THE PERMANENT FUNCTION

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1. Introduction. Let X be an n-square matrix with elements in a field F. The permanent of X is defined by

$$\text{per } (X) = \sum_{\sigma} \prod_{k=1}^{n} x_{k\sigma(k)} \tag{1.1}$$

where σ runs over the symmetric group of permutations on $1, 2, \ldots, n$. This function makes its appearance in certain combinatorial applications **(13)**, and is involved in a conjecture of van der Waerden **(6; 9)**. Certain formal properties of per (X) are known **(1)**, and an old paper of Pólya **(12)** shows that for $n > 2$ one cannot multiply the elements of X by constants in any uniform way so as to convert the permanent into the determinant. In a subsequent paper we intend to investigate this problem for more general operations on X.

The purpose of this paper is to characterize those linear operations on matrices which leave the permanent unaltered. This problem and its generalizations have been considered for the determinant function by Frobenius **(3)** and Kantor **(5)**, later by Schur **(14)**, Morita **(11)**, Dieudonné **(2)**, Marcus and Moyls **(8)**, Marcus and Purves **(10)**, and Marcus and May **(7)**. In view of the result of Pólya **(12)**, it does not seem likely that many of the techniques used in the above papers can be used to investigate the permanent function. Most of these rely heavily on certain properties of the determinant function which are no longer valid for the permanent function. For example, it is in general false that per (AB) = per (A) per (B).

In § 2 we introduce some notation and state our main result. In § 3 we prove this result in a sequence of lemmas, some of which may be of interest in themselves.

2. Notation. Let $M_{m,n}$ denote the vector space of all $m \times n$ matrices over F, with the *natural basis* of *unit matrices* E_{ij}, $(i = 1, \ldots, m; j = 1, \ldots, n)$, where E_{ij} is the matrix with 1 in position (i, j) and 0 elsewhere. In the sequel r will denote an integer satisfying $2 \leqslant r \leqslant \min (m, n)$. A convenient notation for dealing with index sets is the following: $Q_{n,r}$ will denote the totality of strictly increasing sequences of integers satisfying $1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n$.

Received September 12, 1960.

*The work of this author was supported by the United States Air Force Contract no. AF49 (638)-776.

†The work of this author was supported by the National Research Council of Canada.

As usual, $\alpha = (i_1, \ldots, i_r) \in Q_{n,r}$ precedes $\beta = (j_1, \ldots, j_r) \in Q_{n,r}$ in the *lexicographic ordering*, $\alpha < \beta$, if there is a t such that $i_t < j_t$ and $i_s \leqslant j_s$ for all $s < t$.

Let $X \in M_{m,n}$. We define the rth *permanental compound* of X, denoted by $P_r(X)$, in $M_{\binom{m}{r},\binom{n}{r}}$ as follows: if $\omega = (i_1, \ldots, i_r) \in Q_{m,r}$ and $\tau = (j_1, \ldots, j_r) \in Q_{n,r}$, than the (ω, τ) entry (in the doubly lexicographic ordering) of $P_r(X)$ is $X_{\omega 1}$ where $X_{\omega 1}$ is the permanent of the matrix in $M_{r,r}$ whose (s, t) entry is

$$x_{i_s j_t}, \qquad (s, t = 1, \ldots, r).$$

If $x_\alpha = (x_{\alpha 1}, \ldots, x_{\alpha n})$, $(\alpha = 1, \ldots, r; r \leqslant n)$, are any vectors over F, we define $x_1 \vee \ldots \vee x_r$ to be the $\binom{n}{r}$-vector whose $\omega = (j_1, \ldots, j_r) \in Q_{n,r}$ co-ordinate is

$$\operatorname{per}\,(x_{\alpha j_\beta}), \qquad (\alpha = 1, \ldots, r;\ \beta = 1, \ldots, r)$$

in the lexicographic ordering. The notation $E_{\omega\tau}$ designates the (ω, τ) unit matrix in $M_{\binom{m}{r},\binom{n}{r}}$.

We denote the *rank* of X by $\rho(X)$, the *transpose* of X by X', the ith row of X by $X_{(i)}$ and the jth column of X by $X^{(j)}$. Let A be an $s \times t$ matrix, $0_{k,q}$ the $k \times 1$ zero matrix; if $k, q > 0$ we define

$$A \dotplus 0_{k,q} = \begin{vmatrix} A & 0_{s,q} \\ 0_{k,t} & 0_{k,q} \end{vmatrix};$$

if $k = 0$ or $q = 0$ we let $A \dotplus 0_{k,q}$ be

$$\begin{vmatrix} A & 0_{s,q} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} A \\ 0_{k,t} \end{vmatrix}$$

respectively. If $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are n-vectors, the symbols $u \perp v$ and $u \,||\, v$ will indicate respectively that $\sum u_i v_i = 0$ and that u and v are linearly dependent. If $C \in M_{m,n}$ and $X \in M_{m,n}$ we define the *Hadamard product* of C and X to be the matrix $Y = C * X \in M_{m,n}$ given by $y_{ij} = c_{ij} x_{ij}$, $(i = 1, \ldots, m; j = 1, \ldots, n)$.

Next, let T be a linear map of $M_{m,n}$ into itself and let P and Q be permutation matrices in $M_{m,m}$ and $M_{n,n}$ respectively. In the sequel we shall have occasion to use maps ϕ obtained from T as follows:

$$\phi(X) = PT(X)Q, \qquad \text{all } X \in M_{m,n}.$$

We shall say that such a map ϕ *is the same as T to within permutation.*

In the case $m = n = 2$ we shall need the special map Δ defined on $M_{2,2}$ as follows:

$$\begin{cases} \Delta(E_{ij}) = E_{ij} \quad \text{if} \quad i \leqslant j \\ \Delta(E_{21}) = -E_{21}. \end{cases} \tag{2.1}$$

Clearly if $X \in M_{2,2}$ then $\det(X) = \operatorname{per}(\Delta(X))$ where $\det(X)$ denotes the determinant of X. Moreover, $\Delta = \Delta^{-1}$, where Δ^{-1} denotes the inverse of Δ.

https://doi.org/10.4153/CJM-1962-013-4 Published online by Cambridge University Press

Our main results are contained in the

THEOREM. *Let T be a linear map of $M_{m,n}$ into itself, and let r be an integer satisfying* $2 \leqslant r \leqslant \min(m, n)$. *Suppose that the ground field F contains at least r elements, and is not of characteristic* 2. *Assume that there exists a non-singular linear map S_r of $M_{\binom{m}{r},\binom{n}{r}}$ into itself such that*

$$P_r(T(X)) = S_r(P_r(X)) \tag{2.2}$$

for all $X \in M_{m,n}$. Then, if $m + n \geqslant 5$, there are permutation matrices $P \in M_{m,m}$, $Q \in M_{n,n}$ and diagonal matrices $D \in M_{m,m}$, $L \in M_{n,n}$ such that for $m \neq n$,

$$T(X) = DPXQL \tag{2.3}$$

for all $X \in M_{m,n}$; if $m = n$ (> 2), T has the form (2.3) *or*

$$T(X) = DPX'QL \tag{2.4}$$

for all $X \in M_{m,n}$. If $m = n = 2$, we have

$$[\Delta T \Delta](X) = AXB \tag{2.5}$$

for all $X \in M_{2,2}$ or else

$$[\Delta T \Delta](X) = AX'B \tag{2.6}$$

for all $X \in M_{2,2}$, where $A \in M_{2,2}$, $B \in M_{2,2}$, and $\det(AB) \neq 0$.

We note here that in case $r = m = n > 2$ and $S_n = 1$, this result tells us that *the only linear operations which hold the permanent fixed*, that is,

$$\mathrm{per}(T(X)) = \mathrm{per}(X) \tag{2.7}$$

for all $X \in M_{n,n}$, must be obtainable (to within taking the transpose) by pre- and post-multiplication of X by diagonal matrices whose product has permanent 1 *together with pre- and post-multiplication of X by permutation matrices.*

3. Proofs.

LEMMA 1. *Let $X \in M_{m,n}$, let $Q \in M_{m,m}$ be a permutation matrix, and let $D \in M_{m,m}$ be a diagonal matrix. Then*

(a) $$P_r(QX) = P_r(Q)P_r(X)$$

(b) $$P_r(DX) = P_r(D)P_r(X)$$

(c) $$P_r(X') = (P_r(X))'$$

Proof. First note that if $x_u = (x_{u1}, \ldots, x_{un})$, $u = 1, 2, \ldots, r$ are any n-vectors, then $x_1 \vee \ldots \vee x_r = x_{\lambda(1)} \vee \ldots \vee x_{\lambda(r)}$ for any permutation λ on $1, 2, \ldots, r$. In particular, if $\omega = (i_1, \ldots, i_r) \in Q_{m,r}$ then

$$X_{(i_1)} \vee \ldots \vee X_{(i_r)} = X_{(\lambda(i_1))} \vee \ldots \vee X_{(\lambda(i_r))}$$

for any permutation λ on $i_1, i_2, \ldots, i_r$. This is an immediate consequence of the fact that the permanent of a matrix is unaltered by a row (or column) permutation. Let σ be the permutation corresponding to Q. The rows of QX are $X_{(\sigma(1))}, \ldots, X_{(\sigma(m))}$. Let e_k denote the unit vector (of appropriate length) with 1 in position k, 0 elsewhere. Now row ω of $P_r(Q)$ is

$$e_{\sigma(i_1)} \vee \ldots \vee e_{\sigma(i_r)}.$$

Let $i_{\alpha_1}, \ldots, i_{\alpha_r}$ be the rearrangement of $i_1, i_2, \ldots, i_r$ such that

$$\sigma(i_{\alpha_1}) < \sigma(i_{\alpha_2}) < \ldots < \sigma(i_{\alpha_r}).$$

Then

$$e_{\sigma(i_1)} \vee \ldots \vee e_{\sigma(i_r)} = e_{\sigma(i_{\alpha_1})} \vee \ldots \vee e_{\sigma(i_{\alpha_r})}$$

is the unit $\binom{m}{r}$-vector with 1 in position

$$(\sigma(i_{\alpha_1}), \ldots, \sigma(i_{\alpha_r})) \in Q_{m,r}$$

and zero elsewhere. Thus row ω of $P_r(Q)P_r(X)$ is

$$X_{(\sigma(i_{\alpha_1}))} \vee \ldots \vee X_{(\sigma(i_{\alpha_r}))} = X_{(\sigma(i_1))} \vee \ldots \vee X_{(\sigma(i_r))},$$

which is obviously row ω of $P_r(QX)$. Thus (a) is established.

Let $\tau = (j_1, \ldots, j_r) \in Q_{n,r}$. Then row τ of $P_r(X')$ is

$$X^{(j_1)} \vee \ldots \vee X^{(j_r)}.$$

On the other hand, row τ of $(P_r(X))'$ is column τ of $P_r(X)$ which is again clearly

$$X^{(j_1)} \vee \ldots \vee X^{(j_r)}.$$

Thus (c) is proved.

Let σ_k be the diagonal element in row k of D. Let $\omega = (i_1, \ldots, i_r) \in Q_{m,r}$. Now $P_r(D)$ is again a diagonal matrix whose diagonal element in row ω is

$$\sigma_{i_1} \cdot \sigma_{i_2} \cdot \ldots \cdot \sigma_{i_r}.$$

Part (b) follows at once from the fact that the permanent function is linear in each row (and column). In particular,

$$(\sigma_{i_1} \cdot \sigma_{i_2} \cdot \ldots \cdot \sigma_{i_r}) X_{(i_1)} \vee \ldots \vee X_{(i_r)} = \sigma_{i_1} X_{(i_1)} \vee \ldots \vee \sigma_{i_r} X_{(i_r)}$$

which is row ω of $P_r(DX)$. The lemma is proved.

COROLLARY. *Let $X \in M_{m,n}$, let Q be a permutation matrix in $M_{n,n}$, and let D be a diagonal matrix in $M_{n,n}$. Then*

(a′) $$P_r(XQ) = P_r(X)P_r(Q)$$

(b′) $$P_r(XD) = P_r(X)P_r(D).$$

Proof. An identical computation proves both (a′) and (b′). We prove (a′).

$$\begin{aligned} P_r(XQ) &= (P_r((XQ)'))' = (P_r(Q'X'))' \\ &= (P_r(Q')P_r(X'))' = ((P_r(Q))'(P_r(X))')' = P_r(X)P_r(Q). \end{aligned}$$

LEMMA 2. *T is non-singular.*

Proof. Suppose that $T(U) = 0$. Then for any $X \in M_{m,n}$ we have, from (2.2),

$$S_r(P_r(U + X)) = P_r(T(U + X)) = P_r(T(U) + T(X)) = P_r(T(X)) = S_r(P_r(X)).$$

Since S_r is non-singular,

$$P_r(U + X) = P_r(X) \tag{3.1}$$

holds for all $X \in M_{m,n}$. For any permutation matrices P and Q of appropriate sizes, Lemma 1 and its corollary tell us that

$$\begin{aligned} P_r(PUQ + PXQ) &= P_r(P(U + X)Q) \\ &= P_r(P)P_r(U + X)P_r(Q) = P_r(P)P_r(X)P_r(Q) \\ &= P_r(PXQ). \end{aligned}$$

Now as X runs over $M_{m,n}$ so does PXQ. It suffices then to show that (3.1) implies $u_{11} = 0$. Choose $X \in M_{m,n}$ such that

$$\begin{aligned} x_{11} &= 0 \\ x_{kk} &= t - u_{kk}, && 2 \leqslant k \leqslant r \\ x_{ij} &= -u_{ij}, && i \neq j \text{ and } 1 \leqslant i, j \leqslant r \\ x_{ij} &= 0, \end{aligned}$$

otherwise. Then the (1, 1) entry of $P_r(U + X)$ is $u_{11}t^{r-1}$. On the other hand, the (1, 1) entry of $P_r(X)$ is a polynomial in t of degree at most $r - 2$. Since F contains at least r elements, we conclude that $u_{11} = 0$.

LEMMA 3. *Let s be an integer satisfying $1 \leqslant s \leqslant \min(m, n)$. Then there is a basis for $M_{\binom{m}{s},\binom{n}{s}}$ of the form*

$$P_s(X), \qquad X \in M_{m,n}.$$

Proof. Let $\omega = (i_1, \ldots, i_s) \in Q_{m,s}$ and let $\tau = (j_1, \ldots, j_s) \in Q_{n,s}$. If $X \in M_{m,n}$ is the matrix with

$$x_{i_t j_t} = 1, \qquad t = 1, \ldots, s$$

and $x_{\alpha\beta} = 0$ otherwise, then clearly $P_r(X) = E_{\omega\tau}$.

LEMMA 4. *There exists a non-singular linear map S_2 of $M_{\binom{m}{2},\binom{n}{2}}$ such that*

$$P_2(T(X)) = S_2(P_2(X)) \tag{3.2}$$

for all $X \in M_{m,n}$. That is, if (2.2) holds for $r > 2$, it holds for $r = 2$ as well.

Proof. Let $Y = T(X)$. Using (2.2) we can write

$$Y_{\omega\tau} = \sum_{\substack{\alpha \in Q_{m,r} \\ \beta \in Q_{n,r}}} s_{\omega,\tau}^{\alpha,\beta} X_{\alpha\beta} \tag{3.3}$$

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for any $\omega \in Q_{m,r}$ and $\tau \in Q_{n,r}$. In (3.3) the scalars

$$s_{\omega,\tau}^{\alpha,\beta}$$

are the entries in the matrix representation of S_r with respect to the natural basis in $M_{\binom{m}{r},\binom{n}{r}}$, ordered doubly lexicographically. Since T is non-singular we may write

$$x_{st} = \sum_{p=1,q=1}^{m,n} g_{s,t}^{p,q} y_{pq}$$

where the scalars

$$g_{s,t}^{p,q}$$

are the entries in the matrix representation of T^{-1} with respect to the natural basis in $M_{m,n}$. Now (3.3) may be regarded as a polynomial identity in the variables y_{ij}.

We compute that

$$\begin{aligned}\frac{\partial Y_{\omega\tau}}{\partial y_{pt}} &= \sum_{\substack{\alpha\in Q_{m,r}\\ \beta\in Q_{n,r}}} s_{\omega,\tau}^{\alpha,\beta} \frac{\partial X_{\alpha\beta}}{\partial y_{pt}} \\ &= \sum_{\substack{\alpha\in Q_{m,r}\\ \beta\in Q_{n,r}}} s_{\omega,\tau}^{\alpha,\beta} \sum_{u=1,v=1}^{m,n} \frac{\partial x_{uv}}{\partial y_{pt}} \frac{\partial X_{\alpha\beta}}{\partial x_{uv}} \\ &= \sum_{\substack{\alpha\in Q_{m,r}\\ \beta\in Q_{n,r}}} \sum_{u=1,v=1}^{m,n} (s_{\omega,\tau}^{\alpha,\beta} \cdot g_{u,v}^{p,t}) \frac{\partial X_{\alpha\beta}}{\partial x_{uv}},\end{aligned}$$

where we take $p \in \omega$ and $t \in \tau$. Now

$$s_{\omega,\tau}^{\alpha,\beta} \cdot g_{u,v}^{p,t},$$

the coefficient of

$$\frac{\partial X_{\alpha\beta}}{\partial x_{uv}}$$

in the last expression of this equation, is a scalar independent of X and Y. We conclude that any $(r-1)$-order permanental minor of $Y = T(X)$ is expressible as a fixed linear combination of the $(r-1)$-order permanental minors of X. In other words, there is a linear map R_0 of $M_{\binom{m}{r-1},\binom{n}{r-1}}$ into itself such that

(3.4) $$P_{r-1}(T(X)) = R_0(P_{r-1}(X))$$

for all $X \in M_{m,n}$. Since T is non-singular, we see from (2.2) that

(3.5) $$P_r(T^{-1}(X)) = S_r^{-1}(P_r(X))$$

for all $X \in M_{m,n}$. By the above reasoning applied to (3.5) we conclude that there is a linear map R^0 of $M_{\binom{m}{r-1},\binom{n}{r-1}}$ into itself such that for all $X \in M_{m,n}$

$$P_{r-1}(T^{-1}(X)) = R^0(P_{r-1}(X)).$$

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That is, for all $X \in M_{m,n}$ we have

$$P_{r-1}(X) = R^0(P_{r-1}(T(X))). \tag{3.6}$$

Combining (3.4) and (3.6) we have

$$P_{r-1}(X) = R^0R_0(P_{r-1}(X)) \tag{3.7}$$

for all $X \in M_{m,n}$. Lemma 3, with $s = r - 1$, tells us that R^0R_0 is the identity map of $M_{\binom{m}{r-1},\binom{n}{r-1}}$ onto itself. Consequently R_0 is non-singular in (3.4), and we set $S_{r-1} = R_0$. Then, using (3.4), we proceed to reduce $r - 1$ to $r - 2$, etc., finally obtaining (3.2).

Let $A \in M_{m,n}$. If A has at most one non-zero row (column), we shall call A a *row* (column) *matrix*. If A is a row (column) matrix, then the number of non-zero entries in A will be denoted by $h(A)$.

LEMMA 5. *Let $A \in M_{m,n}$, and suppose that $P_2(A) = 0$. Then $\rho(A) = 0, 1$, or 2. Moreover, if A has rank 1 then A is a row (or column) matrix; if A has rank 2, then to within permutation of the rows and columns of A, A has the form*

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \dotplus 0_{m-2,n-2}, \tag{3.8}$$

where $\alpha\delta + \beta\gamma = 0$, $\alpha\delta - \beta\gamma \neq 0$.

Proof. Assume that $A \neq 0$. Suppose first that $\rho(A) = 1$. We may assume without loss of generality that row t of A is some multiple of a fixed vector $z = (z_1, z_2, \ldots, z_n)$, say $c_t z$, $t = 1, 2, \ldots, m$. Since $P_2(A) = 0$, we see that $2c_t c_s z_i z_j = 0$ if $t \neq s$ and $i \neq j$. Since F is not of characteristic 2, we have $c_t c_s z_i z_j = 0$ if $t \neq s$ and $i \neq j$. Since $A \neq 0$, some $c_{t_0} \neq 0$ and some $z_{i_0} \neq 0$. If there is $j \neq i_0$ for which $z_{i_0} z_j \neq 0$, then $c_s = 0$ whenever $s \neq t_0$.

Suppose next that $\rho(A) > 1$. By a suitable permutation we may bring A to the form

$$A = \begin{vmatrix} \alpha & \beta & a_1 & a_2 \ldots a_{n-2} \\ \gamma & \delta & b_1 & b_2 \ldots b_{n-2} \\ c_1 & d_1 & & \\ c_2 & d_2 & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \text{H} \\ \cdot & \cdot & & \\ c_{m-2} & d_{m-2} & & \end{vmatrix}$$

where $H \in M_{m-2,n-2}$, $\alpha\delta \neq 0$, $\alpha\delta - \beta\gamma \neq 0$.

We have

$$\left.\begin{aligned} \alpha b_t + \gamma a_t = 0 \\ \beta b_t + \delta a_t = 0 \end{aligned}\right\}, \qquad t = 1, \ldots, n-2$$

$$\left.\begin{aligned} \alpha d_s + \beta c_s = 0 \\ \gamma d_s + \delta c_s = 0 \end{aligned}\right\}, \qquad s = 1, \ldots, m-2$$

But $\alpha\delta - \beta\gamma \neq 0$. Hence $c_s = d_s = 0$, $s = 1, \ldots, m-2$ and $a_t = b_t = 0$, $t = 1, \ldots, n-2$. Therefore $\delta h_{ij} = 0$ for each element h_{ij} of H. Since $\delta \neq 0$ we have $H = 0$. Also, we note that $\alpha\beta\gamma\delta \neq 0$. This proves Lemma 5.

COROLLARY. *Let $F_{ij} = T(E_{ij})$. Then $\rho(F_{ij}) = 1$ or 2.*

Proof. From (3.2) we see that

$$P_2(F_{ij}) = S_2(P_2(E_{ij})) = S_2(0) = 0.$$

Lemma 5, together with its corollary, enables us to describe partially the structure of the images F_{ij} of the unit matrices E_{ij} in $M_{m,n}$.

Lemmas 6 and 7 are devoted to obtaining the exact structure of F_{ij}.

LEMMA 6.* $\rho(F_{ij}) = 1$.

Proof. Since T is non-singular, $F_{ij} \neq 0$. Suppose that $\rho(F_{ij}) = 2$. We lose no generality in assuming that $i = j = 1$ and that F_{ij} has the form (3.8). Consider F_{1t}, $2 \leqslant t \leqslant n$. Since $P_2(E_{11} + \lambda E_{1t}) = 0$, all λ, we have $P_2(F_{11} + \lambda F_{1t}) = 0$, all λ. Since $\alpha\beta\gamma\delta \neq 0$ in (3.8) we see at once that $h(F_{1t}) = 2$ if $\rho(F_{1t}) = 1$, and moreover, F_{1t} is zero outside positions (1, 1), (1, 2), (2, 1), (2, 2). If $\rho(F_{1t}) = 2$ then by letting λ vary over F we see again that F_{1t} is zero outside these same positions. A similar argument leads to the same conclusions concerning F_{s1}, $s = 2, \ldots, m$.

We next show that F_{1t}, $t = 1, \ldots, n$ and F_{s1}, $s = 1, \ldots, m$ all lie in a space spanned by the following three matrices;

$$G_1 = \begin{vmatrix} \alpha & \beta \\ 0 & 0 \end{vmatrix} \dotplus 0_{m-2,n-2}$$

$$G_2 = \begin{vmatrix} 0 & \beta \\ 0 & \delta \end{vmatrix} \dotplus 0_{m-2,n-2}$$

$$G_3 = \begin{vmatrix} \alpha & 0 \\ \gamma & 0 \end{vmatrix} \dotplus 0_{m-2,n-2}$$

Observe that

$$G_4 = \begin{vmatrix} 0 & 0 \\ \gamma & \delta \end{vmatrix} \dotplus 0_{m-2,n-2} = G_2 + G_3 + G_1,$$

and that

$$F_{11} = G_1 + G_4 = G_2 + G_3.$$

First let us assume that $\rho(F_{1t}) = 1$. We may further assume without loss of generality that $b_{11}b_{21} \neq 0$ and $b_{12} = b_{22} = 0$, where

$$F_{1t} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \dotplus 0_{m-2,n-2}.$$

*The authors are indebted to B. N. Moyls for simplifying the original proof of this lemma.

Then $P_2(F_{11} + F_{1t}) = 0$ implies that $b_{11}\delta + b_{21}\beta = 0$, and hence F_{1t} is a multiple of G_3. For $(b_{11}, b_{21}) \perp (\delta, \beta) \perp (\alpha, \gamma)$, whence $(b_{11}, b_{21}) || (\alpha, \gamma)$. Next assume that $\rho(F_{1t}) = 2$. We have $b_{11}b_{12}b_{21}b_{22} \neq 0$ and $P_2(F_{11} + F_{1t}) = 0$ shows that $b_{11}\delta + b_{22}\alpha + b_{12}\gamma + b_{21}\beta = 0$. Now $\alpha\delta + \beta\gamma = b_{11}b_{22} + b_{12}b_{21} = 0$. So there are non-zero constants c and d such that $\gamma = c\alpha$, $\delta = -c\beta$, $b_{21} = d\,b_{11}$, $b_{22} = -db_{12}$. Consequently we have $0 = b_{11}\delta + b_{22}\alpha + b_{12}\gamma + b_{21}\beta = (c-d)(\alpha b_{12} - \beta b_{11})$. Thus either $c = d$ or $\alpha b_{12} = \beta b_{11}$. If $c = d$ we have $(b_{12}, b_{22}) \perp (\gamma, \alpha) \perp (\beta, \delta)$ and $(b_{11}, b_{21}) \perp (\delta, \beta) \perp (\alpha, \gamma)$, whence $(b_{12}, b_{22}) || (\beta, \delta)$ and $(b_{11}, b_{21}) || (\alpha, \gamma)$. Therefore if $c = d$ we can find constants k and λ such that $F_{1t} = kG_2 + \lambda G_3$. In case $\alpha b_{12} = \beta b_{11}$ we conclude similarly that there are constants k_0 and λ_0 such that $F_{1t} = k_0 G_1 + \lambda_0 G_4$. Thus the matrices F_{1t}, $t = 1, \ldots, n$ and similarly, the matrices F_{s1}, $s = 1, \ldots, m$, all lie in a space of dimension 3 spanned by G_1, G_2, and G_3. But $m + n - 1 > 3$. We have thus contradicted Lemma 2. Hence $\rho(F_{ij}) = 1$.

Lemmas 5 and 6 tell us that each F_{ij} is either a row or column matrix.

LEMMA 7. $h(F_{ij}) = 1$.

Proof. We lose no generality in assuming that $i = j = 1$ and that F_{11} is a row matrix with its non-zero row in row 1. By a suitable permutation of columns we may assume that row 1 of F_{11} has the form $(a_1, a_2, \ldots, a_h, 0, 0, \ldots, 0)$ where we have set $h = h(F_{11})$ for brevity. Then

$$\prod_{t=1}^{h} a_t \neq 0.$$

If $h \geqslant 3$, then F_{1t}, $t = 1, \ldots, n$ and F_{s1}, $s = 1, \ldots, m$ would all be row matrices lying in row 1. This is an immediate consequence of Lemma 5, for we have $P_2(F_{11} + F_{1t}) = P_2(F_{11} + F_{s1}) = 0$. Since $m + n - 1 > n$, we have contradicted Lemma 2.

Suppose then that $h = 2$. We have

$$F_{11} = [a_1 a_2] \dotplus 0_{m-1,n-2}$$

with $a_1 a_2 \neq 0$. We first show that F_{12} is a row matrix lying in row 1. If not, then by permuting the last $m - 1$ rows, we can take F_{12} in the form

$$\begin{vmatrix} 0 & 0 \\ b_1 & b_2 \end{vmatrix} \dotplus 0_{m-2,n-2} \tag{3.9}$$

where $b_1 b_2 \neq 0$ and $a_1 b_2 + a_2 b_1 = 0$. We next remark that

$$P_2(T^2(X)) = S_2(P_2(T(X))) = S_2^2(P_2(X)) \tag{3.10}$$

for all $X \in M_{m,n}$. Consequently all our results concerning the nature of T apply equally well to T^2. In particular, $T^2(E_{11})$ is either a row matrix or a column matrix. But

$$T^2(E_{11}) = T(F_{11}) = T(a_1 E_{11} + a_2 E_{12}) = a_1 F_{11} + a_2 F_{12} = \begin{vmatrix} a_1^2 & a_1 a_2 \\ a_2 b_1 & a_2 b_2 \end{vmatrix} \dotplus 0_{m-2,n-2}.$$

However, $b_2a_1a_2b_1 \neq 0$. Thus F_{12} is a row matrix lying in row 1. Consider now F_{1t}, $t > 2$. If F_{1t} is not a row matrix lying in row 1, we may clearly assume that F_{1t} has the form (3.9). Then again, $(a_1, a_2) \perp (b_2, b_1)$. From $P_2(F_{12} + F_{1t}) = 0$ we see immediately that F_{12} has the form

$$[c_1c_2] \dotplus 0_{m-1,n-2}$$

with $c_1c_2 \neq 0$. So $(c_1, c_2) \perp (b_2, b_1)$. But this implies that F_{1t} is a multiple of F_{11} and we contradict Lemma 2.

Now by Lemma 2, F_{21} cannot lie in row 1. We may assume, from $P_2(F_{11} + F_{21}) = 0$, that F_{21} has the form (3.9). By an argument exactly analogous to that given above, we see that each of F_{2t}, $t = 1, \ldots, n$ is a row matrix lying in row 2. There are two cases left to consider.

(i) $m = 2, \quad n \geqslant 3$
(ii) $m \geqslant 3$.

In case (i) there is $j_0 > 1$ such that F_{ij_0} has a non-zero entry in column 3. Now from

$$P_2(F_{ij_0} - F_{2j_0}) = 0$$

we see that the non-zero entries of F_{2j_0} lie in precisely the same columns as do those of F_{ij_0}. Moreover,

$$h(F_{ij_0}) = h(F_{2j_0}) \leqslant 2.$$

But

$$P_2(E_{11} + E_{21} + \lambda E_{ij_0} - \lambda E_{2j_0}) = 0,$$

all λ. Consequently

$$P_2(F_{11} + F_{21} + \lambda F_{ij_0} - \lambda F_{2j_0}) = 0,$$

all λ. This contradicts Lemma 5.

If $m \geqslant 3$, case (ii), then note that $P_2(E_{11} + E_{21} + \lambda E_{31}) = 0$, all λ. Then $P_2(F_{11} + F_{21} + \lambda F_{31}) = 0$, all λ shows that, by Lemma 5, F_{31} lies in the first two rows. This contradicts Lemma 2 once again. Thus $h = 1$.

Lemma 7 tells us that if $m + n \geqslant 5$, we have $T(E_{ij}) = c_{ij}E_{i'j'}$. By Lemma 2, $c_{ij} \neq 0$, and, moreover, $(i, j) \neq (s, t)$ implies that $(i', j') \neq (s', t')$. We set $i' = \mu(i, j)$ and $j' = \lambda(i, j)$, so that $T(E_{ij}) = c_{ij}E_{\mu(i,j)\lambda(i,j)}$.

LEMMA 8. *Let $m + n \geqslant 5$. If $m \neq n$, then there are permutation matrices $P \in M_{m,m}$ and $Q \in M_{n,n}$, and a matrix $C = (c_{ij}) \in M_{m,n}$ with $c_{ij} \neq 0$ all i, j, such that for all $X \in M_{m,n}$*

$$T(X) = C*(PXQ). \tag{3.11}$$

If $m = n$ (> 2) then T has the form (3.11) *or else*

$$T(X) = C*(PX'Q) \tag{3.12}$$

for all $X \in M_{m,n}$.

Proof. We may assume without loss of generality that $m \leqslant n$. Now by a suitable permutation of T we may assume that $\mu(1, 1) = 1$ and $\lambda(1, 1) = 1$. Now $P_2(E_{11} + E_{22}) \neq 0$ shows that $P_2(F_{11} + F_{22}) \neq 0$ and so $\mu(2, 2) > 1$, $\lambda(2, 2) > 1$. By a suitable permutation of the last $n - 1$ rows and $n - 1$ columns, we may assume that $\mu(2, 2) = 2$ and $\lambda(2, 2) = 2$. Similarly $P_2(E_{11} + E_{33}) \neq 0$ and $P_2(E_{22} + E_{33}) \neq 0$ shows that $\mu(3, 3) > 2$, $\lambda(3, 3) > 2$ and we may assume that $\mu(3, 3) = 3$, $\lambda(3, 3) = 3$. Proceeding in this fashion, it is clear that we may assume $\mu(i, i) = \lambda(i, i) = i$, $i = 1, 2, \ldots, m$.

Fix $\alpha \leqslant m$, $\beta \leqslant m$ so that $\alpha \neq \beta$. Now $P_2(E_{\alpha\alpha} + E_{\alpha\beta}) = 0$ implies that $\mu(\alpha, \beta) = \alpha$ or $\lambda(\alpha, \beta) = \alpha$. Also $P_2(E_{\beta\beta} + E_{\alpha\beta}) = 0$ shows that $\mu(\alpha, \beta) = \beta$ or $\lambda(\alpha, \beta) = \beta$. Therefore either

$$\mu(\alpha, \beta) = \alpha \quad \text{and} \quad \lambda(\alpha, \beta) = \beta, \tag{3.13}$$

or

$$\mu(\alpha, \beta) = \beta \quad \text{and} \quad \lambda(\alpha, \beta) = \alpha \tag{3.14}$$

for the non-singularity of T shows that we cannot have $\mu(\alpha, \beta) = \lambda(\alpha, \beta)$.

Suppose first that (3.13) holds. Let $\gamma \leqslant n$, $\gamma \neq \alpha$, $\gamma \neq \beta$. From $P_2(E_{\alpha\beta} + E_{\alpha\gamma}) = 0$ we have $\mu(\alpha, \gamma) = \alpha$ or $\lambda(\alpha, \gamma) = \beta$. From $P_2(E_{\alpha\alpha} + E_{\alpha\gamma}) = 0$ we have $\mu(\alpha, \gamma) = \alpha$ or $\lambda(\alpha, \gamma) = \alpha$. It follows that $\mu(\alpha, \gamma) = \alpha$. We see that if $\gamma \neq \alpha$, $\gamma \neq \beta$, $\gamma \leqslant n$ then $\mu(\alpha, \gamma) = \alpha$, under the hypothesis (3.13). If in addition we have $\gamma \leqslant m$ then $P_2(E_{\alpha\gamma} + E_{\gamma\gamma}) = 0$ shows that $\mu(\alpha, \gamma) = \gamma$ or $\lambda(\alpha, \gamma) = \gamma$. Hence $\lambda(\alpha, \gamma) = \gamma$.

Let $k \neq \alpha$ and consider $E_{k\beta}$. From $P_2(E_{\alpha\beta} + E_{k\beta}) = 0$ we conclude that $\mu(k, \beta) = \alpha$ or $\lambda(k, \beta) = \beta$. Now $\mu(k, \beta) \neq \alpha$ because $\mu(\alpha, t) = \alpha$, $t = 1, \ldots, n$ and T is non-singular. Hence $\lambda(k, \beta) = \beta$. But $P_2(E_{kk} + E_{k\beta}) = 0$ shows that $\mu(k, \beta) = k$ or $\lambda(k, \beta) = k$. Hence $\mu(k, \beta) = k$. Consequently $u(k, \beta) = k$, $\lambda(k, \beta) = \beta$. If we repeat this argument now with k replacing α in (3.13) we conclude that

$$\mu(i, j) = i, \qquad \lambda(i, j) = j, \qquad (i = 1, \ldots, m; j = 1, \ldots, m) \tag{3.15}$$

Moreover, if $j > m$, the non-singularity of T ensures that $\lambda(i, j) > m$. Now we already know that $\mu(i, j) = i$ for such j. Furthermore,

$$P_2(E_{i_1 j} + E_{i_2 j}) = 0$$

shows that $\lambda(i_1, j) = \lambda(i_2, j)$. Thus, if (3.13) holds, T may be reduced to the form (3.11) by a suitable permutation of the last $n - m$ columns of X.

Suppose next that (3.14) holds. We show that actually $m = n$ and that

$$\mu(i, j) = j, \qquad \lambda(i, j) = i, \qquad (1 \leqslant i, j \leqslant m). \tag{3.16}$$

From $P_2(E_{\alpha\beta} + E_{\alpha k}) = 0$ we have $\mu(\alpha, k) = \beta$ or $\lambda(\alpha, k) = \alpha$. Also $P_2(E_{\alpha\alpha} + E_{\alpha k}) = 0$ shows that $\mu(\alpha, k) = \alpha$ or $\lambda(\alpha, k) = \alpha$. It follows that $\lambda(\alpha, k) = \alpha$, $k = 1, \ldots, n$, because $\alpha \neq \beta$. Thus $m = n$, for T maps row α, an n-dimensional space, into column α, an m-dimensional space.

https://doi.org/10.4153/CJM-1962-013-4 Published online by Cambridge University Press

We conclude also, from $P_2(E_{\alpha k} + E_{kk}) = 0$, that $\mu(\alpha, k) = k$ or $\lambda(\alpha, k) = k$. Since $\lambda(\alpha, k) = \alpha$ it follows that $\mu(\alpha, k) = k$, $k = 1, \ldots, m$. Thus (3.16) is established and T has the form (3.12).

LEMMA 9. $\rho(C) = 1$.

Proof. Let $1 \leqslant i < s \leqslant m$, $1 \leqslant j < t \leqslant n$. If (3.11) holds, choose X so that $PXQ = E_{ij} + E_{it} - E_{sj} + E_{st}$, and if (3.12) holds, choose X so that $PX'Q = E_{ij} + E_{it} - E_{sj} + E_{st}$. In either case, $P_2(X) = 0$ shows that $P_2(T(X)) = P_2(c_{ij}E_{ij} + c_{it}E_{it} - c_{sj}E_{sj} + c_{st}E_{st}) = 0$. Hence $c_{ij}c_{st} - c_{it}c_{sj} = 0$. Thus each second-order subdeterminant of C vanishes.

Using Lemma 9 we can write that $c_{ij} = d_i q_j$, $(i = 1, \ldots, m; j = 1, \ldots, n)$. We set $D = \operatorname{diag}(d_1, \ldots, d_m) \in M_{m,m}$, $L = \operatorname{diag}(q_1, \ldots, q_n) \in M_{n,n}$. By Lemma 8 we can write (2.3) for $m \neq n$ and (2.3) or (2.4) for $m = n$ (> 2). The proof of the theorem is complete for the case $m + n \geqslant 5$.

Suppose that $m = n = 2$. Then (2.2) reduces to the equation

$$\operatorname{per}(T(X)) = \alpha \operatorname{per}(X)$$

for all $X \in M_{2,2}$, where α is some non-zero scalar in F. Using (2.1) we see that

$$\begin{aligned} \det[\Delta T \Delta(X)] &= \operatorname{per}[\Delta^2 T \Delta(X)] = \operatorname{per}[T\Delta(X)] \\ &= \alpha \operatorname{per}[\Delta(X)] = \alpha \det[X] \text{ for all } X \in M_{2,2}. \end{aligned}$$

Now $\det[\Delta T \Delta(X)] = \alpha \det[X]$ for all $X \in M_{2,2}$ shows that $\Delta T \Delta$ preserves the rank of each matrix in $M_{2,2}$; moreover, $(\Delta T \Delta)^{-1} = \Delta T^{-1} \Delta$ exists and has the same property. Consequently, we may appeal to a theorem of Jacob **(4)** to conclude that $\Delta T \Delta$ has the desired form. The proof of the theorem is complete.

We note that if $m \neq n$, we have

$$P_r(T(X)) = P_r(DPXQL) = P_r(D)P_r(P)P_r(X)P_r(Q)P_r(L) = S_r(P_r(X))$$

for all $X \in M_{m,n}$. By Lemma 3 it follows that

$$S_r(Y) = P_r(D)P_r(P)YP_r(Q)P_r(L) = D_0P_0YQ_0L_0 \tag{3.17}$$

for all $Y \in M_{\binom{m}{r},\binom{n}{r}}$, and S_r has the same form as T. If $m = n$ (> 2) and if T has the form (2.4), then $S_r(Y) = D_0P_0Y'Q_0L_0$ for all Y in $M_{\binom{m}{r},\binom{n}{r}}$. Consequently if $m = n$ (> 2) then S_r has the form (3.17) or else

$$S_r(Y) = D_0P_0Y'Q_0L_0 \tag{3.18}$$

for all $Y \in M_{\binom{m}{r},\binom{n}{r}}$.

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