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Homology of the Fermat Tower and Universal Measures for Jacobi Sums

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Abstract. We give a precise description of the homology group of the Fermat curve as a cyclic module over a group ring. As an application, we prove the freeness of the profinite homology of the Fermat tower. This allows us to define measures, an equivalent of Anderson's adelic beta functions, in a manner similar to Ihara's definition of ℓ-adic universal power series for Jacobi sums. We give a simple proof of the interpolation property using a motivic decomposition of the Fermat curve.

1 Introduction

Let X_N be the Fermat curve over $\mathbb{C}$ defined by $x_0^N + y_0^N = z_0^N$ and put $G_N = (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}$. By identifying $\mathbb{Z}/N\mathbb{Z}$ with the group of N-th roots of unity, G_N acts on X_N and the homology group $H_1(X_N, \mathbb{Z})$ becomes a module over the group ring $\mathbb{Z}[G_N]$. Rohrlich [15] proved that it is a cyclic module generated by the Pochhammer contour (written $\kappa^{r,s}$ in this paper, see Section 2), and gave generators of the annihilator. As far as the author knows, Guàrdia [7] and Kamata [10] were the first to give a $\mathbb{Z}$-basis of $H_1(X_N, \mathbb{Z})$ and compute its intersection numbers. In this paper, we compute the intersection numbers among $\kappa^{r,s}$ and construct a basis from them (Theorem 4.1). While the papers cited above use topological arguments, we start by computing the cup products of a standard basis of $H^1(X_N, \mathbb{C})$ (written $\omega^{a,b}$, see Section 2) represented by rational 1-forms (this computation was used in [13], [14]). Then we determine the $\mathbb{Z}$-module structure of the annihilator (Proposition 4.4), giving a precise description of $H_1(X_N, \mathbb{Z})$.

If N divides M, the natural maps $X_M \to X_N$ and $G_M \to G_N$ are compatible and the projective limit $H_1(X_\infty, \widehat{\mathbb{Z}}) := \varprojlim_N H_1(X_N, \mathbb{Z}/N\mathbb{Z})$ becomes a module over the completed group ring $\widehat{\mathbb{Z}}[[G_\infty]] := \varprojlim_N \widehat{\mathbb{Z}}[G_N]$. We will prove the following (see Theorem 6.1).

Theorem 1.1 *The group $H_1(X_\infty, \widehat{\mathbb{Z}})$ is a free cyclic module over $\widehat{\mathbb{Z}}[[G_\infty]]$.*

This has an application to the theory of Ihara [8] and Anderson [3] (see also [2, 5, 9]). For a prime number ℓ, the freeness of $H_1(X_{\ell^\infty}, \mathbb{Z}_\ell) := \varprojlim_n H_1(X_{\ell^n}, \mathbb{Z}/\ell^n\mathbb{Z})$ over $\mathbb{Z}_\ell[[G_{\ell^\infty}]] := \varprojlim_n \mathbb{Z}_\ell[G_{\ell^n}]$ was proved by Ihara [8] in his study of the pro-ℓ fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$. The freeness was used to define the ℓ-adic universal

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power series for Jacobi sums. Anderson [3] defined his adelic beta functions by using the freeness over $\widehat{\mathbb{Z}}[[G_\infty]]$ of the profinite relative homology of affine Fermat curves (proved in [1]). While $\mathbb{Z}_\ell[[G_{\ell^\infty}]]$ is isomorphic to a power series ring of two variables (see Remark 6.3), $\widehat{\mathbb{Z}}[[G_\infty]]$ does not have such a simple expression, and we need a precise study of the annihilators.

We use Theorem 1.1 to define universal measures for Jacobi sums, which agree up to "cyclotomic units" with Anderson's adelic beta functions. Since $H_1(X_N, \widehat{\mathbb{Z}})$ is isomorphic to the Tate module of the Jacobian variety, or to the étale homology by Artin's theorem, and X_N is defined over $\mathbb{Q}$, it has an action of the absolute Galois group $G_\mathbb{Q}$. By Theorem 1.1, the $G_\mathbb{Q}$-action is converted to an action of $\widehat{\mathbb{Z}}[[G_\infty]]^\times$, and we obtain a continuous map $G_\mathbb{Q} \to \widehat{\mathbb{Z}}[[G_\infty]]^\times$; $\sigma \mapsto F_\sigma$. This map is in fact a cocycle in the sense of group cohomology. Each F_σ is naturally regarded as a $\widehat{\mathbb{Z}}$-valued measure on G_∞. Then it defines a function $(\mathbb{Q}/\mathbb{Z})^{\oplus 2} \to (\mathbb{Q}^{\mathrm{ab}} \otimes \widehat{\mathbb{Z}})^\times$ by integrating a character of G_∞ indexed by $(\alpha, \beta) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ (see Section 7). Our measures F_σ interpolate all the Jacobi sums $j_N^{a,b}(v)$ (see Section 8 for the definition). The following is a special case of Theorem 8.1.

Theorem 1.2 *Let $v \nmid N$ be a prime of $\mathbb{Q}(\mu_N)$ and let $\sigma \in G_{\mathbb{Q}(\mu_N)}$ be a lift of Frobenius at v. For any $a, b \in \mathbb{Z}/N\mathbb{Z}$ with $a, b, a+b \neq 0$, we have*

$$\pi_\ell\left(F_\sigma\left(\frac{a}{N}, \frac{b}{N}\right)\right) = j_N^{a,b}(v) \otimes 1$$

for any $\ell \neq \mathrm{char}(\mathbb{F}_v)$, where $\pi_\ell \colon \mathbb{Q}^{\mathrm{ab}} \otimes \widehat{\mathbb{Z}} \to \mathbb{Q}^{\mathrm{ab}} \otimes \mathbb{Z}_\ell$ denotes the projection.

When N is a power of ℓ, the ℓ-part of the formula above is due to Ihara [8]. The general case can be deduced from Anderson's result [3] on values of his hyperadelic gamma functions (see Remark 8.2). So our novelty lies in the simplicity of the proof. Instead of the Tate modules of the Jacobians, we will use the motivic decomposition of the Fermat curves studied in [12], whose essential idea goes as far back as Weil [16]. An advantage here is that we only need to treat projective curves and the usual homology (without boundary).

This paper proceeds as follows. In Section 2, we introduce the cycles $\kappa^{r,s}$ and also cycles $\gamma^{r,s}$ with $\mathbb{Q}$-coefficients. Then we give a basis of $H_1(X_N, \mathbb{Q})$ and its annihilator. In Section 3, we compute the cup products among $\omega^{a,b}$ and the intersection numbers among $\kappa^{r,s}$ and $\gamma^{r,s}$. In Section 4, we give a basis of the integral homology $H_1(X_N, \mathbb{Z})$, reprove Rohrlich's cyclicity, and determine the structure of the annihilator. In Section 5, we study the homology and relative homology of affine Fermat curves. In Section 6, we let N vary and obtain the freeness of the projective limit. In Section 7, we define our measures and discuss how they relate to the Ihara–Anderson theory. In Section 8, we prove the interpolation of Jacobi sums.

2 Cycles and Periods

Fix an integer $N > 2$, and let $X = X_N$ and $G = G_N$ be as in the introduction. By the Riemann–Hurwitz formula, the genus of X is $(N-1)(N-2)/2$. We write an element $(r, s) \in G$ also as $g^{r,s}$ and the addition multiplicatively, *i.e.*, $g^{r,s} g^{r',s'} = g^{r+r',s+s'}$. We fix a primitive N-th root of unity $\zeta = \exp(2\pi i/N)$ and let G act on X by

$$g^{r,s}(x_0 : y_0 : z_0) = (\zeta^r x_0 : \zeta^s y_0 : z_0).$$

The commutative group ring $\mathbb{Z}[G]$ acts on the homology (resp., cohomology) groups by the push-forward (resp., pull-back). Define a path $\delta: [0,1] \to X$ by

$$\delta(t) = \left(t^{1/N} : (1-t)^{1/N} : 1\right) \in X(\mathbb{R}).$$

If we put

$$\kappa^{r,s} = (1 - g^{r,0})(1 - g^{0,s})\delta,$$

it becomes a cycle and defines a class in $H_1(X, \mathbb{Z})$, which we denote by the same letter. By definition, $\kappa^{r,0} = \kappa^{0,s} = 0$. Define elements of $H_1(X, \mathbb{Q})$ by[1]

$$\gamma = \frac{1}{N^2} \sum_{(r,s)\in G} \kappa^{r,s}, \quad \gamma^{r,s} = g^{r,s}\gamma \ ((r,s) \in G).$$

Define elements of $\mathbb{Z}[G]$ by

$$\mathbf{t} = \sum_{r,s} g^{r,s}, \quad \mathbf{v} = \sum_s g^{0,s}, \quad \mathbf{h} = \sum_r g^{r,0}, \quad \mathbf{d} = \sum_r g^{r,r}.$$

There are obvious relations:

$$\begin{aligned} &\mathbf{t} = \mathbf{vh} = \mathbf{hd} = \mathbf{dv}, \quad \mathbf{v}^2 = N\mathbf{v}, \quad \mathbf{h}^2 = N\mathbf{h}, \quad \mathbf{d}^2 = N\mathbf{d}, \\ &\mathbf{t}g^{r,s} = \mathbf{t}, \quad \mathbf{v}g^{r,s} = \mathbf{v}g^{r,0}, \quad \mathbf{h}g^{r,s} = \mathbf{h}g^{0,s}, \quad \mathbf{d}g^{r,s} = \mathbf{d}g^{r-s,0}. \end{aligned} \tag{2.1}$$

Define an element of $\mathbb{Q}[G]$ by

$$\mathbf{p} = \frac{1}{N^2} \sum_{r,s} (1 - g^{r,0})(1 - g^{0,s}) = \frac{1}{N^2}(N^2 - N\mathbf{h} - N\mathbf{v} + \mathbf{t}).$$

Then, by definition, $\gamma = \mathbf{p}\delta$. Using (2.1), one easily sees that $\mathbf{p}$ is an idempotent, *i.e.*, $\mathbf{p}^2 = \mathbf{p}$. In particular, $\mathbf{p}\gamma = \gamma$ and $\mathbf{p}\gamma^{r,s} = \gamma^{r,s}$. We can recover $\kappa^{r,s}$ from γ; since $(1 - g^{r,0})(1 - g^{0,s}) = (1 - g^{r,0})(1 - g^{0,s})\mathbf{p}$ by (2.1), we have

$$\kappa^{r,s} = (1 - g^{r,0})(1 - g^{0,s})\gamma. \tag{2.2}$$

For $a \in \mathbb{Z}/N\mathbb{Z}$, let $\langle a \rangle \in \{0, 1, \dots, N-1\}$ denote its representative. For each $(a,b) \in G$, define a rational differential 1-form on X by

$$\omega^{a,b} = x^{\langle a \rangle} y^{\langle b \rangle - N} \frac{dx}{x} = -x^{\langle a \rangle - N} y^{\langle b \rangle} \frac{dy}{y} = -u^{N - (\langle a \rangle + \langle b \rangle)} v^{\langle b \rangle - N} \frac{du}{u}, \tag{2.3}$$

where $x = x_0/z_0$, $y = y_0/z_0$, $u = z_0/x_0$, $v = y_0/x_0$. It is an eigenform for the G-action: $g^{r,s}\omega^{a,b} = \zeta^{ar+bs}\omega^{a,b}$. Define a subset of G by

$$I = \left\{ (a,b) \in G \mid a, b, a+b \neq 0 \right\}.$$

It is clear from the above expression that $\omega^{a,b}$ is a differential of the second kind if and only if $(a,b) \in I$, and is of the first kind if and only if $\langle a \rangle + \langle b \rangle < N$. For $(a,b) \in I$, we denote the cohomology class of $\omega^{a,b}$ in $H^1(X, \mathbb{C})$ by the same letter. Almost by definition, we have for $(a,b) \in I$,

$$\Omega^{a,b} := \int_\delta \omega^{a,b} = \frac{1}{N} B\left(\tfrac{\langle a \rangle}{N}, \tfrac{\langle b \rangle}{N}\right)$$

[1]The use of γ was suggested by Kenichi Bannai.

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where $B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ is the beta function ([15, 17]).

Lemma 2.1 (i) *For any $(a,b)\in I$, we have $\int_\gamma \omega^{a,b}=\Omega^{a,b}$.*
(ii) *The set $\{\omega^{a,b}\mid(a,b)\in I\}$ is a basis of $H^1(X,\mathbb{C})$.*

Proof By adjointness, we have $\int_\gamma \omega^{a,b}=\int_\delta \mathbf{p}\omega^{a,b}$ and (i) follows, since

$$\mathbf{p}\omega^{a,b}=\frac{1}{N^2}\sum_{r,s}(1-\zeta^{ar})(1-\zeta^{bs})\omega^{a,b}=\omega^{a,b}.$$

Since $\Omega^{a,b}>0$, $\omega^{a,b}\in H^1(X,\mathbb{C})$ is non-trivial. Since $\omega^{a,b}$ are proper vectors for the G-action with different characters and $|I|=\dim H^1(X,\mathbb{C})$, we obtain (ii). ∎

Let $\mathrm{Ann}(X,\mathbb{Q})\subset\mathbb{Q}[G]$ be the annihilator of $H_1(X,\mathbb{Q})$. Put

$$\mathbf{v}_r=\mathbf{v}g^{r,0},\quad \mathbf{h}_s=\mathbf{h}g^{0,s},\quad \mathbf{d}_t=\mathbf{d}g^{t,0}\quad (r,s,t\in\mathbb{Z}/N\mathbb{Z}).$$

Note that $\mathbf{t}=\sum_r\mathbf{v}_r=\sum_s\mathbf{h}_s=\sum_t\mathbf{d}_t$.

Proposition 2.2 (i) *The group $H_1(X,\mathbb{Q})$ is a cyclic $\mathbb{Q}[G]$-module generated by γ.*
(ii) *The set $\{\mathbf{t},\mathbf{v}_r,\mathbf{h}_s,\mathbf{d}_t\mid r,s,t\neq 0\}$ is a basis of* $\mathrm{Ann}(X,\mathbb{Q})$.
(iii) *The set $\{\gamma^{r,s}\mid(r,s)\in I\}$ is a basis of $H_1(X,\mathbb{Q})$.*

Proof Since $\int_{\gamma^{r,s}}\omega^{a,b}=\zeta^{ar+bs}\Omega^{a,b}$, the set

$$\Big\{\frac{1}{N^2}\sum_{r,s}\zeta^{-(ar+bs)}\gamma^{r,s}\;\Big|\;(a,b)\in I\Big\}$$

is a basis of $H_1(X,\mathbb{C})$ dual to $\{(\Omega^{a,b})^{-1}\omega^{a,b}\mid(a,b)\in I\}$, hence (i) follows. We show that $\mathbf{t},\mathbf{v}_r,\mathbf{h}_s,\mathbf{d}_t\in\mathrm{Ann}(X,\mathbb{Q})$. Using (2.1), one sees easily that $\mathbf{tp}=\mathbf{v}_r\mathbf{p}=\mathbf{h}_s\mathbf{p}=0$, hence $\mathbf{t}\gamma=\mathbf{v}_r\gamma=\mathbf{h}_s\gamma=0$. On the other hand, since

$$\int_{\mathbf{d}_t\gamma}\omega^{a,b}=\zeta^{at}\sum_r\zeta^{(a+b)r}\Omega^{a,b}=0$$

for any $(a,b)\in I$, we have $\mathbf{d}_t\in\mathrm{Ann}(X,\mathbb{Q})$. Since the numbers of elements in the set of (ii) and $\{g^{r,s}\mid(r,s)\in I\}$ sum up to $N^2=\dim\mathbb{Q}[G]$, we are left to show their independence. Suppose that

$$a\mathbf{t}+\sum_r b_r\mathbf{v}_r+\sum_s c_s\mathbf{h}_s+\sum_t d_t\mathbf{d}_t=\sum_{r,s}e^{r,s}g^{r,s}\quad(a,b_r,c_s,d_t,e^{r,s}\in\mathbb{Q})$$

with $b_0=c_0=d_0=0$ and $e^{r,s}=0$, unless $(r,s)\in I$. First, $e^{0,0}=0$ (resp., $e^{r,0}=0$, $e^{0,r}=0$, $e^{r,-r}=0$) implies $a=0$ (resp., $b_r+d_r=0$, $c_r+d_{-r}=0$, $b_r+c_{-r}+d_{2r}=0$), so we have $d_{2r}=2d_r$ for all r. Let $N=2^eM$ with odd M. If $M=1$, we have $2^ed_r=d_{2^er}=d_0=0$, hence $d_r=0$. If $M>1$, let f be a positive integer such that $2^f\equiv 1 \mod M$, so that $2^{e+f}\equiv 2^e \mod N$. Then we have $2^fd_{2^er}=d_{2^{e+f}r}=d_{2^er}$, which implies $d_{2^er}=2^ed_r=0$. In any case, we have $d_r=0$, and hence $b_r=c_r=0$ for all r. Therefore, $e^{r,s}=0$ for all r, s, and we are done. ∎

Corollary 2.3 *For any $m\in\mathbb{Z}/N\mathbb{Z}$, we have $\sum_{r\in\mathbb{Z}/N\mathbb{Z}}\kappa^{r,r+m}=N\gamma$. In particular, $N\gamma\in H_1(X,\mathbb{Z})$.*

Proof Since $\sum_{r\in\mathbb{Z}/N\mathbb{Z}}(1-g^{r,0})(1-g^{0,r+m}) = N - \mathbf{v} - \mathbf{h} + \mathbf{d}_{-m}$, this follows from (2.2) and Proposition 2.2 (ii). ∎

3 Intersection Numbers

The general reference for this section is [6, Chap. 0]. For $u, v \in H_1(X,\mathbb{Z})$, let $u \sharp v \in \mathbb{Z}$ denote the intersection number. Let

$$\langle\cdot,\cdot\rangle: H_1(X,\mathbb{C}) \times H^1(X,\mathbb{C}) \longrightarrow \mathbb{C}$$

be the canonical pairing. By the Poincaré duality, for any $u \in H_1(X,\mathbb{Z})$, there is a unique element $\eta_u \in H^1(X,\mathbb{C})$ satisfying $\langle u,\omega\rangle = \deg(\eta_u \cup \omega)$ for all $\omega \in H^1(X,\mathbb{C})$, where $\cup$ is the cup product and

$$\deg: H^2(X,\mathbb{C}) \xrightarrow{\simeq} \mathbb{C}, \quad \varphi \longmapsto \frac{1}{2\pi i}\int_X \varphi$$

is the degree map. Then we have

$$u \sharp v = \frac{1}{2\pi i}\deg(\eta_u \cup \eta_v) = \frac{1}{2\pi i}\langle u, \eta_v\rangle. \tag{3.1}$$

If ω, η are differential forms of the second kind, then we have

$$\deg(\omega \cup \eta) = \sum_{P\in X} \operatorname{Res}_P\Big(\eta \int \omega\Big). \tag{3.2}$$

Here, $\int \omega$ is a primitive function of ω on a small neighbourhood of P, and Res_P denotes the residue at P, which does not depend on the choice of $\int \omega$.

Proposition 3.1 *For any $(a,b),(c,d) \in I$, we have $\deg(\omega^{a,b} \cup \omega^{c,d}) = 0$ unless $(c,d) = (-a,-b)$, and*

$$\deg(\omega^{a,b} \cup \omega^{-a,-b}) = \frac{1}{\frac{\langle a\rangle}{N} + \frac{\langle b\rangle}{N} - 1}.$$

Proof The first assertion is clear, since

$$\omega^{a,b} \cup \omega^{c,d} = g^{r,s}\omega^{a,b} \cup g^{r,s}\omega^{c,d} = \zeta^{(a+c)r+(b+d)s}\omega^{a,b} \cup \omega^{c,d}$$

for all $(r,s) \in G$. By (2.3), both $\omega^{a,b}$ and $\omega^{-a,-b}$ are holomorphic except at $R_n = (1:\xi\zeta^n:0)$ $(n \in \mathbb{Z}/N\mathbb{Z})$, where we put $\xi = \exp(\pi i/N)$. At R_n, $u = z_0/x_0$ is a local parameter, and we have

$$\int \omega^{a,b} = \frac{-1}{N - (\langle a\rangle + \langle b\rangle)} u^{N-(\langle a\rangle+\langle b\rangle)} v^{\langle b\rangle - N}, \quad \omega^{-a,-b} = -u^{\langle a\rangle+\langle b\rangle-N} v^{-\langle b\rangle}\frac{du}{u}.$$

Therefore, we have

$$\operatorname{Res}_{R_n}\Big(\omega^{-a,-b}\int \omega^{a,b}\Big) = \frac{1}{N-(\langle a\rangle+\langle b\rangle)}(\xi\zeta^n)^{-N} = \frac{-1}{N-(\langle a\rangle+\langle b\rangle)}$$

for each n, hence the second assertion follows using (3.2). ∎

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Lemma 3.2 *Let $\gamma \in H_1(X, \mathbb{Q})$ be as defined in Section 2. Then we have*

$$\eta_\gamma = \sum_{(a,b)\in I} \Big(\tfrac{\langle a \rangle}{N} + \tfrac{\langle b \rangle}{N} - 1 \Big) \Omega^{-a,-b} \omega^{a,b}.$$

Proof By Lemma 2.1, we can write $\eta_\gamma = \sum_{(a,b)\in I} c^{a,b} \omega^{a,b}$. By Proposition 3.1 and (3.1), we have

$$\Omega^{-a,-b} = \langle \gamma, \omega^{-a,-b} \rangle = c^{a,b} \deg(\omega^{a,b} \cup \omega^{-a,-b}) = \frac{c^{a,b}}{\frac{\langle a \rangle}{N} + \frac{\langle b \rangle}{N} - 1}.$$

Hence, the lemma follows. ∎

Proposition 3.3 *For any $(r,s) \in G$, we have:*

$$2N^2(\gamma^{r,s} \sharp \gamma) = \begin{cases} 0 & \text{if } r = 0, s = 0, \\ (N-1)(2\langle s \rangle - N) & \text{if } r = 0, s \neq 0, \\ (N-1)(2\langle r \rangle - N) & \text{if } r \neq 0, s = 0, \\ 2(N - \langle r \rangle - \langle s \rangle) & \text{if } r \neq 0, s \neq 0. \end{cases}$$

Proof First, we have

$$\gamma^{r,s} \sharp \gamma = \frac{1}{2\pi i} \langle \gamma^{r,s}, \eta_\gamma \rangle = \frac{1}{2\pi i} \sum_{(a,b)\in I} c^{a,b} \langle \gamma^{r,s}, \omega^{a,b} \rangle$$

with $c^{a,b}$ as in the proof of the lemma above. Then, by the compatibility of the G-action on homology and cohomology under the pairing, we have

$$\langle \gamma^{r,s}, \omega^{a,b} \rangle = \langle \gamma, g^{r,s} \omega^{a,b} \rangle = \zeta^{ar+bs} \langle \gamma, \omega^{a,b} \rangle = \zeta^{ar+bs} \Omega^{a,b}.$$

Hence, we have

$$\gamma^{r,s} \sharp \gamma = \frac{1}{2\pi i} \sum_{(a,b)\in I} \zeta^{ar+bs} \Big(\tfrac{\langle a \rangle}{N} + \tfrac{\langle b \rangle}{N} - 1 \Big) \Omega^{-a,-b} \Omega^{a,b}.$$

By using the functional equations

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad \Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \quad \Gamma(1 - \alpha)\Gamma(\alpha) = \frac{\pi}{\sin \pi\alpha},$$

we have

$$\begin{aligned} \Big(\tfrac{\langle a \rangle}{N} + \tfrac{\langle b \rangle}{N} - 1 \Big) \Omega^{-a,-b} \Omega^{a,b} &= -\frac{\pi}{N^2} \frac{\sin\big(\frac{\langle a \rangle}{N}\pi + \frac{\langle b \rangle}{N}\pi \big)}{\sin\big(\frac{\langle a \rangle}{N}\pi \big) \sin\big(\frac{\langle b \rangle}{N}\pi \big)} \\ &= \frac{2\pi i}{N^2} \frac{1 - \zeta^{a+b}}{(1 - \zeta^a)(1 - \zeta^b)} = \frac{\pi i}{N^2} \Big(\frac{1 + \zeta^a}{1 - \zeta^a} + \frac{1 + \zeta^b}{1 - \zeta^b} \Big). \end{aligned}$$

Therefore, we have

$$\begin{aligned} 2N^2(\gamma^{r,s} \sharp \gamma) &= \sum_{(a,b)\in I} \zeta^{ar+bs} \Big(\frac{1 + \zeta^a}{1 - \zeta^a} + \frac{1 + \zeta^b}{1 - \zeta^b} \Big) = \sum_{a,b \neq 0} \zeta^{ar+bs} \Big(\frac{1 + \zeta^a}{1 - \zeta^a} + \frac{1 + \zeta^b}{1 - \zeta^b} \Big) \\ &= \sum_{b \neq 0} \zeta^{bs} \sum_{a \neq 0} \zeta^{ar} \frac{1 + \zeta^a}{1 - \zeta^a} + \sum_{a \neq 0} \zeta^{ar} \sum_{b \neq 0} \zeta^{bs} \frac{1 + \zeta^b}{1 - \zeta^b}. \end{aligned}$$

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Now the result follows, since

$$\sum_{a\neq 0}\zeta^{ar}\frac{1+\zeta^a}{1-\zeta^a}=\begin{cases}0 & (r=0),\\ 2\langle r\rangle - N & (r\neq 0).\end{cases}$$ ■

Corollary 3.4 (i) *For any* $(r,s),(k,l)\in G$, *we have*

$$2N^2(\gamma^{r,s}\sharp\gamma^{k,l})=\begin{cases}0 & \text{if } r=k, s=l,\\ (N-1)(2\langle s-l\rangle-N) & \text{if } r=k, s\neq l,\\ (N-1)(2\langle r-k\rangle-N) & \text{if } r\neq k, s=l,\\ 2(N-\langle r-k\rangle-\langle s-l\rangle) & \text{if } r\neq k, s\neq l.\end{cases}$$

(ii) *For any* $r,s,k,l\neq 0$, *we have*

$$\kappa^{r,s}\sharp\kappa^{k,l}=\begin{cases}1 & \text{if } \langle r\rangle\leq\langle k\rangle,\langle s\rangle<\langle l\rangle \text{ or } \langle r\rangle<\langle k\rangle,\langle s\rangle\leq\langle l\rangle,\\ -1 & \text{if } \langle r\rangle\geq\langle k\rangle,\langle s\rangle>\langle l\rangle \text{ or } \langle r\rangle>\langle k\rangle,\langle s\rangle\geq\langle l\rangle,\\ 0 & \text{otherwise},\end{cases}$$

i.e., *for a fixed* (k,l),

$$(\kappa^{r,s}\sharp\kappa^{k,l})_{r,s}=\begin{array}{c}\\ \\ \\ k\\ \\ \\ \\ \end{array}\begin{array}{c} \qquad\qquad\quad l \\ \begin{bmatrix}1&\cdots&1&1&0&\cdots&0\\ \vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 1&\cdots&1&1&0&\cdots&0\\ 1&\cdots&1&0&-1&\cdots&-1\\ 0&\cdots&0&-1&-1&\cdots&-1\\ \vdots&\ddots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&\cdots&0&-1&-1&\cdots&-1\end{bmatrix}\end{array}.$$

Proof Since $\gamma^{r,s}\sharp\gamma^{k,l}=(g^{-k,-l}\gamma^{r,s})\sharp(g^{-k,-l}\gamma^{k,l})=\gamma^{r-k,s-l}\sharp\gamma$, (i) follows from the proposition, from which (ii) follows using $\kappa^{r,s}=\gamma^{0,0}-\gamma^{r,0}-\gamma^{0,s}+\gamma^{r,s}$. ■

4 Basis of the Integral Homology and the Annihilator

Theorem 4.1 *For any* $i\neq 0$, *any one of the sets*

$$\{\kappa^{r,s}\mid r,s\neq 0, r\neq i\}\quad \text{or}\quad \{\kappa^{r,s}\mid r,s\neq 0, s\neq i\}$$

is a basis of $H_1(X,\mathbb{Z})$.

Proof We only consider the set $\{\kappa^{r,s}\mid r,s\neq 0, s\neq N-1\}$; the other cases are similar. By the existence of a symplectic basis, it suffices to show that the determinant of the intersection matrix is 1. Renumber the elements by the lexicographic order:

$$\kappa_1=\kappa^{1,1},\dots,\kappa_{N-2}=\kappa^{1,N-2},\kappa_{N-1}=\kappa^{2,1},\dots,\ \kappa_{(N-1)(N-2)}=\kappa^{N-1,N-2}.$$

Let $\mathbf{M}=(\kappa_i\sharp\kappa_j)_{i,j}$ be the intersection matrix. By Corollary 3.4(ii), we have $\mathbf{M}=\mathbf{U}-{}^t\mathbf{U}$, where we put an $(N-2)\times(N-2)$ matrix (resp., $(N-1)\times(N-1)$ block

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matrix) as:

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} U & U & \cdots & U \\ 0 & U & \ddots & \vdots \\ \vdots & \ddots & \ddots & U \\ 0 & \cdots & 0 & U \end{bmatrix}.$$

The determinant $|\mathbf{M}|$ is computed as follows. Put

$$J = -U^{-1} \cdot {}^t U = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -1 & \cdots & -1 & -1 \end{bmatrix}.$$

By the Cayley–Hamilton theorem, we have $\sum_{i=0}^{N-2} J^i = O$ (the zero matrix). By the multilinearity of the determinant, we have (I is the unit matrix and the blank is O)

$$|\mathbf{M}| = \begin{vmatrix} I+J & I & \cdots & I \\ J & I+J & \ddots & \vdots \\ \vdots & \ddots & \ddots & I \\ J & \cdots & J & I+J \end{vmatrix} = \begin{vmatrix} I+J & I & \cdots & I \\ -I & J & & \\ & \ddots & \ddots & \\ & & -I & J \end{vmatrix}$$

$$= \begin{vmatrix} O & \cdots & O & \sum_{i=0}^{N-1} J^i \\ -I & J & & \\ & \ddots & \ddots & \\ & & -I & J \end{vmatrix} = \left| \sum_{i=0}^{N-1} J^i \right| = \left| I + J \sum_{i=0}^{N-2} J^i \right| = 1.$$

Hence, the theorem is proved. ∎

Remark 4.2 For a specific N, one easily finds a symplectic basis of $H_1(X, \mathbb{Z})$ by using the computation of $\mathbf{M}$ as above. See Guàrdia [7] and Kamata [10] for related works.

Corollary 4.3 (Rohrlich [15]) *If $r, s \in (\mathbb{Z}/N\mathbb{Z})^\times$, then $H_1(X, \mathbb{Z})$ is a cyclic $\mathbb{Z}[G]$-module generated by $\kappa^{r,s}$.*

Proof Let $\overline{\kappa}^{k,l}$ denote the class of $\kappa^{k,l}$ in $H_1(X, \mathbb{Z})/\mathbb{Z}[G]\kappa^{r,s}$. Since

$$g^{k,l}\kappa^{r,s} = \kappa^{k+r,l+s} - \kappa^{k+r,l} - \kappa^{k,l+s} + \kappa^{k,l}$$

for any $(k, l), (r, s) \in G$, we have

$$g^{mr,ns}\kappa^{r,s} = \kappa^{(m+1)r,(n+1)s} - \kappa^{(m+1)r,ns} - \kappa^{mr,(n+1)s} + \kappa^{mr,ns}.$$

Therefore, $\overline{\kappa}^{(m+1)r,ns} = \overline{\kappa}^{mr,(n+1)s} = \overline{\kappa}^{mr,ns} = 0$ implies $\overline{\kappa}^{(m+1)r,(n+1)s} = 0$. By induction starting with $\overline{\kappa}^{r,0} = \overline{\kappa}^{0,s} = 0$, we obtain $\overline{\kappa}^{mr,ns} = 0$ for any m and n. Since r, s are invertible, it follows that $\overline{\kappa}^{k,l} = 0$ for any $(k, l) \in G$. ∎

We determine the annihilator $\mathrm{Ann}(X, \mathbb{Z}) \subset \mathbb{Z}[G]$ of $H_1(X, \mathbb{Z})$. By Corollary 4.3, for any $r, s \in (\mathbb{Z}/N\mathbb{Z})^\times$, we have an exact sequence

$$0 \longrightarrow \mathrm{Ann}(X, \mathbb{Z}) \longrightarrow \mathbb{Z}[G] \xrightarrow{\cdot\kappa^{r,s}} H_1(X, \mathbb{Z}) \longrightarrow 0.$$

By Proposition 2.2(ii), the set $\{\mathbf{t}, \mathbf{v}_r, \mathbf{h}_s, \mathbf{d}_t \mid r, s, t \neq 0\}$ generate a submodule $L \subset \mathrm{Ann}(X, \mathbb{Z})$ of finite index. If we put

$$\mathbf{s} = \sum_{0 \le r < s < N} g^{r,s} \in \mathbb{Z}[G],$$

then one easily verifies

$$N\mathbf{s} = -\sum_{0<r<N} r(\mathbf{v}_r - \mathbf{h}_r - \mathbf{d}_r). \tag{4.1}$$

In particular, $\mathbf{s} \in \mathbb{Z}[G] \cap \mathrm{Ann}(X, \mathbb{Q}) = \mathrm{Ann}(X, \mathbb{Z})$. Rohrlich [15, Note 1] stated that $\mathrm{Ann}(X, \mathbb{Z})$ is generated by L and $\mathbf{s}$. A proof is provided by Lim [11, Proposition 4.1]. We prove the following refinement.

Proposition 4.4 *The group* $\mathrm{Ann}(X, \mathbb{Z})/L$ *is cyclic of order* N*, generated by* $\mathbf{s}$.

Proof By Proposition 2.2(ii) and (4.1), $\mathbf{s}$ is of exact order N mod L. If $\mathbf{g} \in \mathrm{Ann}(X, \mathbb{Z})$, there exists an integer n such that $n\mathbf{g} \in L$; let

$$n\mathbf{g} = a\mathbf{t} + \sum_r b_r \mathbf{v}_r + \sum_s c_s \mathbf{h}_s + \sum_t d_t \mathbf{d}_t$$

with $b_0 = c_0 = d_0 = 0$. By comparing the coefficients of $g^{0,0}$, we have $a \equiv 0 \pmod n$. Similarly, we obtain $b_r + c_s + d_{r-s} \equiv 0 \pmod n$ for all $r, s \in \mathbb{Z}/N\mathbb{Z}$. This implies $b_r \equiv -c_r \equiv -d_r \pmod n$ and hence $b_r \equiv b_s + b_{r-s} \pmod n$. It follows that $b_r \equiv rb_1 \pmod n$, hence $Nb_1 \equiv 0 \pmod n$. Therefore, we have

$$n\mathbf{g} \equiv b_1 \sum_r r(\mathbf{v}_r - \mathbf{h}_r - \mathbf{d}_r) \equiv -Nb_1\mathbf{s} \pmod{nL},$$

hence $\mathbf{g} \equiv -\frac{Nb_1}{n}\mathbf{s} \pmod L$. ∎

5 Affine Curves and Relative Homology

Let $Y \subset U$ be open subvarieties of X defined respectively by

$$Y : x_0 y_0 z_0 \neq 0, \quad U : z_0 \neq 0,$$

and put $Z = X - Y$, $V = X - U$, $W = U - Y$. These are stable under the G-action. Put

$$P_n = (0 : \zeta^n : 1), \quad Q_n = (\zeta^n : 0 : 1), \quad R_n = (1 : \xi\zeta^n : 0),$$

where $\xi = \exp(\pi i/N)$ as before. Then we have

$$V = \{R_n \mid n \in \mathbb{Z}/N\mathbb{Z}\}, \quad W = \{P_n, Q_n \mid n \in \mathbb{Z}/N\mathbb{Z}\}, \quad Z = V \cup W.$$

5.1 Homology of U

First, consider the exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow H_2(X, \mathbb{Z}) \longrightarrow H_0(V, \mathbb{Z}) \longrightarrow H_1(U, \mathbb{Z}) \longrightarrow H_1(X, \mathbb{Z}) \longrightarrow 0. \tag{5.1}$$

It follows that $H_1(U, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $(N-1)^2$. The cycles $\kappa^{r,s}$ and hence γ, $\gamma^{r,s}$ are already defined on U. By abuse of notation, we also write their classes in $H_1(U, \mathbb{Z})$ or $H_1(U, \mathbb{Q})$ by the same letter. Let $\mathrm{Ann}(U, \mathbb{Q}) \subset \mathbb{Q}[G]$ be the annihilator of $H_1(U, \mathbb{Q})$.

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Proposition 5.1 (i) *The group $H_1(U,\mathbb{Q})$ is a cyclic $\mathbb{Q}[G]$-module generated by γ.*
(ii) *The set $\{\mathbf{t},\mathbf{v}_r,\mathbf{h}_s \mid r,s\neq 0\}$ is a basis of* $\mathrm{Ann}(U,\mathbb{Q})$.
(iii) *The set $\{\gamma^{r,s} \mid r,s\neq 0\}$ is a basis of $H_1(U,\mathbb{Q})$.*

Proof One shows as in Lemma 2.1 that $H^1(U,\mathbb{C})$ is generated by $\{\omega^{a,b} \mid a,b\neq 0\}$. Note that, even if $a+b=0$, $\omega^{a,b}$ $(a,b\neq 0)$ is holomorphic on U with logarithmic poles along V and $\Omega^{a,b}=\int_\gamma \omega^{a,b}$ is a positive real number. One proves similarly as before that $\{\gamma^{r,s} \mid (r,s)\in G\}$ generate $H_1(U,\mathbb{Q})$, from which (i) follows. The proofs of (ii) and (iii) are also similar (and easier). ∎

Proposition 5.2 (i) *The set $\{\kappa^{r,s} \mid r,s\neq 0\}$ is a basis of $H_1(U,\mathbb{Z})$.*
(ii) *If $r,s\in(\mathbb{Z}/N\mathbb{Z})^\times$, then $H_1(U,\mathbb{Z})$ is a cyclic $\mathbb{Z}[G]$-module generated by $\kappa^{r,s}$.*

Proof (i) The image of $H_0(V,\mathbb{Z})\to H_1(U,\mathbb{Z})$ is generated by the classes $\varepsilon(R_n)$ of small loops around R_n $(n\in\mathbb{Z}/N\mathbb{Z})$ in the positive orientation. By (5.1) and Theorem 4.1, it suffices to show that $\varepsilon(R_n)$ is a $\mathbb{Z}$-linear combination of $\kappa^{r,s}$'s. First, one computes

$$\int_{\varepsilon(R_n)} \omega^{a,b} = \begin{cases} 2\pi i(\xi\zeta^n)^{\langle b\rangle} & (a+b=0), \\ 0 & (a+b\neq 0). \end{cases}$$

On the other hand, by Propositions 2.2(ii) and 5.2(ii), $\mathrm{Ker}(H_1(U,\mathbb{Q})\to H_1(X,\mathbb{Q}))$ is generated by $\{\mathbf{d}_t\gamma \mid t\in\mathbb{Z}/N\mathbb{Z}\}$, and

$$\int_{\mathbf{d}_t\gamma} \omega^{a,b} = \zeta^{at}\sum_r \zeta^{(a+b)r}\Omega^{a,b} = \begin{cases} \zeta^{-bt}\pi/\sin\frac{\langle b\rangle\pi}{N} & (a+b=0), \\ 0 & (a+b\neq 0). \end{cases}$$

Comparing these, we have $\int_{\varepsilon(R_n)}\omega^{a,b} = \int_{\mathbf{d}_{-n-1}\gamma-\mathbf{d}_{-n}\gamma}\omega^{a,b}$ for any $a,b\neq 0$, hence $\varepsilon(R_n)=\mathbf{d}_{-n-1}\gamma-\mathbf{d}_{-n}\gamma$. Since

$$\mathbf{d}_{-n-1}-\mathbf{d}_{-n} = \sum_r\big\{(1-g^{-n-1+r,0})(1-g^{0,r})-(1-g^{-n+r,0})(1-g^{0,r})\big\},$$

we have $\mathbf{d}_{-n-1}\gamma-\mathbf{d}_{-n}\gamma=\sum_r(\kappa^{-n-1+r,r}-\kappa^{-n+r,r})$ by (2.2), and (i) is proved. Then (ii) follows from (i) exactly as in the proof of Corollary 4.3. ∎

Remark 5.3 In a similar manner to the proof of Proposition 4.4, we can show that the annihilator $\mathrm{Ann}(U,\mathbb{Z})\subset\mathbb{Z}[G]$ of $H_1(U,\mathbb{Z})$ is a free $\mathbb{Z}$-module generated by $\{\mathbf{t},\mathbf{v}_r,\mathbf{h}_s \mid r,s\neq 0\}$.

5.2 Relative Homology of (U,W)

Second, consider the exact sequence of $\mathbb{Z}[G]$-modules:

$$0\longrightarrow H_1(U,\mathbb{Z})\longrightarrow H_1(U,W;\mathbb{Z})\xrightarrow{\partial} H_0(W,\mathbb{Z})\xrightarrow{\deg} H_0(U,\mathbb{Z})\longrightarrow 0.$$

It follows that $H_1(U,W;\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank N^2. The path δ defines a class in $H_1(U,W;\mathbb{Z})$, also denoted by the same letter, such that $\partial(\delta)=Q_0-P_0$.

Proposition 5.4 (Anderson [1, Theorem 6]) *$H_1(U,W;\mathbb{Z})$ is a free cyclic $\mathbb{Z}[G]$-module generated by δ.*

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Proof Since $\partial(g^{r,s}\delta) = Q_r - P_s$, it follows that $\mathbb{Z}[G]\delta$ surjects to $\mathrm{Ker}(\deg)$. By Proposition 5.2, $H_1(U,\mathbb{Z})$ is generated by $\kappa^{r,s} \in \mathbb{Z}[G]\delta$. Therefore, $H_1(U,W;\mathbb{Z})$ is generated by δ as a $\mathbb{Z}[G]$-module, and it is free for the rank reason. ■

Remark 5.5 The above sequence with $\mathbb{Q}$-coefficients is an exact sequence of mixed Hodge structures, and $H_1(U,\mathbb{Q}) \simeq W_{-1}H_1(U,W;\mathbb{Q})$, where $W_\bullet$ is the weight filtration. Therefore, $\mathbf{p}$ gives a retraction of the filtration, sending δ to γ, and induces an isomorphism $\mathrm{Gr}^W_{-1} H_1(U,W;\mathbb{Q}) \simeq H_1(X,\mathbb{Q})$.

5.3 Homology of Y

Finally, consider the exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow H_0(W,\mathbb{Z}) \xrightarrow{\varepsilon} H_1(Y,\mathbb{Z}) \longrightarrow H_1(U,\mathbb{Z}) \longrightarrow 0,$$

where the map ε sends the class of $P \in W$ to the class $\varepsilon(P)$ of a small loop around P in the positive orientation. It follows that $H_1(Y,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank N^2+1. The following is immediate from Proposition 5.2.

Proposition 5.6 *For each $(r,s) \in G$, choose a lifting $\widetilde{\kappa}^{r,s} \in H_1(Y,\mathbb{Z})$ of $\kappa^{r,s} \in H_1(U,\mathbb{Z})$.*

(i) *The set $\{\varepsilon(P_n),\varepsilon(Q_n),\widetilde{\kappa}^{r,s} \mid n,r,s \in \mathbb{Z}/N\mathbb{Z}, r,s \neq 0\}$ is a basis of $H_1(Y,\mathbb{Z})$.*
(ii) *If $r,s \in (\mathbb{Z}/N\mathbb{Z})^\times$, then $H_1(Y,\mathbb{Z})$ is generated by $\{\varepsilon(P_0),\varepsilon(Q_0),\widetilde{\kappa}^{r,s}\}$ as a $\mathbb{Z}[G]$-module.*

If we work with $\mathbb{Q}$-coefficients, there is a canonical choice of the lifting. Since $\mathbf{p}$ annihilates $H_0(W,\mathbb{Q})$, the map

$$\widetilde{\mathbf{p}}\colon H_1(U,\mathbb{Q}) \longrightarrow H_1(Y,\mathbb{Q}); \quad c \longmapsto \mathbf{p}\widetilde{c},$$

where $\widetilde{c} \in H_1(Y,\mathbb{Q})$ is any lifting of c, is well defined and gives a section of the $\mathbb{Q}[G]$-homomorphism $H_1(Y,\mathbb{Q}) \to H_1(U,\mathbb{Q})$.

Proposition 5.7

(i) *As a $\mathbb{Q}[G]$-module, $H_1(Y,\mathbb{Q})$ is generated by $\{\varepsilon(P_0),\varepsilon(Q_0),\widetilde{\mathbf{p}}\gamma\}$.*
(ii) *The set $\{\varepsilon(P_n),\varepsilon(Q_n),\widetilde{\mathbf{p}}\gamma^{r,s} \mid n,r,s \in \mathbb{Z}/N\mathbb{Z}, r,s \neq 0\}$ is a basis of $H_1(Y,\mathbb{Q})$.*
(iii) *The set $\{dx/x, dy/y, \omega^{a,b} \mid (a,b) \neq (0,0)\}$ is a basis of $H^1(Y,\mathbb{C})$.*

Proof Part (i) is obvious, and the others follow from the following table of the canonical pairing $H_1(Y,\mathbb{Q}) \times H^1(Y,\mathbb{C}) \to \mathbb{C}$:

	$\frac{dx}{x}$	$\omega^{0,b}$ $(b \neq 0)$	$\frac{dy}{y}$	$\omega^{a,0}$ $(a \neq 0)$	$\omega^{a,b}$ $(a,b \neq 0)$
$\varepsilon(P_n)$	$2\pi i$	$2\pi i\zeta^{bn}$	0	0	0
$\varepsilon(Q_n)$	0	0	$2\pi i$	$2\pi i\zeta^{an}$	0
$\widetilde{\mathbf{p}}\gamma^{r,s}$ $(r,s \neq 0)$	0	0	0	0	$\zeta^{ar+bs}\Omega^{a,b}$

■

6 Projective Limits

Now, we let N vary and study the projective systems of homology groups. All the objects studied earlier are written with the subscript $_N$, *e.g.*, X_N, $g_N^{r,s} \in G_N$, ζ_N, δ_N, $\kappa_N^{r,s}$, $\gamma_N^{r,s}$. If N divides M, we have a morphism

$$\pi_{M/N}: X_M \longrightarrow X_N; \quad (x_0 : y_0 : z_0) \longmapsto \left(x_0^{\frac{M}{N}} : y_0^{\frac{M}{N}} : z_0^{\frac{M}{N}} \right).$$

It is compatible with the natural surjection $G_M \to G_N$ and respects all the subvarieties defined in Section 5. Since $\pi_{M/N} \circ \pi_{L/M} = \pi_{L/N}$, the curves X_N form a projective system. For a ring R, we define $H_1(X_\infty, R) = \varprojlim_N H_1(X_N, R)$. It has the natural structure of a module over the completed group ring $R[[G_\infty]] := \varprojlim_N R[G_N]$. We apply similar notations for other (relative) homology groups.

For $N \mid M$, the image of δ_M under $\pi_{M/N}$ is clearly δ_N. The ring homomorphism $\mathbb{Q}[G_M] \to \mathbb{Q}[G_N]$ sends $g_M^{r,s}$ (resp., $\mathbf{p}_M$) to $g_N^{r,s}$ (resp., $\mathbf{p}_N$). Hence, we have elements

$$\delta_\infty := (\delta_N)_N \in H_1(U_\infty, W_\infty; \mathbb{Z}), \quad \kappa_\infty^{r,s} := (\kappa_N^{r,s})_N \in H_1(U_\infty, \mathbb{Z}),$$
$$\gamma_\infty := (\gamma_N)_N, \ \gamma_\infty^{r,s} = (\gamma_N^{r,s})_N \in H_1(U_\infty, \mathbb{Q}),$$

where $r, s \in \widehat{\mathbb{Z}} := \varprojlim_N \mathbb{Z}/N\mathbb{Z}$. By abuse of notation, the images of $\kappa_\infty^{r,s}$, γ_∞, $\gamma_\infty^{r,s}$ in the homology of X_∞ are written by the same letters. By Proposition 5.4, for any ring R, $H_1(U_\infty, W_\infty; R)$ is a free cyclic $R[[G_\infty]]$-module generated by δ_∞.

Theorem 6.1 *Let R be a profinite ring.*

(i) *For any $r, s \in \widehat{\mathbb{Z}}^\times$, $H_1(X_\infty, R)$ is a free cyclic $R[[G_\infty]]$-module generated by $\kappa_\infty^{r,s}$.*
(ii) *The natural maps $H_1(Y_\infty, R) \to H_1(U_\infty, R) \to H_1(X_\infty, R)$ are isomorphisms of $R[[G_\infty]]$-modules.*

Proof Let $R = \varprojlim_\lambda R_\lambda$ with R_λ finite. Since $H_1(X_N, R) = \varprojlim_\lambda H_1(X_N, R_\lambda)$, we can assume that R is finite. (i) Consider the exact sequence

$$0 \longrightarrow \mathrm{Ann}(X_N, R) \longrightarrow R[G_N] \xrightarrow{\cdot \kappa_N^{r,s}} H_1(X_N, R) \longrightarrow 0.$$

Since $H_1(X_N, R)$ are finite, the functor $\varprojlim_N$ preserves the exactness, so we are reduced to showing that $\varprojlim_N \mathrm{Ann}(X_N, R) = 0$. By Proposition 4.4, $\mathrm{Ann}(X_N, R)$ is generated by $\mathbf{t}_N$, $\mathbf{v}_{N,r}$, $\mathbf{h}_{N,s}$, $\mathbf{d}_{N,t}$ and $\mathbf{s}_N$. If $M = dN$, the homomorphism $\mathbb{Z}[G_M] \to \mathbb{Z}[G_N]$ sends $\mathbf{t}_M$, $\mathbf{v}_{M,r}$, $\mathbf{h}_{M,s}$, $\mathbf{d}_{M,t}$, $\mathbf{s}_M$ to $d^2\mathbf{t}_N$, $d\mathbf{v}_{N,r}$, $d\mathbf{h}_{N,s}$, $d\mathbf{d}_{N,t}$, $d\mathbf{s}_N + \frac{1}{2}(d-1)d\mathbf{t}_N$, respectively. Hence, we have

$$\bigcap_{d \in \mathbb{Z}_{>0}} \mathrm{Im}\big[\mathrm{Ann}(X_{dN}, R) \longrightarrow \mathrm{Ann}(X_N, R)\big] = \bigcap_{n \in \mathbb{Z}_{>0}} n\,\mathrm{Ann}(X_N, R) = 0,$$

and the assertion follows. (ii) We prove the second isomorphism; the other one is similarly proved. By (5.1), we have the following commutative diagram with exact

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rows for $M = dN$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Big(\bigoplus_{m\in\mathbb{Z}/M\mathbb{Z}} R\varepsilon(R_m)\Big)/R\Delta_M & \longrightarrow & H_1(U_M,R) & \longrightarrow & H_1(X_M,R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Big(\bigoplus_{n\in\mathbb{Z}/N\mathbb{Z}} R\varepsilon(R_n)\Big)/R\Delta_N & \longrightarrow & H_1(U_N,R) & \longrightarrow & H_1(X_N,R) & \longrightarrow & 0,
\end{array}$$

where we put $\Delta_N = \sum_n \varepsilon(R_n)$. Since the ramification index of $X_M \to X_N$ at each R_m is d, the class $\varepsilon(R_m)$ maps to $d\varepsilon(R_n)$. Therefore, by applying the functor $\varprojlim_N$, we obtain the isomorphism similarly as above. ∎

For $r,s \in \widehat{\mathbb{Z}}$, write

$$g_\infty^{r,s} = (g_N^{r,s})_N \in \varprojlim_N G_N.$$

Since $H_1(U_\infty,\widehat{\mathbb{Z}}) = (1-g_\infty^{r,0})(1-g_\infty^{0,s})H_1(U_\infty,W_\infty;\widehat{\mathbb{Z}})$ for any $r,s\in\widehat{\mathbb{Z}}^\times$ and it is free by Theorem 6.1, we obtain the following corollary.

Corollary 6.2 *For any $r,s\in\widehat{\mathbb{Z}}^\times$, $(1-g_\infty^{r,0})(1-g_\infty^{0,s}) \in \widehat{\mathbb{Z}}[[G_\infty]]$ is not a zero-divisor.*

Remark 6.3 For a prime ℓ, the same proof as above applies to the ℓ-adic version. For any $r,s\in\mathbb{Z}_\ell^\times$, we have isomorphisms

$$H_1(Y_{\ell^\infty},\mathbb{Z}_\ell) \xrightarrow{\simeq} H_1(U_{\ell^\infty},\mathbb{Z}_\ell) \xrightarrow{\simeq} H_1(X_{\ell^\infty},\mathbb{Z}_\ell)$$

of free cyclic $\mathbb{Z}_\ell[[G_{\ell^\infty}]]$-modules generated by $\kappa_{\ell^\infty}^{r,s} = (\kappa_{\ell^n}^{r,s})_n$. This gives an alternative proof of the freeness of $H_1(X_{\ell^\infty},\mathbb{Z}_\ell)$ due to Ihara [8]. In [3, Section 13.4], Anderson gives another proof starting with the freeness of $H_1(U_{\ell^\infty},W_{\ell^\infty};\mathbb{Z}_\ell)$. Similarly as the standard Iwasawa algebra of one variable, we have an isomorphism

$$\mathbb{Z}_\ell[[G_{\ell^\infty}]] \xrightarrow{\simeq} \mathbb{Z}_\ell[[S,T]] \tag{6.1}$$

sending $g_{\ell^\infty}^{r,s}$ to $(1-S)^r(1-T)^s$. Since $(1-g_{\ell^\infty}^{1,0})(1-g_{\ell^\infty}^{0,1})$ corresponds to ST, which is not a zero-divisor, the freeness of $H_1(U_{\ell^\infty},\mathbb{Z}_\ell)$ follows. This argument is particular to the ℓ-adic case.

7 Definition of Measures

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and put $\mathbb{Q}^{\mathrm{ab}} = \bigcup_N \mathbb{Q}(\mu_N)$. For a subfield $K\subset\overline{\mathbb{Q}}$, let $G_K = G(\overline{\mathbb{Q}}/K)$ denote the absolute Galois group.

Recall Artin's isomorphism between étale and singular homology [4, Exposé XI]

$$\iota\colon H_1^{\text{ét}}(X_N,\mathbb{Z}/M\mathbb{Z}) \xrightarrow{\simeq} H_1(X_N,\mathbb{Z}/M\mathbb{Z}).$$

Since X_N is defined over $\mathbb{Q}$, $G_\mathbb{Q}$ acts on the left member. Since the ι are compatible with $\pi_{M/N}$, $G_\mathbb{Q}$ acts on the limit

$$H_1^{\text{ét}}(X_\infty,\widehat{\mathbb{Z}}) := \varprojlim_{N,M} H_1^{\text{ét}}(X_N,\mathbb{Z}/M\mathbb{Z}).$$

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Fix $r, s \in \widehat{\mathbb{Z}}^\times$ and put $\kappa_{\text{ét}} = \iota^{-1}(\kappa_\infty^{r,s}) \in H_1^{\text{ét}}(X_\infty, \widehat{\mathbb{Z}})$. By Theorem 6.1, we can define a continuous map

$$F\colon G_{\mathbb{Q}} \longrightarrow \widehat{\mathbb{Z}}[[G_\infty]]^\times; \quad \sigma \longmapsto F_\sigma$$

by $\sigma\kappa_{\text{ét}} = F_\sigma \kappa_{\text{ét}}$. Identifying $G_N \simeq \mu_N{}^{\oplus 2}$, $G_{\mathbb{Q}}$ acts on G_N via the cyclotomic character $\chi\colon G_{\mathbb{Q}} \to \widehat{\mathbb{Z}}^\times$, *i.e.*, $\sigma g_N^{r,s} = g_N^{\chi(\sigma)r, \chi(\sigma)s}$, hence $G_{\mathbb{Q}}$ acts on $\widehat{\mathbb{Z}}[[G_\infty]]$. Then F becomes a cocycle in the sense of group cohomology, *i.e.*,

$$F_{\sigma\tau} = \sigma F_\tau \cdot F_\sigma. \tag{7.1}$$

In particular, the restriction $G_{\mathbb{Q}^{\text{ab}}} \to \widehat{\mathbb{Z}}[[G_\infty]]^\times$ is a continuous homomorphism. This is the adelic generalization of Ihara's ℓ-adic universal power series for Jacobi sums [8] (recall (6.1) for the wording "series").

Using δ_∞ instead of $\kappa_\infty^{r,s}$, a cocycle

$$B\colon G_{\mathbb{Q}} \longrightarrow \widehat{\mathbb{Z}}[[G_\infty]]^\times; \quad \sigma \longmapsto B_\sigma$$

is similarly defined. This is Anderson's adelic beta function [3]. As is clear from the definitions, we have

$$(1 - g_\infty^{r,0})(1 - g_\infty^{0,s})F_\sigma = (1 - g_\infty^{\chi(\sigma)r,0})(1 - g_\infty^{0,\chi(\sigma)s})B_\sigma. \tag{7.2}$$

By Corollary 6.2, F_σ and B_σ characterize each other. In particular, if $\sigma \in G_{\mathbb{Q}^{\text{ab}}}$, then $F_\sigma = B_\sigma$ and F_σ does not depend on the choice of $r, s \in \widehat{\mathbb{Z}}^\times$.

Following Anderson, we regard an element of $\widehat{\mathbb{Z}}[[G_\infty]]$ as a function on $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ as follows. It is natural to identify an element $F \in \widehat{\mathbb{Z}}[[G_\infty]]$ with a $\widehat{\mathbb{Z}}$-valued measure μ_F on G_∞; if $F = (\sum_{r,s \in \mathbb{Z}/N\mathbb{Z}} c_N^{r,s} g_N^{r,s})_N$, then for any $g_\infty^{r,s} \in G_\infty$,

$$\mu_F(g_\infty^{r,s} + NG_\infty) = c_N^{r,s}.$$

For $\alpha, \beta \in \mathbb{Q}/\mathbb{Z}$, define a character by

$$\varphi^{\alpha,\beta}\colon G_\infty \longrightarrow (\mathbb{Q}^{\text{ab}})^\times; \quad \varphi^{\alpha,\beta}(g_\infty^{r,s}) = \exp 2\pi i(r\alpha + s\beta).$$

We define a function $F\colon (\mathbb{Q}/\mathbb{Z})^{\oplus 2} \to \mathbb{Q}^{\text{ab}} \otimes \widehat{\mathbb{Z}}$, written by abuse of notation using the same letter, by

$$F(\alpha, \beta) = \int_{G_\infty} \varphi^{\alpha,\beta}\, d\mu_F.$$

If $N\alpha = N\beta = 0$, then $\varphi^{\alpha,\beta}$ factors through G_N and defines a ring homomorphism $\varphi^{\alpha,\beta}\colon \widehat{\mathbb{Z}}[G_N] \to \mathbb{Q}(\mu_N) \otimes \widehat{\mathbb{Z}}$. Then $F(\alpha, \beta)$ is nothing but the image of F under $\varphi^{\alpha,\beta}$ composed with the natural projection $\widehat{\mathbb{Z}}[[G_\infty]] \to \widehat{\mathbb{Z}}[G_N]$. Moreover, the restriction

$$G_{\mathbb{Q}(\mu_N)} \longrightarrow \big(\mathbb{Q}(\mu_N) \otimes \widehat{\mathbb{Z}}\big)^\times; \quad \sigma \longmapsto F_\sigma(\alpha, \beta)$$

is a homomorphism. We also have

$$(\tau F)(\alpha, \beta) = F\big(\chi^{-1}(\tau)\alpha, \chi^{-1}(\tau)\beta\big), \quad \tau\big(F(\alpha, \beta)\big) = F\big(\chi(\tau)\alpha, \chi(\tau)\beta\big)$$

for any $\tau \in G_{\mathbb{Q}}$. Here, τF means the function corresponding to $\tau F \in \widehat{\mathbb{Z}}[[G_\infty]]$.

If $\alpha = 0$ or $\beta = 0$, then $B_\sigma(\alpha,\beta) = 1 \otimes 1$ for any $\sigma \in G_{\mathbb{Q}}$ [3, Theorem 1 (I)]. Therefore by (7.2), we have for $\alpha = a/N, \beta = b/N \in \mathbb{Q}/\mathbb{Z} - \{0\}$

$$
\begin{aligned}
F_\sigma(\alpha,\beta) &= \left(\frac{1-\zeta_N^{\chi(\sigma)ar}}{1-\zeta_N^{ar}} \frac{1-\zeta_N^{\chi(\sigma)bs}}{1-\zeta_N^{bs}} \otimes 1 \right) B_\sigma(\alpha,\beta),\\
F_\sigma(\alpha,0) &= \frac{1-\zeta_N^{\chi(\sigma)ar}}{1-\zeta_N^{ar}} \otimes \chi(\sigma), \quad F_\sigma(0,\beta) = \frac{1-\zeta_N^{\chi(\sigma)bs}}{1-\zeta_N^{bs}} \otimes \chi(\sigma),\\
F_\sigma(0,0) &= 1 \otimes \chi^2(\sigma).
\end{aligned}
$$

Note for example that, for a positive integer $c \equiv \chi(\sigma) \pmod N$, we have

$$
\varphi^{\alpha,0}\left(\frac{1-g_N^{0,\chi(\sigma)s}}{1-g_N^{0,s}} \right) \equiv \varphi^{\alpha,0}(1 + g_N^{0,s} + \cdots + g_N^{0,(c-1)s}) \equiv c \pmod N.
$$

8 Interpolation of Jacobi Sums

Let $K \subset \mathbb{Q}^{\mathrm{ab}}$ be a subfield containing μ_N and let $v \nmid N$ be a prime of K. Let $\mathbb{F}_v$ denote the residue field at v and put $N(v) = |\mathbb{F}_v|$. Then the N-th power residue character $\chi_v\colon \mathbb{F}_v^\times \to \mu_N$ is defined by $\chi_v(x) \equiv x^{(N(v)-1)/N} \pmod v$. For $a,b \in \mathbb{Z}/N\mathbb{Z}$, the Jacobi sum at v is defined by

$$
j_N^{a,b}(v) = -\sum_{x,y \in \mathbb{F}_v^\times, x+y=1} \chi_v^a(x)\chi_v^b(y) \in \mathbb{Q}(\mu_N).
$$

If only one of a,b is 0, then $j_N^{a,b}(v) = 1$. If $a \neq 0$, then $j_N^{a,-a}(v) = \chi_v^a(-1)$. Otherwise, we have $|j_N^{a,b}(v)|^2 = j_N^{a,b}(v) j_N^{-a,-b}(v) = N(v)$. For a prime number ℓ, let $\pi_\ell\colon \mathbb{Q}^{\mathrm{ab}} \otimes \widehat{\mathbb{Z}} \to \mathbb{Q}^{\mathrm{ab}} \otimes \mathbb{Z}_\ell$ denote the projection.

Theorem 8.1 (Anderson [3]) *Let $a,b \in \mathbb{Z}/N\mathbb{Z}$ satisfy $a,b,a+b \neq 0$. Let K be a number field, $v \nmid N$ be a prime of K and $\sigma \in G_K$ be a lift of Frobenius at v. Let w be a prime of $K(\mu_N)$ above v and put $q = N(v)$, $f = [\mathbb{F}_w : \mathbb{F}_v]$. Then we have*

$$
\pi_\ell\left(\prod_{i=0}^{f-1} F_\sigma\left(\frac{q^i a}{N}, \frac{q^i b}{N} \right) \right) = j_N^{a,b}(w) \otimes 1
$$

for any $\ell \neq \mathrm{char}(\mathbb{F}_v)$.

Remark 8.2 When N is a power of ℓ, the formula for the ℓ-component was proved by Ihara [8, Theorem 7]. The theorem can also be derived from [3, Theorem 6] using a decomposition of B_σ (resp., Jacobi sum) into hyperadelic gamma functions (resp., Gauss sums), the cocycle property (7.1) for B_σ, the relation (7.2) between F_σ and B_σ, and the Davenport–Hasse relation.

Proof By (7.1), we have $F_{\sigma^f} = \prod_{i=0}^{f-1} \sigma^i F_\sigma$. Since

$$
(\sigma^i F_\sigma)(\alpha,\beta) = F_\sigma\big(\chi^{-1}(\sigma^i)\alpha, \chi^{-1}(\sigma^i)\beta\big) = F_\sigma(q^{-i}\alpha, q^{-i}\beta),
$$

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we are reduced to the case where $f = 1$, *i.e.*, $\mu_N \subset K$. For any $a, b \in \mathbb{Z}/N\mathbb{Z}$, define a projector to the $\varphi^{\frac{a}{N},\frac{b}{N}}$-isotropic component by

$$\mathbf{e}_N^{a,b} = \frac{1}{N^2}\sum_{r,s}\zeta_N^{-(ar+bs)}g_N^{r,s} \in \mathbb{Q}(\mu_N)[G_N].$$

We regard an element of G_N as an algebraic correspondence from X_N (over $\mathbb{Q}(\mu_N)$) to itself by taking the graph. The pair $(X_N, \mathbf{e}_N^{a,b})$ defines a Chow motive over $\mathbb{Q}(\mu_N)$ with coefficients in $\mathbb{Q}(\mu_N)$ (see [12, Section 2.1]). Then we have a decomposition in the same category

$$h^1(X_N) = \bigoplus_{a,b,a+b\neq 0}(X_N, \mathbf{e}_N^{a,b})$$

[12, Section 2.8], which yields an isomorphism of $G_{\mathbb{Q}(\mu_N)}$-modules

$$H_1^{\text{ét}}(X_N, \mathbb{Z}_\ell) \otimes \mathbb{Q}(\mu_N) = \bigoplus_{a,b,a+b\neq 0}\mathbf{e}_N^{a,b}\left(H_1^{\text{ét}}(X_N, \mathbb{Z}_\ell) \otimes \mathbb{Q}(\mu_N)\right).$$

By [12, Section 3.4], each component of the right member is one-dimensional over $\mathbb{Q}(\mu_N)$, and the Frobenius at $v \nmid N$ acts on it as multiplication by $j_N^{a,b}(v)$ (see the remark below). Hence the theorem follows. ∎

Remark 8.3 By the Poincaré duality, we have an isomorphism $H_1^{\text{ét}}(X_N, \mathbb{Z}_\ell) \simeq H^1_{\text{ét}}(X_N, \mathbb{Z}_\ell(1))$. Here, $\mathbb{Z}_\ell(1)$ denotes the Tate twist, on which $G_\mathbb{Q}$ acts via the cyclotomic character. These are also isomorphic to the ℓ-adic Tate module of the Jacobian of X_N. By [12, Section 3.4], the geometric (=inverse) Frobenius at v acts on $\mathbf{e}_N^{a,b}(H^1_{\text{ét}}(X_N, \mathbb{Z}_\ell) \otimes \mathbb{Q}(\mu_N))$ by $j_N^{a,b}(v)$. Since the action of $g \in G_N$ on the homology corresponds to the action of g^{-1} on the cohomology, the Frobenius at v acts on the homology by $N(v)/j_N^{-a,-b}(v) = j_N^{a,b}(v)$.

Recall that, by Weil [16], for $a, b \in \mathbb{Z}/N\mathbb{Z}$ such that $a, b, a+b \neq 0$, $j_N^{a,b}$ defines a Hecke (quasi-) character $j_N^{a,b}\colon G_{\mathbb{Q}(\mu_N)} \to \mathbb{Q}(\mu_N)^\times$ of conductor dividing N^2 which sends any Frobenius at $v \nmid N$ to $j_N^{a,b}(v)$.

Corollary 8.4 *Let $a, b \in \mathbb{Z}/N\mathbb{Z}$ satisfy $a, b, a+b \neq 0$. Then the homomorphism*

$$G_{\mathbb{Q}(\mu_N)} \longrightarrow \left(\mathbb{Q}(\mu_N) \otimes \widehat{\mathbb{Z}}\right)^\times; \quad \sigma \longmapsto F_\sigma\left(\frac{a}{N}, \frac{b}{N}\right)$$

coincides with the Jacobi-sum Hecke character $j_N^{a,b}$.

Proof For each prime ℓ, the coincidence of the ℓ-component on a Frobenius at v is proved in Theorem 8.1 for all primes v of $\mathbb{Q}(\mu_N)$ not dividing ℓN. By the Chebotarev density theorem, we obtain the coincidence on the whole $G_{\mathbb{Q}(\mu_N)}$. ∎

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References

[1] G. W. Anderson, *Torsion points on Fermat Jacobians, roots of circular units and relative singular homology.* Duke Math. J. **54**(1987), no. 2, 501–561. http://dx.doi.org/10.1215/S0012-7094-87-05422-6

[2] ———, *The hyperadelic gamma function: a précis.* In: Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), Adv. Stud. Pure Math., 12, North-Holland, Amsterdam, 1987, pp. 1-19.

[3] ———, *The hyperadelic gamma function.* Invent. Math. **95**(1989), no. 1, 63–131. http://dx.doi.org/10.1007/BF01394145

[4] M. Artin, A. Grothendieck, and J.-L. Verdier (Eds.), *Théorie des topos et cohomologie étale des schémas.* (SGA 4), Tome 3, Lecture Notes in Mathematics, 305, Springer-Verlag, Berlin-New York, 1973.

[5] R. F. Coleman, *Anderson–Ihara theory: Gauss sums and circular units.* In: Algebraic number theory, Adv. Stud. Pure Math., 17, Academic Press, Boston, MA, 1989, pp. 55–72.

[6] P. Griffiths and J. Harris, *Principles of algebraic geometry. Pure and applied mathematics.* Wiley-Interscience, New York, 1978.

[7] J. Guàrdia, *A fundamental domain for the Fermat curves and their quotients. Contributions to the algorithmic study of problems of arithmetic moduli (Spanish).* Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.) **94**(2000), no. 3, 391–396.

[8] Y. Ihara, *Profinite braid groups, Galois representations, and complex multiplications.* Ann. of Math. **123**(1986), no. 1, 43–106. http://dx.doi.org/10.2307/1971352

[9] Y. Ihara, *Braids, Galois groups, and some arithmetic functions.* In: Proceedings of the International Congress of Mathematicians (Kyoto, 1990), Mathematical Society of Japan, Tokyo, 1991,pp. 99–120.

[10] Y. Kamata, *The algorithm to calculate the period matrix of the curve* $x^m + y^n = 1$. Tsukuba J. Math. **26**(2002), no. 1, 15–37.

[11] C.-H. Lim, *Endomorphisms of Jacobian varieties of Fermat curves.* Compositio Math. **80**(1991), no. 1, 85–110.

[12] N. Otsubo, *On the regulator of Fermat motives and generalized hypergeometric functions.* J. Reine Angew. Math. **660**(2011), 27–82.

[13] ———, *Certain values of Hecke L-functions and generalized hypergeometric functions.* J. Number Theory **131**(2011), no. 4, 648–660. http://dx.doi.org/10.1016/j.jnt.2010.10.002

[14] ———, *On the Abel-Jacobi maps of Fermat Jacobians.* Math. Z. **270**(2012), 423–444. http://dx.doi.org/10.1007/s00209-010-0805-3

[15] D. E. Rohrlich, *Appendix to: B. H. Gross, On the Periods of abelian integrals and a formula of Chowla and Selberg.* Invent. Math. **45**(1978), 193–211. http://dx.doi.org/10.1007/BF01390273

[16] A. Weil, *Jacobi sums as "Größencharaktere".* Trans. Amer. Math. Soc. **73**(1952), 487–495

[17] ———, *Sur les périodes des intégrales Abéliennes.* Comm. Pure App. Math. **29**(1976), no. 6, 813–819. http://dx.doi.org/10.1002/cpa.3160290620

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