, then clearly $B_2 \neq 0$. As φ_2 is faithful, $\varphi_2 \neq 0$, and hence, $B_2 \neq \operatorname{Ker}\varphi_2 = N' \cap B$. It then follows that $B = B' \oplus B_2 \neq B' \oplus (N' \cap B)$.

Conversely assume that $N' \cap B$ is not a non-zero complementary summand of B' in B. If $N' \cap B = 0$, then φ_2 is a monomorphism. On the other hand, $N' \cap B \neq 0$ implies $B \neq B' \oplus N' \cap B$. This gives $B_2 \neq N' \cap B = \operatorname{Ker}\varphi_2$, and hence, $\varphi_2 \neq 0$. In both cases φ_2 is faithful. □

Corollary 2.9. *B is A-injective if and only if for any complement C of B in $M = A \oplus B$ we have $M = C \oplus B$.*

Corollary 2.10. *B is A-ojective if and only if for any complement C of B in $M = A \oplus B$, we have $M = C \oplus A' \oplus B'$ with $A' \leq A$, $B' \leq B$.*

We end this section by proving that A-mixed injectivity passes to summands of A. The main idea of the proof is suggested in [**6**, Proposition 1.5]. We were not able to give a proof using the characterization of mixed injectivity given in Theorem 2.8 (cf. [**8**, Proposition 8]).

Proposition 2.11. *Let A and B be modules and let $A^* \leq^{\oplus} A$. If B is A-mixed injective, then B is A^*-mixed injective.*

Proof. Let $A = A^* \oplus A^{**}$. Given a homomorphism $\varphi : A^* \geq X \longrightarrow B$, define $\Psi : X \oplus A^{**} \longrightarrow B$ by $\Psi|_X = \varphi$ and $\Psi|_{A^{**}} = 0$. As B is A-mixed injective, we get decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, together with homomorphisms $\varphi_1 : A_1 \longrightarrow B_1$ and $\varphi_2 : B_2 \longrightarrow A_2$ with φ_2 faithful such that the following diagram commutes:

$$\begin{array}{ccccc}
A_2 & \xleftarrow{\pi_{A_2}} & X \oplus A^{**} & \xrightarrow{\pi_{A_1}} & A_1 \\
\uparrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \varphi_1} \\
B_2 & \xleftarrow[\pi_{B_2}]{} & B & \xrightarrow[\pi_{B_1}]{} & B_1
\end{array}$$

Clearly $\pi_{A_2}(A^{**}) = 0$, and so $A^{**} \leq A_1$. Hence, $A_1 = A^{**} \oplus (A_1 \cap A^*)$. It follows that $A = A^{**} \oplus (A_1 \cap A^*) \oplus A_2$, and consequently, $A^* = (A_1 \cap A^*) \oplus [(A_2 \oplus A^{**}) \cap A^*]$. Let $A_1^* = A_1 \cap A^*$ and $A_2^* = (A_2 \oplus A^{**}) \cap A^*$. Now $A = A_1^* \oplus A_2^* \oplus A^{**} = A_1^* \oplus A^{**} \oplus A_2^* = A_1 \oplus A_2^*$. Let λ denote the natural projection of A onto A_2^* along A_1, and let $\eta = \lambda|_{A_2}$. Clearly η is a monomorphism, and hence, $\eta\varphi_2$ is faithful. Let π_1 and π_2 denote the natural projections of A^* onto A_1^* and A_2^*, respectively. Now we have the diagram

$$\begin{array}{ccccc}
A_2^* & \xleftarrow{\pi_2} & X & \xrightarrow{\pi_1} & A_1^* \\
\uparrow{\scriptstyle \eta} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
A_2 & \xleftarrow{\pi_{A_2}} & X \oplus A^{**} & \xrightarrow{\pi_{A_1}} & A_1 \\
\uparrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \varphi_1} \\
B_2 & \xleftarrow[\pi_{B_2}]{} & B & \xrightarrow[\pi_{B_1}]{} & B_1
\end{array}$$

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Given $x \in X$, then $x = a_1^* + a_2^*$ with $a_1^* \in A_1^*$ and $a_2^* \in A_2^*$. Then $a_2^* = a_2 + a^{**}$ for $a_2 \in A_2$ and $a^{**} \in A^{**}$. Hence, $x = (a_1^* + a^{**}) + a_2$, and $a_2^* = \lambda(a_2^*) = \lambda(a_2 + a^{**}) = \lambda(a_2) = \eta(a_2)$. Assume that $\varphi(x) = b_1 + b_2$. Then, $\varphi_1(a_1^*) = \varphi_1(a_1^* + a^{**}) = \varphi_1\pi_{A_1}(x) = \pi_{B_1}\Psi(x) = \pi_{B_1}\varphi(x) = b_1$; $\eta\varphi_2(b_2) = \eta\varphi_2\pi_{B_2}\varphi(x) = \eta\varphi_2\pi_{B_2}\Psi(x) = \eta\pi_{A_2}(x) = \eta(a_2) = a_2^*$. □

3. Symmetric injectivity. B is *A-essential injective* if for any $\varphi : A \geq X \longrightarrow B$ with essential kernel, there exists a homomorphism $\varphi_1 : A \longrightarrow B$ that extends φ (cf. [**9**]). We note that essential injectivity behaves like injectivity concerning direct sums and summands.

Proposition 3.1. (*cf.* [**9**, *Lemma* 4]). *B is A-essential injective if and only if for any submodule N of $M = A \oplus B$ with $N \cap B = 0$ and $N \cap A \leq^e A$, we have $M = N' \oplus B$ with $N \leq N'$.*

Corollary 3.2. *B is A-essential injective if and only if for any complement C of B in $M = A \oplus B$ with $C \cap A \leq^e A$, we have $M = C \oplus B$.*

Lemma 3.3. *Let $M = A \oplus B$. If B is A-mixed injective, then B is A-essential injective.*

Proof. Let $N \leq M$ with $N \cap B = 0$ and $N \cap A \leq^e A$. As B is A-mixed injective, by Theorem 2.8, we get $M = N' \oplus A' \oplus B'$ with $N \leq N'$, $A' \leq A$ and $B' \leq B$. Now $(N \cap A) \cap A' = N \cap (A \cap A') = N \cap A' = 0$. Hence, $A' = 0$, and therefore, $M = N' \oplus B'$. This implies $B = B' \oplus N' \cap B$. Hence, $N' \cap B = 0$, by (3) of Theorem 2.8. It then follows that $M = N' \oplus B$. □

By Theorem 2.8, B is A-ojective if and only if for any submodule N of $M = A \oplus B$ with $N \cap B = 0$, we have $M = N' \oplus A' \oplus B'$ with $A' \leq A$, $B' \leq B$, $N \leq N'$ and $N' \cap B = 0$. We modify this characterization to give equal attention to both A and B. We say that A and B are *symmetrically injective* if for any submodule N of $M = A \oplus B$ with $N \cap (A \cup B) = 0$, we have $M = N' \oplus A' \oplus B'$ with $A' \leq A$, $B' \leq B$, $N \leq N'$ and $N' \cap (A \cup B) = 0$. (Note that for submodules X, Y and Z of M, $X \cap (Y \cup Z) = 0$ if and only if $X \cap Y = 0$ and $X \cap Z = 0$.)

Theorem 3.4. *The following are equivalent:*

(1) *A and B are symmetrically injective.*

(2) *For any monomorphism $\varphi : A \geq X \longrightarrow B$, there exists $D \in [[\varphi : A \geq X \longrightarrow B]]$, with φ_1 and φ_2 being monomorphisms.*

(3) *For any monomorphism $\Psi : B \geq Y \longrightarrow A$, there exists $D' \in [[\Psi : B \geq Y \longrightarrow A]]$, with Ψ_1 and Ψ_2 being monomorphisms.*

Proof. (1)⇔(2): The proof is almost the same as in Theorem 2.8. We only need to note the following observations:

(1)⇒(2): φ is a monomorphism, as $N \cap A = 0$ (Lemma 2.6), and it is easy to check that $\operatorname{Ker}\varphi_1 = N' \cap A$ and $\operatorname{Ker}\varphi_2 = N' \cap B$, and therefore, φ_1 and φ_2 are monomorphisms if and only if $N' \cap (A \cup B) = 0$.

(2)⇒(1): For a monomorphism $\varphi : A \geq X \longrightarrow B$, $\langle\varphi\rangle \cap A = 0$ (Lemma 2.6), and clearly $\langle\varphi\rangle \cap B = 0$. Hence, $\langle\varphi\rangle \cap (A \cup B) = 0$.

(1)⇔(3): Follows by symmetry. □

Remark. Let X, Y and Z be submodules of a module M with $Z \cap (X \cup Y) = 0$. By Zorn's lemma, we can find a submodule Z' of M maximal with respect to the property

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that $Z \leq Z'$ and $Z' \cap (X \cup Y) = 0$. Clearly Z' is a closed submodule of M. An example of such a submodule is a complement C of X with $C \cap Y = 0$ (or a complement C of Y with $C \cap X = 0$).

The following corollary is analogous to Corollary 2.10.

COROLLARY 3.5. *A and B are symmetrically injective if and only if for any submodule K of $M = A \oplus B$ maximal with $K \cap (A \cup B) = 0$, we have $M = K \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$.*

THEOREM 3.6. *Let $M = A \oplus B$ with A extending. Then the following are equivalent:*

(1) *B is A-ojective.*

(2) *B is A-mixed injective and $\overline{A}$ and B are symmetrically injective for every $\overline{A} \leq^{\oplus} A$.*

(3) *B is A-essential injective and for every closed submodule K of M with $K \cap (A \cup B) = 0$, we have $M = K \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$.*

Proof. (1)⇒(2): That B is A-mixed injective is trivial. Also B is $\overline{A}$-ojective by [**8**, Proposition 8], and $\overline{A}$ is extending. Hence, there is no loss of generality if we assume that $\overline{A} = A$. Let K be a submodule of M maximal with $K \cap (A \cup B) = 0$. Then K is a closed submodule of M with $K \cap B = 0$. As A is extending, we get by [**8**, Lemma 9] that $M = K \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$. Hence, A and B are symmetrically injective.

(2)⇒(3): B is A-essential injective by Lemma 3.3. Let K be a closed submodule of M with $K \cap (A \cup B) = 0$. Now $K \oplus B = (K \oplus B) \cap A \oplus B$. Since A is extending, $(K \oplus B) \cap A \leq^e A_1$, where $A_1 \leq^{\oplus} A$. Let $A = A_1 \oplus A_2$ and $N = A_1 \oplus B$. Then $K \leq N$ and $K \oplus B \leq^e N$. Hence, K is a complement of B in N. As $K \cap A_1 = 0$, K is maximal in N such that $K \cap (A_1 \cup B) = 0$. Since A_1 and B are symmetrically injective, we get $N = K \oplus A'_1 \oplus B'$ with $A'_1 \leq A_1$ and $B' \leq B$. Hence, $M = A_2 \oplus N = K \oplus (A_2 \oplus A'_1) \oplus B'$.

(3)⇒(1): Let C be a complement of B in M. Since A is extending, $C \cap A \leq^e A^*$, where $A = A^* \oplus A^{**}$. Let $N = A^* \oplus B$ and $C^* = C \cap N$. Then by [**8**, Lemma 2], C^* is a complement of B in N. Now $C^* \cap A^* = C \cap N \cap A^* = C \cap A^* = C \cap A \cap A^* = C \cap A \leq^e A^*$. As B is A-essential injective, B is A^*-essential injective. Hence, $N = C^* \oplus B$, by Corollary 3.2. This gives $M = C^* \oplus A^{**} \oplus B = C^* \oplus L$, where $L = A^{**} \oplus B$. Let $C^{**} = C \cap L$. Then $C = C^* \oplus C^{**}$. Clearly C^{**} is a closed submodule in M. Also $C^{**} \cap A = L \cap C \cap A \leq L \cap A^* = 0$. Then $C^{**} \cap (A \cup B) = 0$. The hypothesis then implies that $M = C^{**} \oplus A' \oplus B'$ with $A' \leq A$ and $B' \leq B$. Hence, $L = C^{**} \oplus (A' \cap L) \oplus B'$, and consequently $M = C^* \oplus L = C^* \oplus C^{**} \oplus (A' \cap L) \oplus B' = C \oplus (A' \cap L) \oplus B'$. Hence, by Theorem 2.8 (2), B is A-ojective. □

COROLLARY 3.7. *Let A and B be extending, and A be B-ojective. Then B is A-ojective if and only if B is A-mixed injective (if and only if B is A-essential injective.)*

THEOREM 3.8. *Let $M = A \oplus B$ such that A and B are extending. Then the following are equivalent:*

(1) *M is extending and the decomposition is exchangeable.*

(2) *A is B-ojective and B is A-essential injective.*

(3) *B is A-ojective and A is B-essential injective.*

(4) *A is B-ojective and B is A-mixed injective.*

(5) *B is A-ojective and A is B-mixed injective.*

Proof. Corollary 3.7 and [**8**, Theorem 10*]. □

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