RATIONAL APPROXIMATION WITH SERIES

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The Siegel conjecture on the rational approximation to algebraic numbers was proved a few years ago by K. F. Roth [1] with the following theorem:

Let α be any algebraic number, not rational. If

$$\left|\alpha - \frac{h}{q}\right| < \frac{1}{q^\kappa}$$

has an infinity of solutions in integers h and q $(q > 0)$, then $\kappa \leqq 2$.

This result, which gives a best-possible bound for κ, improved on earlier results of Liouville, Thue, Siegel, and Dyson.

The analogous problem of approximating to algebraic functions, with degree replacing absolute value, was considered by B. P. Gill [2], who obtained a result corresponding to that of Siegel. In this paper we improve on Gill's result by proving the analogue of Roth's theorem, so obtaining a best-possible result.

Let $\mathfrak{k}$ denote an arbitrary field of zero characteristic and z an indeterminate. Then the set $\mathfrak{K}$ of all formal Laurent series

$$x = \alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots,$$

where

$$\alpha_d, \alpha_{d-1}, \cdots \in \mathfrak{k},$$

is a field. Further, the sets $\mathfrak{T} = \mathfrak{k}[z]$ and $\mathfrak{R} = \mathfrak{k}(z)$ form a subring and a subfield of $\mathfrak{K}$ respectively, with

$$\mathfrak{K} \supseteqq \mathfrak{R} \supseteqq \mathfrak{T}.$$

If $x \in \mathfrak{K}$ then we denote by $\deg x$ the degree of x, i.e.

$$\begin{aligned} \deg x &= -\infty \text{ if } x \equiv 0, \\ &= d \text{ if } \alpha_d \text{ is the leading non-zero coefficient in } x \not\equiv 0. \end{aligned}$$

The n-dimensional space of all vectors $(x_1, x_2, \cdots x_n)$, where the $x_k \in \mathfrak{K}$, is denoted by $\mathfrak{P}_n$.

We prove the following theorem.

THEOREM 1.1. Let $t \in \mathfrak{K}$ be algebraic over $\mathfrak{T}$ but not in $\mathfrak{R}$. If

$$\deg\left(t - \frac{u}{v}\right) < -\nu \deg v \tag{1.2}$$

for infinitely many $u/v \in \mathfrak{R}$, then $\nu \leqq 2$.
[Note. It is taken throughout this paper that, in such a representation u/v of an element of $\mathfrak{R}$, the u, v are relatively prime elements of $\mathfrak{T}$.]

This result is clearly best-possible for ν. For if d is a positive integer there exists a non-trivial set $\alpha_d, \alpha_{d-1}, \cdots \alpha_0$ of $d+1$ elements of $\mathfrak{k}$ such that, if v is the polynomial

$$\alpha_d z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_0,$$

then the coefficients of z^{-i} in the product vt vanish for $i = 1, \cdots d$. Putting u equal to the polynomial part of vt we then have

$$\deg (vt - u) < -d \leqq -\deg v,$$

i.e.

$$\deg\left(t - \frac{u}{v}\right) < -2 \deg v.$$

Since t is not rational it follows that, by allowing d to range over all positive integers, we obtain an infinity of distinct solutions u/v of (1.2) with $\nu = 2$.

For the case where the ground field $\mathfrak{k}$ is of positive characteristic p Mahler [3] has shown that the equivalent bound for ν is $\nu \leqq p$, which is again best-possible.

In the proof of our theorem certain details are omitted because of the essential similarity between our case and that of Roth.

2. Let $x_1, \cdots x_m$ be m indeterminates and let

$$F = F(x_1, \cdots x_m) \in \mathfrak{K}[x_1, \cdots x_m],$$

i.e. F is a sum of terms of the form

$$a(i_1, \cdots i_m) x_1^{i_1} \cdots x_m^{i_m},$$

where the $a = a(i_1, \cdots i_m) \in \mathfrak{K}$. We extend the notation $\deg x$ defined above for $x \in \mathfrak{K}$ to include

$$\begin{aligned} \deg F &= -\infty \text{ if } F \equiv 0; \\ &= \max \{\deg a(i_1, \cdots i_m)\} \text{ if } F \not\equiv 0, \end{aligned}$$

where the maximum is taken over all non-zero a. Clearly this is consistent with our earlier notation, since it merely means that $z^{\deg F}$ is the largest power of z that occurs with non-zero coefficient in F. If F itself is in $\mathfrak{K}$ then the two notations agree.

Obviously, if

$$F'(x_1, \cdots x_m) \in \mathfrak{K}[x_1, \cdots x_m],$$

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then

$$\deg (F \pm F') \leqq \max \{\deg F, \deg F'\}$$

and

$$\deg (FF') \leqq \deg F + \deg F'.$$

We consider differential operators of the form

$$\Delta = \frac{\partial^{i_1+\cdots+i_m}}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}}$$

and call $(i_1 + \cdots + i_m)$ the order of Δ.

For a positive integer h, let

$$\phi_\beta(x_1, \cdots x_m) \in \mathfrak{R}[x_1, \cdots x_m] \qquad (\beta = 0, 1, \cdots h-1),$$

and let Δ_α, $(\alpha = 0, 1, \cdots h-1)$, be operators on $x_1, \cdots x_m$ of order at most α. Then we call the determinant

$$G(x_1, \cdots x_m) = \{\Delta_\alpha \phi_\beta(x_1, \cdots x_m)\}_{\alpha, \beta=0,1,\cdots h-1}$$

a generalized Wronskian of $\phi_0, \phi_1, \cdots \phi_{h-1}$.

LEMMA 2.1. The necessary and sufficient condition that

$$\phi_\beta(x_1, \cdots x_m) \in \mathfrak{R}[x_1, \cdots x_m] \qquad (\beta = 0, 1, \cdots h-1)$$

are linearly independent over $\mathfrak{T}$ is that at least one of their generalized Wronskians is non-zero.

LEMMA 2.2. Let $R(x_1, \cdots x_m)$ be a polynomial in $m \geqq 2$ variables, with coefficients in $\mathfrak{T}$, which is not identically zero. Let R be of degree at most r_j in x_j, $(j = 1, \cdots m)$. Then there exists an integer h satisfying

$$1 \leqq h \leqq r_m + 1$$

and there exist differential operators Δ_λ, $(\lambda = 0, 1, \cdots h-1)$, on the variables $x_1, \cdots x_{m-1}$, of order at most λ, such that, if

$$F(x_1, \cdots x_m) = \det \left\{\Delta_\lambda \frac{\partial^\mu R}{\partial x_m^\mu}\right\}_{\lambda, \mu=0,1,\cdots h-1}$$

then (i) F has coefficients in $\mathfrak{T}$ and is not identically zero;

(ii) $$F(x_1, \cdots x_m) = U(x_1, \cdots x_{m-1}) \cdot V(x_m),$$

where U and V have coefficients in $\mathfrak{T}$, U is of degree at most hr_j in x_j, $(j = 1, \cdots m-1)$, and V is of degree at most hr_m in x_m.

The proofs of these two lemmas are omitted as they are very similar to those of Roth.

With F, h and R as defined above we prove the following inequality.

LEMMA 2.3. $\deg F \leqq h \cdot \deg R$

PROOF. Put

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$$R_{\lambda,\mu} = \Delta_\lambda \frac{\partial^\mu R}{\partial x_m^\mu} \qquad (\lambda, \mu = 0, 1, \cdots h-1).$$

Now R is the sum of terms of the form

$$a(s_1, \cdots s_m) x_1^{s_1} \cdots x_m^{s_m}.$$

Differentiation with respect to any $x_1, \cdots x_m$ of such a term will not increase the degree of the coefficient $a(s_1, \cdots s_m)$. Hence

$$\deg R_{\lambda,\mu} \leqq \deg R \qquad (\lambda, \mu = 0, 1, \cdots h-1).$$

On the other hand, F is the sum of $h!$ terms, each of which is a product of the form

$$\pm R_{\lambda_0, 0} R_{\lambda_1, 1} \cdots R_{\lambda_{h-1}, h-1}.$$

It follows that

$$\deg F \leqq h \cdot \max \{\deg R_{\lambda,\mu}\},$$

where the maximum is taken over $\lambda, \mu = 0, 1, \cdots h-1$. Hence the assertion.

3. Let $P(x_1, \cdots x_m) \in \mathfrak{R}[x_1, \cdots x_m]$ and, further, let $a_1, \cdots a_m \in \mathfrak{R}$ and let $r_1, \cdots r_m$ be any positive numbers.

Definition 3.1.
The index

$$\theta\{P\} = \theta\{P; (a_1, \cdots a_m); r_1, \cdots r_m\}$$

of P at the point $(a_1, \cdots a_m) \in \mathfrak{P}_m$ relative to $r_1, \cdots r_m$ is put equal to $+\infty$ if $P \equiv 0$, otherwise

$$\theta\{P\} = \min \left\{ \frac{j_1}{r_1} + \cdots + \frac{j_m}{r_m} \right\}$$

for all sets of non-negative integers $j_1, \cdots j_m$ for which

$$\left\{ \frac{\partial^{j_1 + \cdots + j_m} P}{\partial x_1^{j_1} \cdots \partial x_m^{j_m}} \right\}_{(a_1, \cdots a_m)} \neq 0.$$

The index then has the following properties [$Q(x_1, \cdots x_m)$ being a second polynomial in $x_1, \cdots x_m$, and the indices being evaluated at $(a_1, \cdots a_m)$ relative to $r_1, \cdots r_m$].

(3.2) $\qquad \theta\{P\} \geqq 0, = 0$ if and only if $P(a_1, \cdots a_m) \neq 0.$

(3.3) $$\theta\{P + Q\} \geqq \min(\theta\{P\}, \theta\{Q\}).$$

(3.4) $$\theta\{PQ\} = \theta\{P\} + \theta\{Q\}$$

(3.5) If

$$Q = \frac{\partial^{k_1 + \cdots + k_m} P}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}} \text{ for some } k_1, \cdots k_m \geqq 0,$$

then

$$\theta\{Q\} \geqq \theta\{P\} - \left(\frac{k_1}{r_1} + \cdots + \frac{k_m}{r_m}\right).$$

Also, if P is actually a function of less than m of the variables $x_1, \cdots x_m$, say P is independent of x_m, then

$$\theta\{P; (a_1, \cdots a_m); r_1, \cdots r_m\} = \theta\{P; (a_1, \cdots a_{m-1}); r_1, \cdots r_{m-1}\}.$$

Hence, in particular, if P is a function of $x_1, \cdots x_{m-1}$ only and Q is a function of x_m only, then, from (3.4),

$$\begin{aligned} &\theta\{PQ; (a_1, \cdots a_m); r_1, \cdots r_m\} \\ &= \theta\{P; (a_1, \cdots a_{m-1}); r_1, \cdots r_{m-1}\} + \theta\{Q; (a_m); r_m\}. \end{aligned} \tag{3.6}$$

4. Let $r_1, \cdots r_m$ be m positive integers and let ρ be a non-negative number. We denote by

$$\mathfrak{B}_m = \mathfrak{B}_m(\rho; r_1, \cdots \ _m)$$

the set of all polynomials $R(x_1, \cdots x_m) \in \mathfrak{T}[x_1, \cdots x_m]$ which satisfy the conditions

(i) $R \not\equiv 0$;
(ii) R is of degree at most r_j in x_j, $(j = 1, \cdots m)$;
(iii) $\deg R \leqq \rho$.

Let $v_1, \cdots v_m \in \mathfrak{T}$ be of positive degree. We put

$$\Theta_m\{\rho; v_1, \cdots v_m; r_1, \cdots r_m\} = \sup \theta\left\{R; \left(\frac{u_1}{v_1}, \cdots \frac{u_m}{v_m}\right); \ r_1, \cdots r_m\right\},$$

where the supremum is taken over all $R \in \mathfrak{B}_m$ and over $u_1, \cdots u_m \in \mathfrak{T}$ satisfying $(u_i, v_i) = 1$, $(i = 1, \cdots m)$.

We now obtain an upper bound for Θ_m, by induction with respect to m. For $m = 1$ we have the following inequality.

LEMMA 4.1.

$$\Theta_1\{\rho; v_1; r_1\} \leqq \frac{\rho}{r_1 \cdot \deg v_1}.$$

Proof. By the definition of θ, the polynomial $R(x_1)$ is divisible by $(x_1 - u_1/v_1)r_1 \cdot \theta\{R\}$. Applying Gauss's theorem on factorization, we have

$$R(x_1) = (v_1 x_1 - u_1)^{r_1 \cdot \theta\{R\}} Q(x_1)$$

where $Q(x_1) \in \mathfrak{T}[x_1]$. The leading coefficient of R is therefore divisible by $v_1^{r_1 \cdot \theta\{R\}}$, whence

$$r_1 \theta\{R\} \deg v_1 \leqq \deg R \leqq \rho,$$

and the assertion follows.

LEMMA 4.2. Let $m \geqq 2$ be an integer and let $r_1, \cdots r_m$ be positive integers

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satisfying

$$r_m > 10\delta^{-1}, \quad r_{j-1} > r_j \delta^{-1} \quad \text{for } j = 2, \cdots m,$$

where $0 < \delta < 1$. Also, let $v_1, \cdots v_m \in \mathfrak{T}$ be of positive degree. Then

$$\Theta_m\{\rho; v_1, \cdots v_m; r_1, \cdots r_m\} \leqq 2 \cdot \max (\Phi + \Phi^{\frac{1}{2}} + \delta^{\frac{1}{2}}) \tag{4.3}$$

where the maximum is taken over all integers h satisfying

$$1 \leqq h \leqq r_m + 1,$$

and where

$$\Phi = \Theta_1\{h\rho; v_m; hr_m\} + \Theta_{m-1}\{h\rho; v_1, \cdots v_{m-1}; hr_1, \cdots hr_{m-1}\}. \tag{4.4}$$

We again omit the proof because of its similarity to that of Roth. Note that if

$$F(x_1, \cdots, x_m) = U(x_1, \cdots, u_{m-1}) \cdot V(x_m)$$

is the function defined in lemma 2.2 then

$$\max (\deg U, \deg V) \leqq \deg F \leqq h \deg R \leqq h\rho,$$

by lemma 2.3. It follows from this and lemma 2.2 that

$$U(x_1, \cdots x_{m-1}) \in \mathfrak{B}_{m-1}(h\rho; hr_1, \cdots hr_{m-1})$$

and

$$V(x_m) \in \mathfrak{B}_1(h\rho; hr_m).$$

We now restrict $\delta, v_1, \cdots v_m, r_1, \cdots r_m$, give ρ a particular value, and obtain an explicit upper bound for $\Theta_m\{\rho; v_1, \cdots v_m; r_1, \cdots r_m\}$ in terms of m and δ.

LEMMA 4.5. Let m be a positive integer and let δ satisfy

$$0 < \delta < m^{-1}.$$

Let $r_1, \cdots r_m$ be positive integers satisfying

$$r_m > 10\delta^{-1}, \quad r_{j-1} > r_j \delta^{-1} \quad \text{for } j = 2, \cdots m.$$

Let $v_1, \cdots v_m \in \mathfrak{T}$ have positive degree and satisfy

$$r_j \deg v_j \geqq r_1 \deg v_1 \qquad (j = 2, \cdots m). \tag{4.6}$$

Then

$$\Theta_m\{\delta r_1 \deg v_1; v_1, \cdots v_m; r_1, \cdots r_m\} < 10^m \delta^{(\frac{1}{2})^m}.$$

PROOF. If $m = 1$ then, by lemma 4.1,

$$\Theta_1\{\delta r_1 \deg v_1; v_1; r_1\} \leqq \frac{\delta r_1 \deg v_1}{r_1 \deg v_1} = \delta < 10\delta^{\frac{1}{2}}.$$

Assume, now, that $m \geqq 2$ and that the lemma holds if m is replaced by $m - 1$. Note that the hypothesis remains valid if we replace m by $m - 1$

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and r_j by hr_j, $(j = 1, \cdots m-1)$. Now, by lemma 4.1,

$$\Theta_1\{\delta hr_1 \deg v_1 ; v_m ; hr_m\} \leqq \frac{\delta hr_1 \deg v_1}{hr_m \deg v_m} \leqq \delta,$$

by (4.6). Hence, if Φ is the sum defined in (4.4), we have, by the inductive hypothesis,

$$\Phi < \delta + 10^{m-1} \cdot \delta^{(\frac{1}{2})^{m-1}} < 2(10^{m-1}\delta^{(\frac{1}{2})^{m-1}}).$$

Now the hypotheses of lemma 4.2 are less stringent than those of lemma 4.5. Hence lemma 4.2 is applicable and, by (4.3),

$$\begin{aligned} &\Theta_m\{\delta r_1 \deg v_1 ; v_1, \cdots v_m ; r_1, \cdots r_m\} \\ &\quad < 2\{2 \cdot 10^{m-1}\delta^{(\frac{1}{2})^{m-1}} + 2^{\frac{1}{2}}\, 10^{(m-1)/2}\delta^{(\frac{1}{2})^m} + \delta^{\frac{1}{2}}\} \\ &\quad < 2\left\{\frac{2}{10} + \frac{2^{\frac{1}{2}}}{10} + \frac{1}{10^2}\right\} 10^m \delta^{(\frac{1}{2})^m} \\ &\quad < 10^m \delta^{(\frac{1}{2})^m}. \end{aligned}$$

Thus lemma 4.5 holds for m and the induction is complete.

5. Lemma 5.1. Let $n \geqq 2$ and let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n, \quad \text{where } a_0 \neq 0,$$

and

$$g(x) = b_0x^s + b_1x^{s-1} + \cdots b_s$$

be two elements of $\mathfrak{T}[x]$, of degree α and β in z respectively. Suppose that d is a non-negative integer such that

$$d \geqq s - n + 1$$

and let $h(x) \in \mathfrak{T}[x]$ be of degree at most $(n-1)$ in x and satisfy

$$a_0^d g(x) \equiv h(x), \qquad \text{mod } f(x).$$

Then $h(x)$ is of degree at most $(\beta + d\alpha)$ in z.

Proof. If $s \leqq n-1$ the lemma is trivial. We complete the proof by induction on s.

Assume that $s \geqq n$, whence $d \geqq 1$, and assume that the lemma holds for $(s-1)$ instead of s.

Put

$$g^*(x) = a_0g(x) - b_0x^{s-n}f(x).$$

Then $g^*(x)$ is of degrees at most $(s-1)$ in x and at most $(\beta+\alpha)$ in z. Also

$$a_0^{d-1}g^*(x) \equiv a_0^d g(x) \equiv h(x), \qquad \text{mod } f(x).$$

Then, by the inductive hypothesis, $h(x)$ is of degree at most

$$(\beta + \alpha) + (d-1)\alpha = \beta + d\alpha$$

in z.

6. Let $t = t(z) \in \mathfrak{K}$ be algebraic, of degree at least 2, over $\mathfrak{T}$, and suppose that the inequality (1.2) is satisfied by infinitely many $u/v \in \mathfrak{R}$. Then we wish to show that $\nu \leqq 2$.

We may assume that t is of negative degree in z. For if not, let t' be the polynomial part of t, and put $t^* = t - t'$. Then t^* is also algebraic and of the same degree as t, and is of negative degree in z. Further u/v satisfies (1.2) if and only if

$$\deg\left(t^* - \frac{u'}{v}\right) < \nu \deg v,$$

where

$$u' = u - vt' \in \mathfrak{T}.$$

Now t is the root of some irreducible polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \in \mathfrak{T}[x],$$

where $a_0 \neq 0$, $n \geqq 2$. Let $f(x)$ be of degree $\alpha \geqq 0$ in z.

We now prove our final lemma.

Let m be a positive integer, and let $\delta, r_1, \cdots r_m, v_1, \cdots v_m$ satisfy the following conditions

(6.1) $$0 < \delta < \min(m^{-1}, \alpha^{-1}),$$

(6.2) $$10^m\delta^{(\frac{1}{4})^m} + 2(1+\delta)nm^{\frac{1}{2}} < \tfrac{1}{2}m,$$

(6.3) $$r_m > 10\delta^{-1}, \quad r_{j-1} > r_j\delta^{-1} \qquad (j = 2, \cdots m),$$

(6.4) $$\delta^2 \deg v_1 > m,$$

(6.5) $$r_j \deg v_j \geqq r_1 \deg v_1 \qquad (j = 2, \cdots m).$$

Note that these conditions are stricter than those of lemma 4.5. Define the integer ρ' by

$$\rho' \leqq \delta r_1 \deg v_1 < \rho' + 1,$$

whence, by (6.4),

(6.6) $$\rho' + 1 > \delta^{-1}r_1m.$$

Define the numbers λ, γ, η by

(6.7) $$\lambda = 4(1+\delta)nm^{\frac{1}{2}}$$

(6.8) $$\gamma = \tfrac{1}{2}(m - \lambda)$$

(6.9) $$\eta = 10^m\delta^{(\frac{1}{4})^m}$$

Note that (6.2) is then equivalent to

$$\eta < \gamma.$$

Lemma 6.10. Suppose that the conditions (6.1)—(6.5) are satisfied, and suppose that $u_1, \cdots u_m \in \mathfrak{T}$ are relatively prime to $v_1, \cdots v_m$ respectively. Then there exists a polynomial

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$$Q(x_1, \cdots x_m) \in \mathfrak{T}[x_1, \cdots x_m]$$

of degree at most r_j in x_j, $(j = 1, \cdots m)$, such that

(i) $\theta\{Q; (t, \cdots t); r_1, \cdots r_m\} \geqq \gamma - \eta$;

(ii) $Q\left(\frac{u_1}{v_1}, \cdots \frac{u_m}{v_m}\right) \neq 0$;

(iii) for all

$$Q_{i_1, \cdots i_m}(x_1, \cdots x_m) = \frac{\partial^{i_1+\cdots+i_m}}{\partial x_1^{i_1} \cdots \partial x_m^{i_m}} Q, \tag{6.11}$$

where $i_1, \cdots i_m$ are non-negative integers,

$$Q_{i_1, \cdots i_m}(t, \cdots t)$$

is of degree at most ρ' in z.

PROOF. We consider polynomials $W(x_1, \cdots x_m) \in \mathfrak{T}[x_1, \cdots x_m]$ of the form

$$W(x_1, \cdots x_m) = \sum_{d_0=0}^{\rho'} \sum_{d_1=0}^{r_1} \cdots \sum_{d_m=0}^{r_m} \xi(d_0, d_1, \cdots d_m) z^{d_0} x_1^{d_1} \cdots x_m^{d_m}.$$

Here the total number of coefficients $\xi(d_0, d_1, \cdots d_m) \in \mathfrak{k}$ is

$$(\rho' + 1)(r_1 + 1) \cdots (r_m + 1), = M \text{ say.} \tag{6.12}$$

Denote by $j^{(i)}$, $(i = 1, \cdots D)$, the D sets of integers $j_1, \cdots j_m$ satisfying

$$0 \leqq j_1 \leqq r_1, \cdots 0 \leqq j_m \leqq r_m \text{ and } \frac{j_1}{r_1} + \cdots + \frac{j_m}{r_m} \leqq \tfrac{1}{2}(m - \lambda).$$

By a result of Roth, ([1], lemma 8)

$$\begin{aligned} D &\leqq 2m^{\frac{1}{2}} \lambda^{-1}(r_1 + 1) \cdots (r_m + 1), \\ &= 2m^{\frac{1}{2}} \lambda^{-1}(\rho' + 1)^{-1} M \text{ by (6.12).} \end{aligned} \tag{6.13}$$

For $i = 1, \cdots D$, put

$$W_{j^{(i)}}(x_1, \cdots x_m) = \frac{\partial^{j_1+\cdots+j_m} W}{\partial x_1^{j_1} \cdots \partial x_m^{j_m}},$$

where $j^{(i)} = (j_1, \cdots j_m)$. Then, for each such derivative, we form the polynomial

$$W_{j^{(i)}}(x, \cdots x) \in \mathfrak{T}[x],$$

which is of degree at most $(r_1 + \cdots + r_m)$, $\leqq mr_1$, in x and, also, of degree at most ρ' in z.

Now, let

$$T_{j^{(i)}}(W; x) \in \mathfrak{T}[x]$$

be that element, of order at most $n - 1$ in x, which satisfies

$$a_0^{mr_1} W_{j^{(i)}}(x, \cdots x) \equiv T_{j^{(i)}}(W; x), \qquad \text{mod } f(x).$$

Since $mr_1 \geqq \max\{0, (mr_1 - n + 1)\}$, we have, by lemma 5.1,

$$\deg T_{j^{(i)}} \leqq \rho' + mr_1\alpha.$$

Hence, for a given $j^{(i)}$, the polynomial $T_{j^{(i)}}(W; x)$ is defined by at most

$$n(\rho' + mr_1\alpha + 1)$$

elements of $\mathfrak{k}$.
Therefore, for each W, the set $T_{j^{(i)}}(W; x)$, $(i = 1, \cdots D)$, is defined by at most

$$Dn(\rho' + mr_1\alpha + 1)$$

elements of $\mathfrak{k}$. Obviously these elements are combinations of the $\xi(d_0, d_1, \cdots d_m)$, the integers, and the known elements of $\mathfrak{k}$ involved in $f(x)$. However, they are linear and homogeneous in the unknowns $\xi(d_0, d_1, \cdots d_m)$ occurring in W. But

$$\begin{aligned} Dn(\rho' + mr_1\alpha + 1) &\leqq 2m^{\frac{1}{2}}Mn\lambda^{-1}\left(\frac{\rho' + mr_1\alpha + 1}{\rho' + 1}\right) \text{ by (6.13)},\\ &\leqq \frac{M}{2(1+\delta)}\left(1 + \frac{mr_1\alpha}{\rho' + 1}\right) \text{ by (6.7)},\\ &< M \text{ by (6.1) and (6.6)}. \end{aligned}$$

It follows that W may be chosen so that

$$T_{j^{(i)}}(W; x) \equiv 0, \quad \bmod f(x) \quad (i = 1, \cdots D).$$

Since $a_0 \neq 0$ we then have

$$W_{j^{(i)}}(x, \cdots x) \equiv 0, \quad \bmod f(x) \quad (i = 1, \cdots D)$$

and, since $f(t) = 0$ by definition of f, the derivatives $W_{j^{(i)}}(x_1, \cdots x_m)$ satisfy

$$W_{j^{(i)}}(t, \cdots t) = 0 \qquad (i = 1, \cdots D).$$

Hence

$$\begin{aligned} \theta\{W; t, \cdots t; r_1, \cdots r_m\} &\geqq \tfrac{1}{2}(m - \lambda)\\ &= \gamma \text{ by (6.8)}. \end{aligned}$$

Now, also,

$$W(x_1, \cdots x_m) \in \mathfrak{B}_m(\delta r_1 \deg v_1; r_1, \cdots r_m).$$

By lemma 4.5

$$\theta\left\{W; \left(\frac{u_1}{v_1}, \cdots \frac{u_m}{u_m}\right); r_1, \cdots r_m\right\} < \eta,$$

where η is defined in (6.9). Hence, there exists non-negative integers $k_1, \cdots k_m$ such that

$$\frac{k_1}{r_1} + \cdots + \frac{k_m}{r_m} < \eta$$

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and if

$$Q(x_1, \cdots x_m) = \frac{\partial^{k_1+\cdots+k_m} W}{\partial x_1^{k_1} \cdots \partial x_m^{k_m}}$$

then

$$Q\left(\frac{u_1}{v_1}, \cdots \frac{u_m}{v_m}\right) \neq 0.$$

Then, by (3.5),

$$\theta\{Q; (t, \cdots t); r_1, \cdots r_m\} \geqq \gamma - \eta$$

and so Q satisfies parts (i) and (ii) of the lemma.

It also satisfies part (iii). For both $Q(x_1, \cdots x_m)$ and the derivative $Q_{i_1, \cdots i_m}$ defined in (6.11) are clearly elements of $\mathfrak{T}[x_1, \cdots x_m]$ of degree at most ρ' in z. Then, since t is of negative degree,

$$\deg Q_{i_1, \cdots i_m}(t, \cdots t) \leqq \rho'.$$

This completes the proof of lemma 6.10.

7. Proof of Theorem 1.1. We suppose that $\nu > 2$ and that the inequality

$$\deg \left(t - \frac{u}{v}\right) < -\nu \deg v \tag{7.1}$$

has infinitely many solutions $u/v \in \mathfrak{R}$.

We can show (after Gill [2]) that for any integer $\mu \geqq 0$ there is at most one solution u/v of (7.1) for which $\deg v = \mu$. For suppose that r/s is also a solution, with $\deg s = \mu$. Then (7.1) implies

$$\deg (su - rv) < -\nu\mu + \deg s + \deg v = \mu(2 - \nu) \leqq 0$$

since $\mu \geqq 0$ and $\nu > 2$. But $r, s, u, v \in \mathfrak{T}$ whence $su - rv \in \mathfrak{T}$ and so is identically zero. Since $s, v \neq 0$ this implies that r/s and u/v are identical.

From this it follows that an infinity of solutions of (7.1) implies solutions for which $\deg v$ is arbitrarily large. We deduce a contradiction of this.

We first choose m so that

$$m > 4nm^{\frac{1}{2}}, \quad \text{and } 2m(m - 4nm^{\frac{1}{2}})^{-1} > \nu.$$

If δ is sufficiently small we then have

$$m - 4(1 + \delta)nm^{\frac{1}{2}} - 2\eta < 0,$$

which is the same as (6.2). We choose δ to satisfy also the inequality (6.1) and further to satisfy

$$\frac{2m(1 + 2\delta)}{m - 4(1 + \delta)nm^{\frac{1}{2}} - 2\eta} < \nu.$$

This inequality is equivalent to

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$$\frac{m(1+2\delta)}{\gamma-\eta} < \nu. \tag{7.2}$$

Now let u_1/v_1 be a solution of (7.1), with $(u_1, v_1) = 1$, and so that v_1 satisfies (6.4). Let $u_2/v_2, \cdots u_m/v_m$ be further solutions of (7.1) with $(u_i, v_i) = 1$, $(i = 2, \cdots m)$, such that

$$\deg v_j > 2\delta^{-1} \deg v_{j-1} \qquad (j = 2, \cdots m).$$

Now take r_1 to be an integer satisfying

$$r_1 \deg v_1 > 10\delta^{-1} \deg v_m,$$

and define $r_2, \cdots r_m$ by

$$\frac{r_1 \deg v_1}{\deg v_j} \leqq r_j < 1 + \frac{r_1 \deg v_1}{\deg v_j} \qquad (j = 2, \cdots m). \tag{7.3}$$

Then (6.5) is satisfied. Also

$$\frac{r_j \deg v_j}{r_1 \deg v_1} < 1 + \frac{\deg v_j}{r_1 \deg v_1} \leqq 1 + \frac{\deg v_m}{r_1 \deg v_1} < 1 + \frac{\delta}{10} < 1 + \delta. \tag{7.4}$$

The conditions (6.3) are satisfied, since

$$r_m \geqq \frac{r_1 \deg v_1}{\deg v_m} > 10\delta^{-1}$$

and

$$\frac{r_{j-1}}{r_j} > \frac{\deg v_j}{\deg v_{j-1}} \left(1 + \frac{\delta}{10}\right)^{-1} > \delta^{-1}.$$

Now let $Q(x_1, \cdots x_m) \in \mathfrak{T}[x_1, \cdots x_m]$ be the polynomial of lemma 6.10.

Since Q is of degree at most r_j in x_j, $(j = 1, \cdots m)$, and is non-zero for $x_i = u_i/v_i$, $(i = 1, \cdots m)$, we have

$$v_1^{r_1} \cdots v_m^{r_m} Q\left(\frac{u_1}{v_1}, \cdots \frac{u_m}{v_m}\right) \in \mathfrak{T}, \neq 0.$$

Thus,

$$\deg Q\left(\frac{u_1}{v_1}, \cdots \frac{u_m}{v_m}\right) \geqq -(r_1 \deg v_1 + \cdots + r_m \deg v_m) \geqq -m r_1 (1+\delta) \deg v_1, \tag{7.5}$$

by (7.4). On the other hand,

$$Q\left(\frac{u_1}{v_1}, \cdots \frac{u_m}{v_m}\right) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} Q_{i_1, \cdots i_m}(t, \cdots t) \cdot \left(\frac{u_1}{v_1} - t\right)^{i_1} \cdots \left(\frac{u_m}{v_m} - t\right)^{i_m}$$

where, by (i) of lemma 6.10, the terms with

$$\frac{i_1}{r_1} + \cdots + \frac{i_m}{r_m} < \gamma - \eta$$

vanish. In every other term we have

$$\deg\left(\left(\frac{u_1}{v_1}-t\right)^{i_1}\cdots\left(\frac{u_m}{v_m}-t\right)^{i_m}\right) < -\nu(i_1 \deg v_1 + \cdots + i_m \deg v_m)$$
$$\leqq -\nu r_1(\gamma-\eta)\deg v_1, \text{ by (7.3)}.$$

By (iii) of lemma 6.10, it follows that

$$\deg Q\left(\frac{u_1}{v_1},\cdots\frac{u_m}{v_m}\right) \leqq \rho' - \nu r_1(\gamma-\eta)\deg v_1$$
$$\leqq \delta r_1 \deg v_1 - \nu r_1(\gamma-\eta)\deg v_1.$$

Comparing this with (7.5) we have

$$\nu r_1(\gamma-\eta)\deg v_1 \leqq \delta r_1 \deg v_1 + (1+\delta) m r_1 \deg v_1,$$
$$< m(1+2\delta) r_1 \deg v_1$$

since $m \geqq 2$. Now $\deg v_1 \neq 0$, hence

$$\nu < \frac{m(1+2\delta)}{\gamma-\eta}$$

contrary to (7.2), and the proof is complete.

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