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A NEW METHOD IN ARITHMETICAL FUNCTIONS AND CONTOUR INTEGRATION

BY
BRUCE C. BERNDT

1. **Introduction.** If f is a suitable meromorphic function, then by a classical technique in the calculus of residues, one can evaluate in closed form series of the form,

$$\sum_{n=-\infty}^{\infty} f(n) \quad \text{or} \quad \sum_{n=-\infty}^{\infty} (-1)^n f(n).$$

Suppose that $a(n)$ is an arithmetical function. It is natural to ask whether or not one can evaluate by contour integration

$$\sum_{n=-\infty}^{\infty} a(n)f(n) \quad \text{or} \quad \sum_{n=-\infty}^{\infty} (-1)^n a(n)f(n), \tag{1.1}$$

where f belongs to a suitable class of meromorphic functions. We shall give here only a partial answer for a very limited class of arithmetical functions.

Our techniques are applicable to arithmetical functions which have the representation,

$$a(n) = \sum_{d|n} g(d)h(d, n),$$

where g and h are arithmetical functions such that for each fixed d, $h(d, z)$ is a polynomial in z. In fact, more generally, instead of summing over all divisors of n, we may sum instead over any subset of the divisors of n, in particular, the divisors in an arithmetic progression $A(q, a)=\{mq+a:q\geq 1, a\geq 0, (q, a)=1, m\geq 0 \text{ integral}\}$. Thus, our methods are applicable to the arithmetical functions,

$$a(n, q, a) = \sum_{\substack{d \mid n \\ d\in A(q,a)}} g(d)h(d, n).$$

Note that $a(n, 1, 0)=a(n)$. In the proofs of our results, we shall need the supplementary arithmetical functions,

$$a^{(m)}(n, q, a) = \sum_{\substack{d=1 \\ d \mid n \\ d\in A(q,a)}}^{m} g(d)h(d, n).$$

Plainly,

$$\lim_{m\to\infty} a^{(m)}(n, q, a) = a(n, q, a).$$

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Clearly, we also need a growth condition on $a(n)$. Suppose that $a(n)=0(n^b)$ as n tends to ∞, where b is some fixed real number.

In general, we are not able to sum

$$(1.2)\qquad \sum_{n=-\infty}^{\infty} a(n, q, a)f(n) \quad \text{or} \quad \sum_{n=-\infty}^{\infty} (-1)^n a(n, q, a)f(n)$$

in closed form. Instead, our results transform the series of (1.2) into series generally involving an arithmetical function different from $a(n, q, a)$. In another paper [1] we have shown how to evaluate in closed form by the calculus of residues series of the form (1.1) when $a(n)$ is a primitive character.

In the sequel, we make the following assumptions on f. Let f be meromorphic in the extended complex plane. Suppose that $|f(z)| \leq A\ |z|^{-c}$ for some positive numbers A and c, uniformly as $|z|$ tends to ∞. (In the theorems below, more restrictive lower bounds on c will be required.) Let $\{z_1, \ldots, z_l\}$ be the complete set of poles of f, and put $S=\{z_1, \ldots, z_l\} \cup \{z_0\}$, where $z_0=0$. The residue of a meromorphic function g at the pole z' will be denoted by $R\{g, z'\}$.

We shall illustrate our method with four different arithmetical functions, or classes of arithmetical functions. We shall conclude the paper with several examples.

2. **Main results.** For complex z, let

$$S(d, z) = \sum_{j=0}^{d-1} e^{2\pi i z j/d} \quad \text{and} \quad T(d, z) = \sum_{j=-[(d-1)/2]}^{[d/2]} e^{2\pi i z j/d}.$$

THEOREM 1. *Let*

$$A_\nu^{(m)}(z, q, a) = \sum_{\substack{d=1 \\ d \in A(q,a)}}^{m} d^{\nu-1} S(d, z)$$

and

$$B_\nu^{(m)}(z, q, a) = \sum_{\substack{d=1 \\ d \in A(q,a)}}^{m} d^{\nu-1} T(d, z).$$

Then if $c > \sup\{1, \nu+1\}$,

$$(2.1)\qquad \sum_{\substack{n=-\infty \\ n \notin S}}^{\infty} \sigma_\nu(n, q, a)f(n) = -\lim_{m\to\infty} \sum_{z_j \in S} R\left\{\frac{\pi e^{-\pi i z} A_\nu^{(m)}(z, q, a)f(z)}{\sin(\pi z)}, z_j\right\}$$

and

$$\sum_{\substack{n=-\infty \\ n \notin S}}^{\infty} (-1)^n \sigma_\nu(n, q, a)f(n) = -\lim_{m\to\infty} \sum_{z_j \in S} R\left\{\frac{\pi B_\nu^{(m)}(z, q, a)f(z)}{\sin(\pi z)}, z_j\right\}.$$

Note that

$$\sigma_\nu(n, 1, 0) = \sigma_\nu(n) = \sum_{d \mid n} d^\nu.$$

Proof. If N is a positive integer, let C_N denote the square whose sides of length $2N+1$ are parallel to the coordinate axes and whose center is the origin. Assume that N is chosen large enough so that S is contained on the interior of C_N. The residue of $\pi e^{-\pi iz}A_v^{(m)}(z, q, a)f(z)/\sin(\pi z)$ at the integer $n \notin S$ is

$$A_\nu^{(m)}(n, q, a)f(n) = \sigma_\nu^{(m)}(n, q, a)f(n),$$

where we have used the elementary fact that

$$S(d, n) = \begin{cases} d, & \text{if } d \mid n, \\ 0, & \text{if } d \mid n. \end{cases}$$

Hence, by the residue theorem,

$$\begin{aligned} \frac{1}{2\pi i}\int_{C_N} &\frac{\pi e^{-\pi iz}A_\nu^{(m)}(z, q, a)f(z)}{\sin(\pi z)}\,dz \\ &= \sum_{\substack{n=-N \\ n\notin S}}^{N} \sigma_\nu^{(m)}(n, q, a)f(n) + \sum_{z_j\in S} R\left\{\frac{\pi e^{-\pi iz}A_\nu^{(m)}(z, q, a)f(z)}{\sin(\pi z)}, z_j\right\}. \end{aligned} \tag{2.2}$$

Now, there exists a constant $M=M(m, \nu)$, independent of N, such that for all z on C_N,

$$\left|\frac{e^{-\pi iz}A_\nu^{(m)}(z, q, a)}{\sin(\pi z)}\right| \leq M.$$

Thus,

$$\left|\int_{C_N} \frac{e^{-\pi iz}A_\nu^{(m)}(z, q, a)f(z)}{\sin(\pi z)}\,dz\right| \leq \frac{4(2N+1)MA}{(N+\frac{1}{2})^c},$$

which tends to 0 as N tends to ∞ since $c>1$. Thus, upon letting N tend to ∞, we find that from (2.2),

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} \sigma_\nu^{(m)}(n, q, a)f(n) = -\sum_{z_j\in S} R\left\{\frac{\pi e^{-\pi iz}A_\nu^{(m)}(z, q, a)f(z)}{\sin(\pi z)}, z_j\right\}. \tag{2.3}$$

We now take the limit of both sides of (2.3) as m tends to ∞. We have [2, p. 260]

$$\sigma_\nu^{(m)}(n, q, a) \leq d(n)n^\nu = 0(n^{\nu+\varepsilon})$$

for every $\varepsilon>0$, where $d(n)=\sigma_0(n)$. Hence, since $c>\nu+1$, by the dominated convergence theorem we may take the limit on m inside the summation sign on the left side of (2.3). This concludes the proof of (2.1). The proof of the second part of Theorem 1 follows along the same lines.

Let r, s and t be positive integers with $s\leq 2$. Let $\mu_r(n)$ denote Klee's generalization [3] of the Möbius function, i.e., if $n=\prod_{i=1}^k p_i^{a_i}$ is the canonical factorization of n into primes p_i, $1\leq i\leq k$, then

$$\mu_r(n) = \begin{cases} 1, & n = 1, \\ (-1)^k, & a_i = r, \quad 1 \leq i \leq k \\ 0, & \text{otherwise.} \end{cases}$$

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The next theorem concerns the wide class of arithmetical functions,

$$\varphi_{r,s,t}(n) = \sum_{d \mid n} \mu_r^s(d)(n/d)^t.$$

Several well known arithmetical functions are special cases of the above. Thus, $\varphi_{1,1,1}(n)=\varphi(n)$, Euler's φ-function, and $\varphi_{1,2,1}(n)=\psi(n)$, Dedekind's ψ-function. For arbitrary t, $\varphi_{1,1,t}(n)=J_t(n)$, Jordan's totient function, and $\varphi_{1,2,t}(n)=\psi_t(n)$, an extension of $\psi(n)$ by Suryanarayana [4]. For arbitrary r, $\varphi_{r,1,1}(n)=\Phi_r(n)$, Klee's totient function [3], and $\varphi_{r,2,1}(n)=\Psi_r(n)$, another extension of $\psi(n)$ by Suryanarayana [4].

THEOREM 2. *Define*

$$C_{r,s,t}^{(m)}(z, q, a) = z^t \sum_{\substack{d=1 \\ d\in A(q,a)}}^{m} \frac{\mu_r^s(d)}{d^{t+1}} S(d, z)$$

and

$$D_{r,s,t}^{(m)}(z, q, a) = z^t \sum_{\substack{d=1 \\ d\in A(q,a)}}^{m} \frac{\mu_r^s(d)}{d^{t+1}} T(d, z).$$

Then if $c>t+1$,

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} \varphi_{r,s,t}(n, q, a)f(n) = -\lim_{m\to\infty} \sum_{z_j\in S} R\left\{\frac{\pi e^{-\pi i z}C_{r,s,t}^{(m)}(z, q, a)f(z)}{\sin(\pi z)}, z_j\right\} \tag{2.4}$$

and

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} (-1)^n \varphi_{r,s,t}(n, q, a)f(n) = -\lim_{m\to\infty} \sum_{z_j\in S} R\left\{\frac{\pi D_{r,s,t}^{(m)}(z, q, a)f(z)}{\sin(\pi z)}, z_j\right\}.$$

Proof. Proceed exactly as in the proof of Theorem 1 with $A_v^{(m)}(z,q,a)$ and $B_v^{(m)}(z, q, a)$ replaced by $C_{r,s,t}^{(m)}(z, q, a)$ and $D_{r,s,t}^{(m)}(z, q, a)$, respectively. Observe that

$$|\varphi_{r,s,t}^{(m)}(n, q, a)| \le d(n)n^t = 0(n^{t+\varepsilon}),$$

for every $\varepsilon>0$. Thus, since $c>t+1$, we may again apply the dominated convergence theorem to obtain

$$\lim_{m\to\infty} \sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} \varphi_{r,s,t}^{(m)}(n, q, a)f(n) = \sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} \varphi_{r,s,t}(n, q, a)f(n).$$

We shall state our last two theorems for only the case $q=1$, $a=0$.

Let $r(n)$ denote the number of representations of n as the sum of two squares. Then [2, p. 242],

$$r(n) = 4\sum_{d \mid n} \chi(d),$$

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where

$$\chi(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ 1, & \text{if } n \equiv 1 \pmod 4, \\ -1, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Define $r(-n)=r(n)$ if n is a positive integer.

THEOREM 3. *Let*

$$E^{(m)}(z) = 4\sum_{d=1}^{m} \frac{\chi(d)}{d} S(d, z)$$

and

$$F^{(m)}(z) = 4\sum_{d=1}^{m} \frac{\chi(d)}{d} T(d, z).$$

Then if $c>1$,

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} r(n)f(n) = -\lim_{m\to\infty} \sum_{z_j\in S} R\left\{\frac{\pi e^{-\pi i z}E^{(m)}(z)f(z)}{\sin(\pi z)}, z_j\right\}$$

and

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} (-1)^n r(n)f(n) = -\lim_{m\to\infty} \sum_{z_j\in S} R\left\{\frac{\pi F^{(m)}(z)f(z)}{\sin(\pi z)}, z_j\right\}. \tag{2.5}$$

Proof Proceed as in Theorem 1. Observe that

$$|r^{(m)}(n)| \leq 4d(n) = 0(n^{\varepsilon})$$

for every $\varepsilon>0$. Thus, since $c>1$, we may again apply the dominated convergence theorem.

Recall the definition of $\Lambda(n)$:

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, \\ 0, & \text{if } n \neq p^k, \end{cases}$$

where p is an arbitrary prime and k is a positive integer. Clearly,

$$\sum_{d\mid n} \Lambda(d) = \log n.$$

THEOREM 4. *Let*

$$G^{(m)}(z) = \sum_{d=1}^{m} \frac{\Lambda(d)}{d} S(d, z)$$

and

$$H^{(m)}(z) = \sum_{d=1}^{m} \frac{\Lambda(d)}{d} T(d, z).$$

Then if $c>1$,

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} \log |n| f(n) = -\lim_{m\to\infty} \sum_{z_j\in S} R\left\{\frac{\pi e^{-\pi i z}G^{(m)}(z)f(z)}{\sin(\pi z)}, z_j\right\}$$

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and

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{\infty} (-1)^n \log|n| f(n) = -\lim_{m\to\infty} \sum_{z_j\in S} R\left\{\frac{\pi H^{(m)}(z)f(z)}{\sin(\pi z)}, z_j\right\}.$$

3. **Examples.** For brevity, we confine our attention to the case $q=1$, $a=0$.

Let $f(z)=1/(z^2+a^2)$, $a\neq ni$, where n is an arbitrary integer. Apply (2.1). The residues at 0 and $\pm ai$, are respectively,

$$a^{-2}\sum_{d=1}^{m} d^{\nu}$$

and

$$-\frac{\pi}{2a\sinh(\pi a)}\sum_{d=1}^{m} d^{\nu-1}\sum_{j=0}^{d-1} e^{\pm\pi a\mp 2\pi aj/d}.$$

A straightforward calculation yields

$$\sum_{j=0}^{d-1}(e^{\pi a-2\pi aj/d}+e^{-\pi a+2\pi aj/d}) = 2\sinh(\pi a)\coth(\pi a/d). \tag{3.1}$$

Hence, by (2.1) if $\nu<1$,

$$\sum_{n=1}^{\infty}\frac{\sigma_\nu(n)}{n^2+a^2} = \frac{1}{2a^2}\sum_{d=1}^{\infty} d^{\nu}\left\{\frac{\pi a}{d}\coth(\pi a/d)-1\right\}.$$

By calculations similar to the above and each using (3.1), we have

$$\sum_{n=1}^{\infty}\frac{r(n)}{n^2+a^2} = \frac{2}{a^2}\sum_{d=1}^{\infty}\chi(d)\left\{\frac{\pi a}{d}\coth(\pi a/d)-1\right\}$$

and

$$\sum_{n=1}^{\infty}\frac{\log n}{n^2+a^2} = \frac{1}{2a^2}\sum_{d=1}^{\infty}\Lambda(d)\left\{\frac{\pi a}{d}\coth(\pi a/d)-1\right\}. \tag{3.2}$$

If $f(z)=z^{-t}(z^2+a^2)^{-2}$, (2.4) gives

$$\sum_{n=1}^{\infty}\frac{\varphi_{r,s,t}(n)}{n^t(n^2+a^2)} = \frac{1}{2a^2}\sum_{d=1}^{\infty}\frac{\mu_r^s(d)}{d^t}\left\{\frac{\pi a}{d}\coth(\pi a/d)-1\right\}. \tag{3.3}$$

In particular, since $\sum_{d=1}^{\infty}\mu(d)/d=0$,

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{n(n^2+a^2)} = \frac{\pi}{2a}\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}\coth(\pi a/d).$$

On the other hand, (2.5) yields

$$\sum_{n=1}^{\infty}\frac{(-1)^n r(n)}{n^2+a^2} = \frac{2}{a^2}\sum_{d=1}^{\infty}\chi(d)\left\{\frac{\pi a}{d}\operatorname{csch}(\pi a/d)-1\right\}.$$

Identities similar to the previous identity hold for the other arithmetical functions studied here.

We give a few additional miscellaneous examples for our theorems.

If $f(z)=1/z^2$,

$$\sum_{n=1}^{\infty}\frac{\log n}{n^2}=\frac{\pi^2}{6}\sum_{d=1}^{\infty}\frac{\Lambda(d)}{d^2},$$

which is well known [2, p. 253] and can also be obtained from (3.2) by letting a tend to 0. Similarly, if

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s},\qquad \text{Re } s>0,$$

we have

$$\sum_{n=1}^{\infty}\frac{r(n)}{n^2}=\frac{2\pi^2}{3}L(2,\chi),$$

which is again known [2, p. 256], and

$$\sum_{n=1}^{\infty}\frac{(-1)^n r(n)}{n^2}=\frac{-\pi^2}{3}L(2,\chi).$$

If $f(z)=1/z^{t+2}$,

$$\sum_{n=1}^{\infty}\frac{\varphi_{r,s,t}(n)}{n^{t+2}}=\frac{\pi^2}{6}\sum_{d=1}^{\infty}\frac{\mu_r^s(d)}{d^{t+2}},$$

which is well known if $r=s=t=1$ [2, p. 250]. This can also be obtained from (3.3) by letting a tend to 0.

Let $f(z)=1/(z^4+a^4)$, $z\neq\rho^{\pm 1}n$, where $\rho=\exp(\pi i/4)$ and n is an arbitrary integer. Then if $\nu<3$,

$$\sum_{n=1}^{\infty}\frac{\sigma_\nu(n)}{n^4+a^4}=\frac{1}{4a^4}\sum_{d=1}^{\infty}d^\nu\left\{\frac{\pi a\rho}{d}\cot(\pi a\rho/d)+\frac{\pi a\bar\rho}{d}\cot(\pi a\bar\rho/d)-2\right\}.$$

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University of Illinois,
Urbana, Illinois

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