Can. J. Math. Vol. **49** (3), 1997 pp. 617–640

ON THE ZEROS OF SOME GENUS POLYNOMIALS

SAUL STAHL

ABSTRACT. In the genus polynomial of the graph G, the coefficient of x^k is the number of distinct embeddings of the graph G on the oriented surface of genus k. It is shown that for several infinite families of graphs all the zeros of the genus polynomial are real and negative. This implies that their coefficients, which constitute the genus distribution of the graph, are log concave and therefore also unimodal. The geometric distribution of the zeros of some of these polynomials is also investigated and some new genus polynomials are presented.

0. **Preliminaries.** This article is concerned with embeddings of finite graphs on oriented surfaces. As is usual in this context, graphs will be allowed to have multiple edges and loops. Two embeddings are considered to be the equivalent if the counterclockwise cyclic orientations the ambient surfaces induce on the edges emanating from each vertex are identical. It is well known that every embedding is equivalent to one in which all the regions (connected components of the complement of the graph in the surface) are homeomorphic to the open 2-cell. Consequently it will henceforth be assumed that every embedding is such a 2-cell embedding.

The set of edges emanating from a vertex of degree d can be cyclically oriented in $(d-1)!$ ways. Hence, if the degree sequence of a graph is $d_1, d_2, \ldots, d_v$, then it has

$$\prod_{i=1}^{v}(d_i-1)!$$

distinct embeddings. The genus of an embedding is the genus of the ambient surface. Our interest lies in the number $\gamma_G(k)$ of embeddings of a given graph G on an oriented surface of a given genus k. The first theorem of this type is the interpolation theorem of [7] which states that if $m < n$ are integers such that $\gamma_G(m)$ and $\gamma_G(n)$ are positive, then so is $\gamma_G(k)$ positive for each integer k, $m \leq k \leq n$. Some experimental results were presented in [1] and the authors of [11] called for a systematic approach to the subject. They also suggested the possible utility of a genus polynomial of the form

$$\sum_{i=0}^{\infty}\gamma_G(k)x^k.$$

A variety of results regarding these numbers were obtained in [8, 9, 16, 17, 19, 20, 21]. In this paper we focus on the genus polynomial and demonstrate that for several infinite

This research was supported by NSF Grant DMS-9203445.
Received by the editors July 13, 1994; revised August 28, 1995.
AMS subject classification: 05C10, 05A15, 30C15, 26C10.

families of graphs all the zeros of these polynomials are real and negative, thus implying that their coefficients, the numbers of interest to us, are in fact unimodal. We also determine the geometric distribution of the zeros of some of these polynomials and formulate some questions and conjectures.

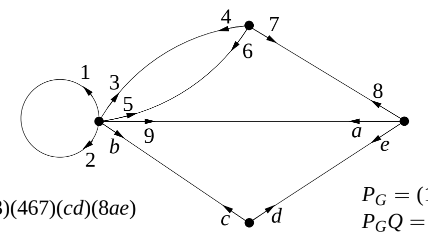
Our results are obtained in the context of permutation-partition pairs which constitute a combinatorial generalization of graphs and graph embeddings. These were first developed in [18] to facilitate the study of the minimum genus of the amalgamation of graphs, but have also proved very useful in this enumerative context. A more geometric, though still equivalent, object was developed in [2, 3] for the same purpose. The original theory of permutation-partition pairs was focused on the number of *regions* of the embedding, and the results could then be translated to their *genera*, whenever necessary. Here, however, it is more convenient to work directly with the genus and consequently some of the machinery of permutation-partition pairs has to be retooled. The following table of contents will give the reader some feeling for the structure of the paper.

The reader is referred to [10] for an explanation of all the terms that are not defined here.

1. **Genus Polynomials.** With every graph G it is possible to associate a pair (P_G, Π_G), consisting of a permutation P_G and a (set theoretic) partition Π_G in the following manner. First convert each edge of G into a pair of oppositely directed arcs and let D_G denote the set of all these arcs. The permutation P_G maps each arc into its opposite and is therefore necessarily an involution (without fixed points). For each vertex v of G, let D_v denote the set of arcs of D_G that emanate from v. Then $\Pi_G = \{D_v \mid v \in V(G)\}$ (see Figure 1). A *permutation-partition pair* (P, Π) consists of an arbitrary permutation P and an arbitrary partition Π, both defined over some common underlying set S. The elements of the underlying set are the *bits* of the pair. For every such pair (P, Π), let $S(\Pi)$ denote the set of all the permutations Q of the underlying set S such that each cycle in the disjoint cycle decomposition of Q is a cyclic permutation of a member of Π. Each permutation $Q \in S(\Pi)$ is called a *rotation system* of (P, Π) and the corresponding pair (P, Q) is called an *embedding* of (P, Π). In the graphical case, where $(P, \Pi) = (P_G, \Pi_G)$ for some graph G, this terminology is consistent with that of topological graph theory. Namely, if the graph G is embedded on the oriented closed surface Σ, then the clockwise sense of Σ induces a cyclic permutation Q_v of D_v at each vertex v of G, and the product of these Q_v's is clearly an element Q of $S(\Pi_G)$. Figure 1 illustrates this with two examples.

If (P, Q) is an embedding of the permutation-partition pair (P, Π), then each cycle in the disjoint cycle decomposition of the composition PQ (read from left to right) is called a *region* of this embedding. It is well known [6, 22, 23] that this terminology too

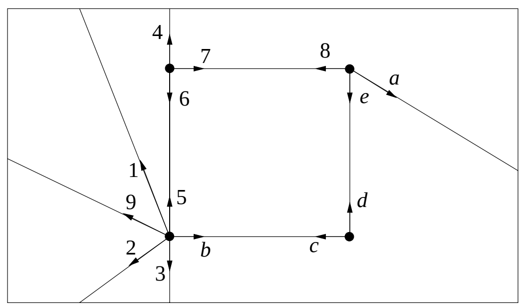
https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press



$Q = (12b953)(467)(cd)(8ae)$

$P_G = (12)(34)(56)(78)(9a)(bc)(de)$
$P_G Q = (1bd84)(2)(36)(57a)(9ec)$

A plane embedding of a graph G



$Q = (1923b5)(467)(8ea)(cd)$

$P_G = (12)(34)(56)(78)(9a)(bc)(de)$
$P_G Q = (136)(2984bda)(57ec)$

A toroidal embedding of a graph G

FIGURE 1

is consistent with that of the embeddings of graphs, as is indicated by the examples of Figure 2. As long as we restrict ourselves to 2-cell embeddings of graphs, there is a one-to-one correspondence between the topological embeddings of the graph G on closed oriented surfaces on the one hand, and the combinatorial embeddings of (P_G, Π_G) defined here. This correspondence is such that the boundaries of the topological regions of the embeddings of G are described by the disjoint cycle decomposition of the corresponding products PQ.

If σ is an arbitrary permutation of the set S, then $\|\sigma\|$ denotes the number of cycles in the disjoint cycle decomposition of σ. The *number of components* of the permutation-partition pair (P, Π), denoted by $c(P, \Pi)$ is the number of orbits that the group generated by P and any rotation system $Q \in \Pi$ determined in the underlying set S.

The *genus* of the embedding (P, Q) of the permutation-partition pair (P, Π) is

$$\gamma(P, Q) = c(P, \Pi) - \frac{1}{2}(\|P\| + \|Q\| + \|PQ\| - n) \tag{2}$$

where n is the cardinality of the underlying set S. For example, in the embeddings of Figure 1, this formula yields

$$1 - \frac{1}{2}(7 + 4 + 5 - 14) = 0 \text{ and } 1 - \frac{1}{2}(7 + 4 + 3 - 14) = 1.$$

It is known [6, 22, 23] that the combinatorial genus of (2) is always a nonnegative integer which, for graph embeddings, agrees with the topological genus.

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

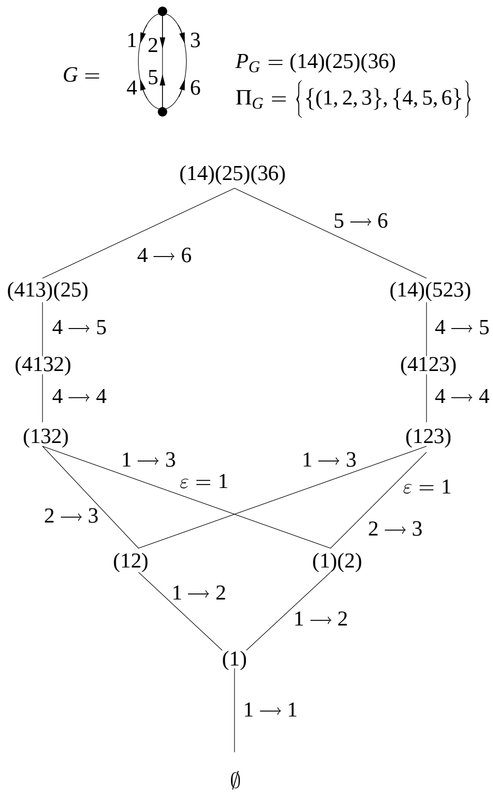


FIGURE 2

For any permutation-partition pair (P, Π) and for any nonnegative integer k, let $\gamma_{(P,\Pi)}(k)$ denote the number of embeddings of (P, Π) that have genus k. The *genus polynomial* of the pair (P, Π) is defined as

$$GP_{(P,\Pi)}(x) = \sum_{k=0}^{\infty} \gamma_{(P,\Pi)}(k) x^k.$$

The investigation of the genus polynomials of graphs was first proposed in [11], and some fairly explicit descriptions have been obtained for several infinite families of graphs [8, 9, 16, 17, 19, 20, 21]. Some more will be described below.

It is our purpose here to investigate the zeros of the genus polynomials of several of these families of graphs as well as those of some new families.

2. **The genus version of the Walkup reduction.** The Walkup reduction is a process that expresses the genera of the embeddings of a pair (P, Π) in terms of those of smaller pairs. It therefore makes possible inductive proofs and recurrence formulas for the genus polynomials. Since this process reduces graphical pairs to nongraphical pairs, these formulas are much harder, if not impossible, to derive when attention is restricted to embeddings of *graphs* only.

Let (P,Π) be a permutation-partition pair with an underlying set S. If $a,b \in S$, then we say that $a \equiv b \pmod{\Pi}$ whenever a and b are *distinct* elements that belong to the same member of Π. A *constraint* on the pair (P,Π) is an ordered pair, denoted by $a \longrightarrow b$, such that there exists a rotation system $Q \in S(\Pi)$ such that $aQ = b$. In other words, either $a \equiv b \pmod{\Pi}$ or else, if $a = b$, then $\{b\} \in \Pi$. Given such a constraint $a \longrightarrow b$, the set of rotation systems $Q \in S(\Pi)$ such that Q maps a to b is denoted by $S(\Pi; a \longrightarrow b)$. More compactly,

$$S(\Pi; a \longrightarrow b) = \{Q \in S(\Pi) \mid aQ = b\}.$$

If b is any element of the underlying set S, then P/b denotes the permutation of $S - \{b\}$ obtained by deleting b from the disjoint cycle decomposition of P. Thus, if $P = (1\ 2\ 3)(4)(5\ 6)$, then $P/1 = (2\ 3)(4)(5\ 6)$, and $P/4 = (1\ 2\ 3)(5\ 6)$. Similarly, Π/b denotes the partition of $S - \{b\}$ induced by Π. For any constraint $a \longrightarrow b$, where $a \neq b$, we denote by $(P,\Pi)/a \longrightarrow b$ the permutation-partition pair $(\bar{P},\bar{\Pi})$, defined over $S - \{b\}$, where

$$\bar{P} = \begin{cases} P(b\ a\ bP)/b & \text{if } a, b, bP \text{ are all distinct} \\ P(b\ bP)/b = P/b & \text{if } bP = a \neq b \\ P/b & \text{if } bP = b \neq a \\ P/b & \text{if } a = b \text{ and } \{b\} \in \Pi. \end{cases}$$

$$\bar{\Pi} = \Pi/b$$

For example, if

$$(P,\Pi) = \big((1\ 2\ 3\ 4)(5)(6\ 7), \{1,2,4 \mid 3,5,6 \mid 7\}\big),$$

then

$$\begin{aligned} (P,\Pi)/4 \longrightarrow 2 &= \big((1\ 4)(3)(5)(6\ 7), \{1,4 \mid 3,5,6 \mid 7\}\big), \\ (P,\Pi)/4 \longrightarrow 1 &= \big((2\ 3)(4)(5)(6\ 7), \{2,4 \mid 3,5,6 \mid 7\}\big), \\ (P,\Pi)/3 \longrightarrow 6 &= \big((3\ 4\ 1\ 2\ 7)(5), \{1,2,4 \mid 3,5 \mid 7\}\big), \\ (P,\Pi)/7 \longrightarrow 7 &= \big((1\ 2\ 3\ 4)(5)(6), \{1,2,4 \mid 3,5,6\}\big). \end{aligned}$$

The reader is referred to [19] for several other examples. If $\{b\} \in \Pi$ then $\bar{P} = P/b$. On the other hand, when $a \equiv b \pmod{\Pi}$ are distinct, the derivation of $\bar{P}$ from P can be described as follows. If a and b are in the same cycle $\sigma = (a\ d \cdots e\ b\ f \cdots g)$ of P, then $\bar{P}$ is obtained from P by splitting σ at a and b into two cycles, and suppressing b, so as to obtain $(a\ d \cdots e)(f \cdots g)$, all the other cycles of P being passed on intact to $\bar{P}$. On the other hand, if a and b belong to distinct cycles $(a\ d \cdots e)$ and $(b\ f \cdots g)$ of P, then $\bar{P}$ is obtained from P by coalescing these two cycles into one, and suppressing b so as to obtain $(a\ d \cdots e\ f \cdots g)$, all the other cycles of P being passed on intact to $\bar{P}$.

The following easy observations are proven in detail as Lemmas 1.1 and 1.3 of [19].

LEMMA 2.1. *Let (P,Π) be a permutation-partition pair with $a \equiv b \pmod{\Pi}$, and let $(P_a,\Pi_a) = (P,\Pi)/a \longrightarrow b$. Then the function*

$$f\colon S(\Pi; a \longrightarrow b) \longrightarrow S(\Pi_a)$$

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

defined by

$$f(Q) = [(a\ b)Q]/b \stackrel{\text{def}}{=} Q_a$$

is a bijection such that

$$\|PQ\| = \begin{cases} \|P_a Q_a\| + 1 & \text{if } a = b\,P \neq b \\ \|P_a Q_a\| & \text{otherwise.} \end{cases} \qquad \blacksquare$$

It is easy to see that if a and b are distinct and $a \equiv b \pmod{\Pi}$, then

$$c(P,\Pi) + 1 \geq c[(P,\Pi)/a \longrightarrow b] \geq c(P,\Pi). \tag{3}$$

Moreover, if a and b belong to distinct cycles of P then $c[(P,\Pi)/a \longrightarrow b] = c(P,\Pi)$. The converse does not hold, as is illustrated by $P = (1\ 2)$, $\Pi = \{1\ 2\}$, $a = 1$, $b = 2$, where $c[(P,\Pi)/a \longrightarrow b] = 1 = c(P,\Pi)$.

LEMMA 2.2. *Let (P,Π) be a permutation-partition pair such that $\{b\} \in \Pi$, and let $(P_b,\Pi_b) = (P,\Pi)/b \longrightarrow b$. Then the function*

$$f\colon S(\Pi) \longrightarrow S(\Pi_b)$$

defined by

$$f(Q) = Q/b \stackrel{\text{def}}{=} Q_b$$

is a bijection such that

$$\|PQ\| = \begin{cases} \|P_b Q_b\| & \text{if } b \neq b\,P \\ \|P_b Q_b\| + 1 & \text{if } b = b\,P. \end{cases} \qquad \blacksquare$$

The content of the next two corollaries is based on a device that the author first encountered in [27], but which had probably been used by many other investigators previously. We shall refer to these two corollaries as the *Walkup reduction* in the sequel.

COROLLARY 2.3. *If (P,Π) is a permutation-partition pair, $b \in S$ and $\{b\} \notin \Pi$, then, if $(P_a,\Pi_a) = (P,\Pi)/a \longrightarrow b$, we have*

$$GP_{(P,\Pi)}(x) = \sum_{a \equiv b \pmod{\Pi}} x^{\varepsilon(a,b)} GP_{(P_a,\Pi_a)}(x)$$

where

$$\varepsilon(a,b) = \begin{cases} 1 & \text{if } c[(P_a,\Pi_a)] = c(P,\Pi) \text{ and } \|P\| = \|P_a\| - 1 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. It suffices to show that for each such a and for each positive integer k,

$$\gamma_{(P,\Pi)}(k) = \sum_{a \equiv b \pmod{\Pi}} \gamma_{(P_a,\Pi_a)}\big(k - \varepsilon(a,b)\big).$$

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

This, in turn, is tantamount to showing that for each such a,

$$\gamma(P,Q)=\gamma(P_a,Q_a)+\varepsilon(a,b). \tag{4}$$

However,

$$\gamma(P,Q)-\gamma(P_a,Q_a)=c(P,\Pi)-c(P_a,\Pi_a)-\frac{1}{2}[(\|P\|-\|P_a\|)+(\|Q\|-\|Q_a\|)\\ +(\|PQ\|-|P_aQ_a\|)-(n-\{n-1\})].$$

Since $\{b\}\not\in\Pi$, it follows that $\|Q\|=\|Q_a\|$. If $a=bP$, then $c(P,\Pi)=c(P_a,\Pi_a)$, $\|P\|=\|P_a\|$, and, by Lemma 2.1, $\|PQ\|=\|P_aQ_a\|+1$. Thus, in this case,

$$\gamma(P,Q)-\gamma(P_a,Q_a)=0=\varepsilon(a,b).$$

Suppose next that $a\neq bP$ but a and b still belong to the same cycle of P. Then, $\|P\|=\|P_a\|-1$ and $\|PQ\|=\|P_aQ_a\|$. Hence,

$$\gamma(P,Q)-\gamma(P_a,Q_a)=c(P,\Pi)-c(P_a,\Pi_a)+1=\varepsilon(a,b).$$

Finally, if $a\neq bP$ and a and b belong to different cycles of P, then $c(P,\Pi)=c(P_a,\Pi_a)$, $\|P\|=\|P_a\|+1$, and $\|PQ\|=\|P_aQ_a\|$. Hence,

$$\gamma(P,Q)-\gamma(P_a,Q_a)=0=\varepsilon(a,b).$$

Thus, (4) has been verified in all cases. ■

COROLLARY 2.4. *Let (P,Π) be a permutation-partition pair and suppose $\{b\}\in\Pi$. If $(P_b,\Pi_b)=(P,\Pi)/b\longrightarrow b$, then*

$$GP_{(P,\Pi)}(x)=GP_{(P_b,\Pi_b)}(x).$$

PROOF. It suffices to show that for each positive integer k

$$\gamma_{(P,\Pi)}(k)=\gamma_{(P_b,\Pi_b)}(k).$$

This, in turn, is tantamount to proving that

$$\gamma(P,Q)=\gamma(P_b,Q_b). \tag{5}$$

However, if $b=bP$, then, because $\{b\}\in\Pi$ we have

$$\gamma(P,Q)-\gamma(P_b,Q_b)=c(P,\Pi)-c(P_b,\Pi_b)-\frac{1}{2}[(\|P\|-\|P_b\|)+(\|Q\|-\|Q_b\|)\\ +(\|PQ\|-\|P_bQ_B\|)-(n-\{n-1\})]\\ =1-\frac{1}{2}(1+1+1-1)=0.$$

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

On the other hand, if $b \neq bP$, then

$$\gamma(P,Q) - \gamma(P_b,Q_b) = c(P,\Pi) - c(P_b,\Pi_b) - \frac{1}{2}[(\|P\| - \|P_b\|) + (\|Q\| - \|Q_b\|) + (\|PQ\| - \|P_bQ_b\|) - (n - \{n-1\})]$$
$$= 0 - \frac{1}{2}(0 + 1 + 0 - 1) = 0.$$

Thus, (5) holds in all cases. ∎

The conclusions of Corollaries 2.3 and 2.4 are the genus version of the Walkup reduction of [19]. Thus, we can associate with each pair (P,Π) a (genus) *reduction diagram* $GT_{(P,\Pi)}$ which differs from the (region) reduction diagram of [19] only in that the edge labels $\delta = \delta_{a,bP}$ and $\delta = \delta_{b,bP}$ are replaced with $\varepsilon = \varepsilon(a,b)$ and $\varepsilon = 0$ respectively. As was the case with the original reduction diagrams of [19], labels of the form $\varepsilon = 0$ will be suppressed. Figure 2 contains the genus version of the reduction diagram of Figure 3 of [19]. For the sake of completeness, we also include an explicit definition of $GT_{(P,\Pi)}$.

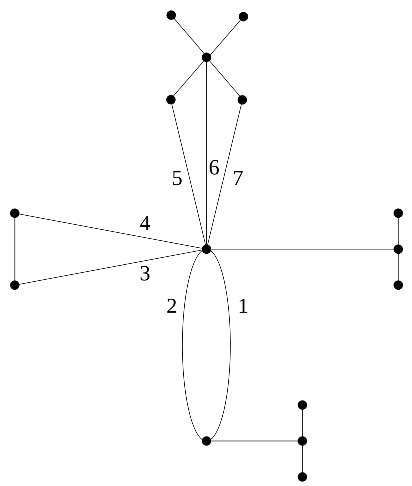


FIGURE 3. A vertex-forest multijoin of type $(2^2, 3)$ and strength 7.

The complete (genus) reduction diagram $GT_{(P,\Pi)}$ of the pair (P,Π) is a ranked weighted directed graph whose vertices are permutation-partition pairs, whose edges picture the above reduction process, and which is constructed as follows. Let $b_1, b_2, \ldots, b_n$ be any linear ordering of the set S that underlies the pair (P,Π) and let $GT^{(0)}_{(P,\Pi)} = \{(P,\Pi)\}$. Assuming that the vertex set $GT^{(i)}_{(P,\Pi)}$ $(0 \le i < n)$ has been defined, let $(\bar{P},\bar{\Pi})$ be any vertex in this set $GT^{(i)}_{(P,\Pi)}$. If $\{b_{i+1}\}$ is a singleton member of $\bar{\Pi}$, then $(\bar{P},\bar{\Pi})$ has only one descendent, namely $(\bar{P},\bar{\Pi})/b_{i+1} \longrightarrow b_{i+1}$. The edge from $(\bar{P},\bar{\Pi})$ to $(\bar{P},\bar{\Pi})/b_{i+1} \longrightarrow b_{i+1}$ is assigned the weight $\varepsilon = 0$. On the other hand, if $\{b_{i+1}\}$ is not a singleton member of $\bar{\Pi}$, then each of the pairs $(\bar{P},\bar{\Pi})/a \longrightarrow b_{i+1}$ $(a \equiv b_{i+1})$ is a descendent of $(\bar{P},\bar{\Pi})$. Each branch from $(\bar{P},\bar{\Pi})$ to any of its descendents $(\bar{P},\bar{\Pi})/a \longrightarrow b_{i+1}$ is assigned the weight $\varepsilon(a, b_{i+1})$.

The vertex set $GT^{(i+1)}_{(P,\Pi)}$ consists of the set of all the descendents of all the vertices in $GT^{(i)}_{(P,\Pi)}$. It is clear that each pair of vertices $(\bar{P},\bar{\Pi})$ and $(\bar{\bar{P}},\bar{\bar{\Pi}})$ in $GT^{(i)}_{(P,\Pi)}$ have $\bar{\Pi}=\bar{\bar{\Pi}}$ and also have the same underlying set $\bar{S}=\{b_{i+1},b_{i+2},\ldots,b_n\}$. The set $GT^{(n)}_{(P,\Pi)}$ consists of only the trivial pair (ϕ,ϕ). The next lemma follows immediately from Corollaries 2.3 and 2.4.

LEMMA 2.5. *The embeddings of the pair (P,Π) are in a one-to-one correspondence with the directed paths of $GT_{(P,\Pi)}$ that start from (P,Π) and end at (ϕ,ϕ). This correspondence is such that the genus of each embedding is given by the sum of the weights along its corresponding path.* ■

3. **Polynomials all of whose zeros are real.** Polynomials all of whose zeros are real, have arisen in a variety of contexts, and there is a fair amount of interest in and an extensive literature on this subject [4, 13, 24, 25, 26]. The geometric distribution of the zeros of families of such polynomials has also been investigated [4]. Our interest in this type of polynomial was originally due to its implication about unimodality [Theorem 3.1]. It seems likely, however, that information about the zeros of the genus polynomial will eventually lead to information about the genus distribution. Lemmas 3.2 and 3.3 below contain minor variations of some well known [25, 26] techniques for proving that all the zeros of a given polynomial are real.

A sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to be the *unimodal* if there is an integer k, $0\le k\le n$, such that

$$a_0\le a_1\le\cdots\le a_k\ge a_{k+1}\ge\cdots\ge a_n.$$

A sequence $a_0, a_1, \ldots, a_n$ is said to be *log concave* if

$$a_{i-1}a_{i+1}\le a_i^2 \qquad i=1,2,\ldots,n-1.$$

The following proposition is well known and easily demonstrated.

PROPOSITION 3.1. *Every log concave sequence of positive real numbers is also unimodal.* ■

There are several techniques for proving that a given sequence is log concave, and one of them is the following. The reader is referred to [5, (p. 270), 24] for a proof.

PROPOSITION 3.2. *(Newton) If all the zeros of the polynomial*

$$a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$$

are real, then the sequence $a_0, a_1, \ldots, a_n$ is log concave. ■

A commonly used technique for proving that all the zeros of a certain polynomial are real calls for locating another polynomial whose zeros must interlace with those of the given polynomial. Moreover, we are concerned here with genus polynomials whose coefficients, being cardinalities of certain sets, are necessarily positive. Hence their real

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

zeros are necessarily negative. Accordingly, we make the following definitions. Suppose the zeros of the polynomials $P(x)$ and $Q(x)$ are simple and can be listed as $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ respectively, where

$$x_n < y_n < x_{n-1} < y_{n-1} < \cdots < x_1 < y_1 < 0.$$

We say that the zeros of $P(x)$ and $Q(x)$ *interlace* and write $P(x) \leq Q(x)$. If, on the other hand, the zeros of the polynomials $P(x)$ and $Q(x)$ are simple and can be listed as x_1, $x_2, \ldots, x_{n-1}$ and $y_1, y_2, \ldots, y_n$ respectively, where

$$y_n < x_{n-1} < y_{n-1} < \cdots < x_1 < y_1 < 0,$$

then we write $P(x) < Q(x)$ and again say that the zeros of these polynomials *interlace*. We shall use the notation $P(x) <, \leq Q(x)$ to denote the fact the zeros of these polynomials interlace in one of these two senses. While this employment of the inequality symbol is useful, the reader should be warned that this relation of interlacing is *not* transitive.

The following two lemmas provide the basis for several inductive proofs in the sequel. As they contain little that is not already known [4, 25, 26], their proofs are omitted.

LEMMA 3.3. *Let $P(x)$ and $Q(x)$ be two polynomials and let a, b c, d be positive reals. If $P(x) \leq Q(x)$, then*
(1.) $P(x) \leq aP(x) + bQ(x) \leq Q(x)$
(2.) $Q(x) < cxP(x) + dQ(x)$
(3.) $Q(x) \leq aQ(x) - bP(x)$ *if* $aQ(0) - bP(0)$ *is positive.*
(4.) $aP(x) + bQ(x) < cxP(x) + dQ(x)$. ■

LEMMA 3.4. *Let $P(x)$ and $Q(x)$ be two polynomials and let a, b c, d be positive reals. If $P(x) < Q(x)$, then*
(1.) $P(x) < aP(x) + bQ(x) \leq Q(x)$
(2.) $Q(x) \leq axP(x) + bQ(x)$
(3.) $Q(x) \leq aQ(x) - bP(x)$ *if* $aQ(0) - bP(0)$ *is positive.*
(4.) $aP(x) + bQ(x) \leq cxP(x) + dQ(x)$. ■

4. **Vertex-forest multijoins.** Let B_q denote the bouquet on q circles, that is, B_q is the graph consisting of a single vertex and q loops. Based on some permutation counting results of [12, 14] it was proved in [9] that for $q > 2$,

$$(q+1)\gamma_{B_q}(k) = 4(2q-1)(2q-3)(q-1)^2(q-2)\gamma_{B_{q-2}}(k-1) + 4(2q-1)(q-1)\gamma_{B_{q-1}}(k). \tag{6}$$

It follows that for $q > 2$,

$$GP_{B_q}(x) = \frac{4(2q-1)(2q-3)(q-1)^2(q-2)}{q+1} xGP_{B_{q-2}}(x) + \frac{4(2q-1)(q-1)}{q+1} GP_{B_{q-1}}(x). \tag{7}$$

Since $GP_{B_1}(x) = 1$ and $GP_{B_2}(x) = 4 + 2x$, we have here a complete description of the genus polynomial of the bouquet. We note that $GP_{B_3}(x) = 40 + 80x$ and $GP_{B_4}(x) = 336(2 + 10x + 3x^2)$.

PROPOSITION 4.1. *The zeros of the genus polynomial of the bouquet B_q on q circles are real, negative, and satisfy the following relations:*

$$GP_{B_q}(x) < GP_{B_{q+1}}(x) \quad \text{if } q \text{ is odd} \qquad (8)$$
$$GP_{B_q}(x) \leq GP_{B_{q+1}}(x) \quad \text{if } q \text{ is even.}$$

PROOF. We proceed by induction on q. The cases $q = 1, 2, 3$ call for the inspection of the polynomials listed above and are easily verified. Assume that the proposition holds for all positive integers $q < k$. It follows from formula (7) that there exist real numbers c and d such that

$$GP_{B_{k+1}}(x) = cxGP_{B_{k-1}}(x) + dGP_{B_k}(x) \qquad (9)$$

If k is odd, then, by the induction hypothesis

$$GP_{B_{k-1}}(x) \leq GP_{B_k}(x)$$

and an application of Lemma 3.3 (2.) to (9) above yields

$$GP_{B_{k+1}}(x) = cxGP_{B_{k-1}}(x) + dGP_{B_k}(x) > GP_{B_k}(x).$$

Similarly, if k is even, then, by the induction hypothesis

$$GP_{B_{k-1}}(x) < GP_{B_k}(x)$$

and an application of Lemma 3.4 (2.) to (9) above yields

$$GP_{B_{k+1}}(x) = cxGP_{B_{k-1}}(x) + dGP_{B_k}(x) \geq GP_{B_k}(x).$$

■

The following corollary was first proved in [9]. Here it is an immediate consequence of the above Proposition and Newton's Theorem 3.2.

COROLLARY 4.2. *The genus distribution of the bouquet on n circles is log concave.* ■

These observations can be used to garner information about other infinite families of graphs as well. A vertex-forest multijoin is a graph G with a vertex u such that $V(G)-\{u\}$ induces a subforest of G. If $T_1, T_2, \ldots, T_t$ are the components of this induced subforest, and if the central vertex u of G is joined to T_i by λ_i edges, then we may assume that the labeling is such that

$$2 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s \text{ and } 0 \leq \lambda_{s+1} \leq \lambda_{s+2} \leq \cdots \leq \lambda_t \leq 1.$$

We refer to the s-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ as the *type* of G and to $n = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ as its *strength*. The region distribution of such a graph G was derived in the proof of Theorem 3.2 of [21] and can be restated in terms of the genus as follows. Let G be a vertex-forest multijoin of type $(\lambda_1, \lambda_2, \ldots, \lambda_s)$ and strength n. Let (P, Π) be any permutation partition pair wherein P consists of s cycles of lengths $\lambda_1, \lambda_2, \ldots, \lambda_s$ respectively, and Π has only one member, namely $\Pi = \{\{1, 2, \ldots, n\}\}$. Then there is an integer a_G such that

$$GP_G(x) = a_G GP_{(P,\Pi)}(x). \tag{10}$$

The value of a_G is easily deduced from the observation that the total number of embeddings of any permutation-partition pair (P, Π), is $\prod_{i=1}^{k}(p_i - 1)!$, where $p_1, p_2, \ldots, p_k$ are the respective cardinalities of the constituent members of Π. All the permutation-partition pairs in this section will be such that Π contains only one set, and it is clear that in this special case $GP_{(P,\Pi)}(x)$ depends only on the cycle structure of P. Hence, for the remainder of this section, all mention of Π will be suppressed and P will be encoded in terms of the number of cycles of each length that it possesses, in the usual manner. Thus, we shall write

$$GP_{(2^n)}(x) \text{ for } GP_{((1\ 2)(3\ 4)\cdots(2n-1\ 2n),\{\{1,2,\ldots,2n\}\})}(x)$$

and

$$GP_{(2^2,3)}(x) \text{ for } GP_{((1\ 2)(3\ 4)(5\ 6\ 7),\{\{1,2,3,4,5,6,7\}\})}(x).$$

If each edge of the bouquet B_q is subdivided once, we obtain a vertex-forest multijoint of type (2^q) and strength $2q$. It therefore follows that for some real b_n $n = 1, 2, 3, \ldots,$

$$b_n GP_{(2^n)}(x) = GP_{B_n}(x).$$

The Walkup reduction can now be used to obtain the genus polynomials corresponding to some other permutations (and so also to some other vertex-forest multijoins). Suppose $P = (1\ 2)(3\ 4)\cdots(2n-1\ 2n)$. Then the branch of the reduction diagram corresponding to the constraint $(2n-1) \longrightarrow 2n$ yields the descendent

$$P_1 = (1\ 2)(3\ 4)\cdots(2n-3\ 2n-2)(2n-1) \text{ of type } (1, 2^{n-1})$$

with $\varepsilon(2n-1, n) = 0$. On the other hand, each of the branches $a \longrightarrow 2n$, $a = 1, 2, \ldots, 2n-2$ yields a descendent of type $(2^{n-2}, 3)$ with $\varepsilon(a, 2n) = 0$. It follows from two applications of Corollary 2.3 that

$$\begin{aligned} GP_{(2^n)}(x) &= GP_{(1,2^{n-1})}(x) + 2(n-1)GP_{(2^{n-2},3)}(x) \\ &= 2(n-1)GP_{(2^{n-1})}(x) + 2(n-1)GP_{(2^{n-2},3)}(x). \end{aligned}$$

Thus,

$$\begin{aligned} GP_{(2^{n-2},3)}(x) &= \frac{1}{2(n-1)}GP_{(2^n)}(x) - GP_{(2^{n-1})}(x) \\ &= \frac{1}{2(n-1)b_n}GP_{B_n}(x) - \frac{1}{b_{n-1}}GP_{B_{n-1}}(x). \end{aligned} \tag{11}$$

PROPOSITION 4.3. *All the zeros of the genus polynomial of any vertex-forest multijoin G of type* $(2^n, 3)$ *are real and negative. Moreover*

$$GP_{B_{n+2}}(x) \le GP_G(x).$$

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

PROOF. In order to apply Lemmas 3.3 (3.) and 3.4 (3.), it will be necessary to know that the constant term of $GP_G(x)$ is not zero, *i.e.*, that G has a plane embedding. This, however, is clear from the fact that G is planar since it is the one point amalgamation of $n+1$ planar graphs. By Proposition 4.1,

$$GP_{B_{n+1}}(x) <, \leq GP_{B_{n+2}}(x)$$

and so, it follows from (11) above and Lemmas 3.3 (3.), 3.4 (3.) that for some real c_n, d_n,

$$GP_G(x) = c_n GP_{B_{n+2}}(x) - d_n GP_{B_{n+1}}(x) \geq GP_{B_{n+2}}(x). \qquad \blacksquare$$

COROLLARY 4.4. *The genus distribution of every vertex-forest multijoin of type* $(2^n, 3)$ *is log concave.* ∎

If the graph G has a bridge e, then it is known [11] that the genus polynomial of G is the product of the genus polynomials of the two components of $G - e$. The genus polynomial is also known for several infinite families of graphs that fall into the two categories discussed in this section and the next. Loosely speaking, one may speak of the graphs of this section as short while characterizing those of the next as long. The author believes that these two categories are significant in that they represent the two opposite ends of a spectrum of types. Possibly the growth of the average genus (logarithmic versus linear in the number of edges) can be used to formalize this spectrum, but as yet not enough information is available. In view of this it would be useful to find other aspects of the genus that differentiate between families in the two categories. It is for this purpose that the following proposition is presented.

PROPOSITION 4.5. *There exists a constant k such that the genus polynomial a of vertex-forest multijoins of type* (2^n) *or* $(2^n, 3)$ *has a zero of magnitude less than* k/n^2, *for* $n = 1, 2, 3, \ldots$.

PROOF. It follows from Propositions 4.1 and 4.3 that it suffices to prove the proposition for graphs of type (2^n) where n is even. Assume therefore, that n is indeed even. By definition, the constant term of $GP_{B_n}(x)$ is $\gamma_{B_n}(0)$ which is not zero because B_n has some obvious plane embeddings. Easy induction arguments based on (6, 7) allow us to conclude that $GP_{B_n}(x)$ has degree $n/2$ and that

$$\frac{\gamma_{B_n}(0)}{\gamma_{B_n}(n/2)} = \frac{2^n}{n!}.$$

Hence, if $r_1, r_2, \ldots, r_{n/2}$ are the zeros of $GP_{B_n}(x)$, which are already known to be real, then

$$\sqrt[n/2]{\prod_{i=1}^{n/2} |r_i|} = \sqrt[n/2]{\frac{2^n}{n!}}$$

and the required upper bound follows from Stirling's formula. ∎

For example, the zeros of $GP_{B_{20}}(x)$, rounded off to 4 decimal places, are

$$\{-7.3700, -.7080, -.2122, -.0904, -.0459, \\ -.0259, -.0156, -.0097, -.0062, -.0039\}.$$

The *dipole* DP_n is the graph that consists of n multiedges joining two distinct vertices. The genus polynomial of the dipole was derived, in equivalent form, in [17]. Specifically, let $s(n,k)$ be the (absolute value of) the Stirling number of the first kind, *i.e.*, let

$$s(n,k) = s(n-1,k-1) + (n-1)s(n-1,k) \quad n,k = 1,2,3,\ldots$$
$$s(n,0) = s(0,k) = 0, \quad \text{except } s(0,0) = 1.$$

Then, if

$$\mathrm{SO}_n(x) = \sum_{k=0}^{[n/2]-1} s(n, n-2k-1)x^k$$

we have

$$GP_{\mathrm{DP}_n}(x) = 2\frac{(n-1)!^2}{(n+1)!}\,\mathrm{SO}_{n+1}(x). \tag{12}$$

Since the dipole DP_n is a vertex-forest multijoin of type (n), it follows from (10) that if G is *any* vertex-forest multijoin of type (n), then its genus polynomial is also given by (12). Since the forest portion of any vertex-forest multijoin is arbitrary, type (n) covers a large set of homeomorphically distinct graphs. Vertex-forest multijoins of type (n) will be referred to as *vertex-tree multijoins*.

PROPOSITION 4.6. *All the zeros of the genus polynomial of each vertex-tree multijoin are real and negative.*

PROOF. Define

$$\mathrm{SE}_n(x) = \sum_{k=0}^{[n/2]} s(n, n-2k)x^k.$$

It is then easily verified from the recursive definition of $s(n,k)$ that

$$\begin{aligned} \mathrm{SO}_n(x) &= \mathrm{SO}_{n-1}(x) + (n-1)\,\mathrm{SE}_{n-1}(x) \quad n = 2,3,4,\ldots \\ \mathrm{SE}_n(x) &= (n-1)x\,\mathrm{SO}_{n-1}(x) + \mathrm{SE}_{n-1}(x). \end{aligned} \tag{13}$$

It now follows from Lemmas 3.3 (4.) and 3.4 (4.) and a straightforward induction that

$$\mathrm{SO}_n(x) <,\leq \mathrm{SE}_n(x). \tag{14}$$

In particular, all the zeros of $\mathrm{SO}_n(x)$ are real, and so, in view of the comments preceding the theorem, we are done. ∎

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

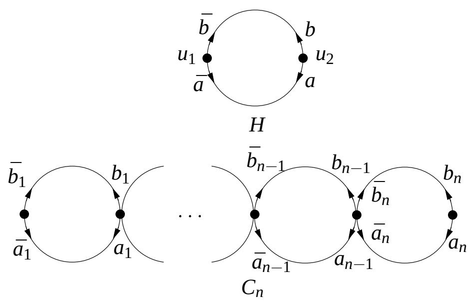


FIGURE 4

COROLLARY 4.7. *The genus distribution of the dipole is log concave.* ∎

COROLLARY 4.8. *All the zeros of the genus polynomial of a vertex-forest multijoin of type* $(2, n)$ *are real and negative.*

PROOF. Let G be a vertex-forest multijoin of type $(2, n)$. Then, by (10) there exist real numbers a_n such that upon applying the Walkup reduction of Corollary 2.3 we get

$$GP_G(x) = a_n GP_{(2,n)}(x) = a_n GP_{(1,n)}(x) + a_n n GP_{(n+1)}(x) \\ = a_n n GP_{(n)}(x) + a_n n GP_{(n+1)}(x). \tag{15}$$

Since $GP_{(n)}(x) = b\,\mathrm{SO}_{n+1}(x)$, it follows from (13, 14), and Lemmas 3.3 (1.) and 3.4 (1.) that

$$GP_{(n)}(x) <, \leq GP_{(n+1)}(x) \tag{16}$$

and so, when Lemmas 3.3 (1.) and 3.4 (1.) are applied to (16) we conclude that

$$GP_n(x) <, \leq GP_G(x) \tag{17}$$

and so all the zeros of $GP_G(x)$ are real and negative. ∎

PROPOSITION 4.9. *There exists a constant* k' *such that the genus polynomial of a vertex-forest multijoin of type* (n) *or* $(2, n)$ *has a zero of magnitude less than* k'/n^2.

PROOF. In view of (17) it suffices to prove this proposition for type (n) only. The constant term of the polynomial $\mathrm{SO}_n(x)$ and its leading coefficient are, respectively, $s(n, n-1) = \binom{n}{2}$ and either $s(n, 1)$ or $s(n, 2)$. Since

$$s(n, 2) \geq s(n, 1) \geq (n-1)!,$$

the desired result now follows by an argument similar to that of Proposition 4.5 above. ∎

5. *H*-Linear families of graphs.

If H is any graph, then the family of graphs obtained by consistently amalgamating additional copies of H is called an *H-linear family of graphs*. A more formal definition appears in [19]. The examples that appear in Figures 4, 7,8, 9 should give the reader a sufficient grasp of the concept.

Let $G = \{G_n\}_{n=1}^{\infty}$ be an H-linear family of graphs. The argument of [19] that was used to obtain expressions for the region distribution of G_n can be easily modified to yield a similar expression for the genus polynomial of G_n.

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

EXAMPLE 5.1. The graph G_n of Figure 4, the cobblestone path of [8], is obtained by successively amalgamating n copies of H. Let G'_n denote the permutation-partition pair obtained from (P_{G_n}, Π_{G_n}) by replacing the two transpositions $(a_n\bar{a}_n)(b_n\bar{b}_n)$ with $(a_n\bar{a}_n b_n\bar{b}_n)$. Apply the genus version of the Walkup reduction to (P_{G_n}, Π_{G_n}) so as to eliminate all the bits in the last (n-th) copy of H. Figure 5, 6 illustrate this process. Since the initial portion of the genus reduction diagram GT_{G_n} displayed in Figure 5 contains 4 paths of weight 0 from node P_{G_n} to node $P_{G_{n-1}}$ and 2 paths of weight 0 to node $P_{G'_{n-1}}$, it follows that

$$GP_{G_n}(x) = 4GP_{G_{n-1}}(x) + 2GP_{G'_{n-1}}(x).$$

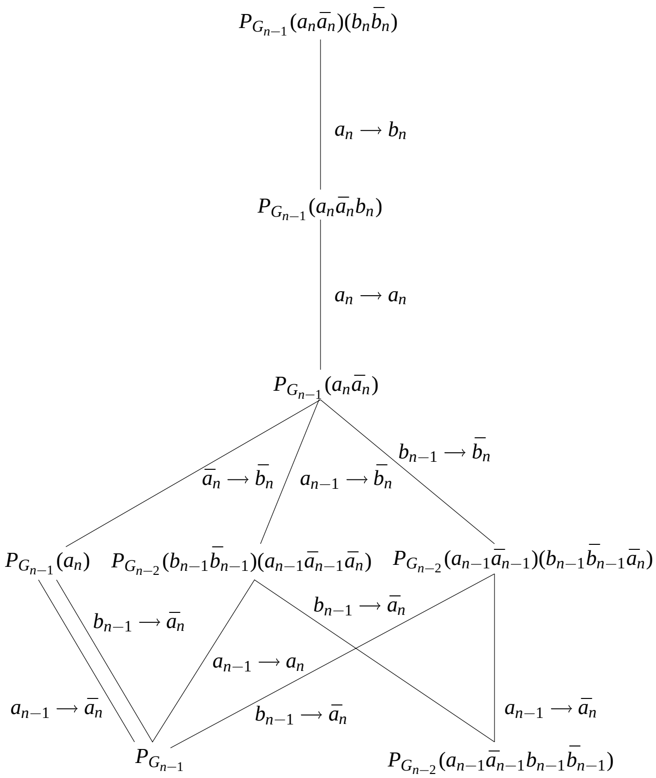


FIGURE 5

Similarly, since the initial portion of the genus reduction diagram $GT_{G'_n}$ displayed in Figure 6 contains 6 paths of weight 1 from the node $P_{G'_n}$ to the node $P_{G_{n-1}}$, it follows that

$$GP_{G'_n}(x) = 6xGP_{G_{n-1}}(x).$$

Thus, if we let $\mathbf{v}_n(x)$ be the column vector $\binom{GP_{G_n}(x)}{GP_{G'_n}(x)}$, then the above discussion is tantamount to the equation

$$\mathbf{v}_n(x) = \begin{pmatrix} 4 & 2 \\ 6x & 0 \end{pmatrix} \mathbf{v}_{n-1}(x) \quad n = 2, 3, 4, \ldots.$$

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

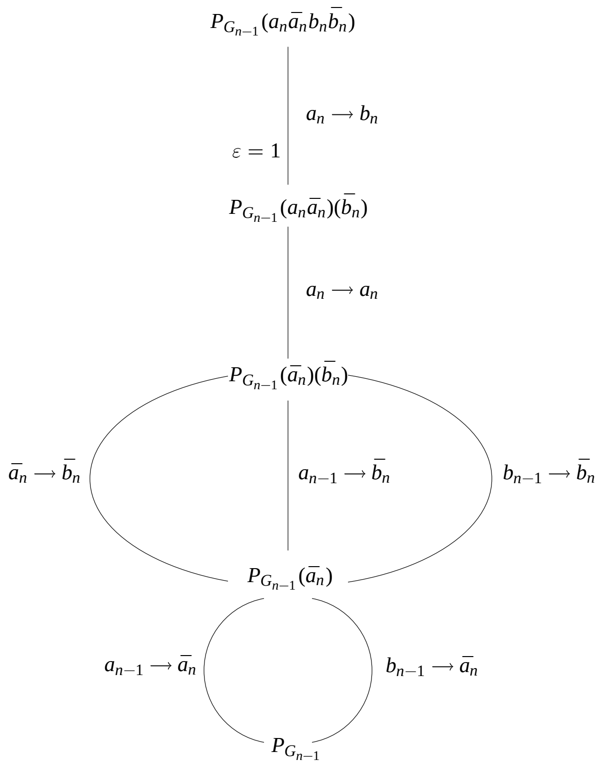


FIGURE 6

Since $(P_{G_1}, \Pi_{G_1}) = \big((a_1\bar{a}_1)(b_1\bar{b}_1), \{a_1, b_1 \mid \bar{a}_1, \bar{b}_1\}\big)$ and $G_1' = \big((a_1\bar{a}_1 b_1\bar{b}_1), \{a_1, b_1 \mid \bar{a}_1, \bar{b}_1\}\big)$, it follows that

$$\mathbf{v}_1(x) = \begin{pmatrix} GP_{G_1}(x) \\ GP_{G_1'}(x) \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix}.$$

Hence,

$$\mathbf{v}_2(x) = \begin{pmatrix} 4+2x \\ 6x \end{pmatrix},$$
$$\mathbf{v}_3(x) = \begin{pmatrix} 16+20x \\ 24x+12x^2 \end{pmatrix},$$
$$\mathbf{v}_4(x) = \begin{pmatrix} 64+128x+24x^2 \\ 96x+120x^2 \end{pmatrix}, \quad etc.$$

This procedure generalizes to the following analog of Theorem 2.6 of [19]. The set of polynomials in x with integer coefficients is denoted here by $\mathbb{Z}[x]$.

PROPOSITION 5.2. *Let $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ be an H-linear family of graphs. Then there exist a positive integer d, a $d \times d$ matrix M and a column d-vector $\mathbf{v}(x)$, with entries in $\mathbb{Z}[x]$, such that the first entry of $M^n\mathbf{v}$ is $GP_{G_n}(x)$.* ∎

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

EXAMPLE 5.3. The ladder L_n of Figure 7. Here,

$$M = \begin{pmatrix} 0 & 4 \\ 2x & 2 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

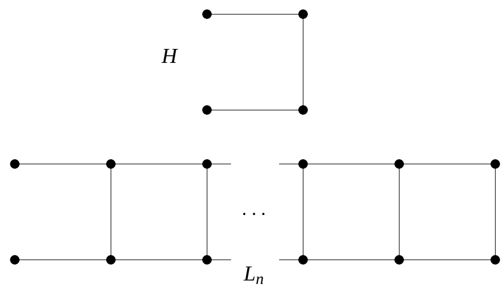


FIGURE 7

EXAMPLE 5.4. The double ladder LL_n of Figure 8. Here,

$$M = 6\begin{pmatrix} 3x & 3 \\ 2x & 1+3x \end{pmatrix} \text{ and } \mathbf{v} = 2\begin{pmatrix} 2 \\ 1+x \end{pmatrix}.$$

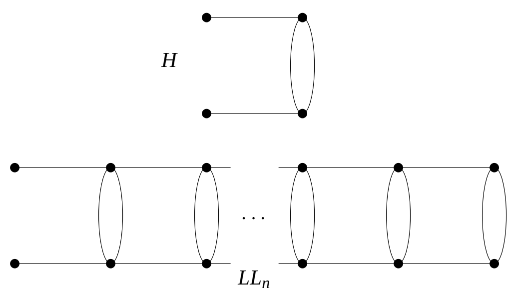


FIGURE 8

EXAMPLE 5.5. The diamond band D_n of Figure 9. Here,

$$M = 4\begin{pmatrix} 2+3x & 1 \\ 4x & 2x \end{pmatrix} \text{ and } \mathbf{v} = 2\begin{pmatrix} 1+x \\ 2x \end{pmatrix}.$$

It is convenient to denote the set of zeros of the polynomial $P(x)$ by $Z\big(P(x)\big)$.

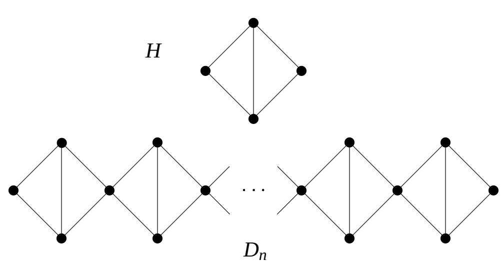


FIGURE 9

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

PROPOSITION 5.6. *Let $C = \{C_n\}_{n=1}^{\infty}$ denote the family of cobblestone paths. Then $\cup_{n=1}^{\infty} Z\big(GP_{C_n}(x)\big)$ is a dense subset of $(-\infty, -\frac{1}{3})$.*

PROOF. Let $M = \begin{pmatrix} 4 & 2 \\ 6x & 0 \end{pmatrix}$ be the generating matrix of the cobblestone paths $C = \{C_n\}_{n=1}^{\infty}$, and let

$$M^n = \begin{pmatrix} A_n(x) & B_n(x) \\ \cdot & \cdot \end{pmatrix}.$$

We first describe the zeros of $B_n(x)$ and only then go on to those of $A_n(x)$ and $GP_{C_n}(x)$. The eigenvalues of M are the distinct functions $\lambda_{1,2} = 2(1 \pm \sqrt{1+3x})$. Since M is similar to the diagonal matrix with the same eigenvalues, it follows that for $n = 1, 2, 3, \ldots$

$$B_n(x) = M_{1,2}\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = 2\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}$$

and

$$\begin{aligned} A_n(x) &= -\lambda_1\lambda_2\frac{\lambda_1^{n-1} - \lambda_2^{n-1}}{\lambda_1 - \lambda_2} + M_{1,1}\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \\ &= 6xB_{n-1}(x) + 2B_n(x) \end{aligned} \tag{18}$$

Hence, the zeros of $B_n(x)$ are those zeros of $\lambda_1^n(x) - \lambda_2^n(x)$ which are not zeros of $\lambda_1(x) - \lambda_2(x)$. These are the solutions of

$$\Big(\frac{1+\sqrt{1+3x}}{1-\sqrt{1+3x}}\Big)^n = 1,$$

with the exception of the value $-\frac{1}{3}$. Consequently the zeros of $B_n(x)$ can be listed as

$$x_{n,k} = \frac{1}{3}\Big[\Big(\frac{1-e^{2\pi ik/n}}{1+e^{2\pi ik/n}}\Big)^2 - 1\Big] \quad k = 1, 2, \ldots, \Big\{\frac{n}{2}\Big\} - 1.$$

The values $k = 0, n/2$ were excluded because $x_{n,0} = -\frac{1}{3}$ and $x_{n,n/2} = \infty$, which are not zeros of $B_n(x)$. The values $k = \{n/2\}, \ldots, n-1$ are redundant since $x_{n,n-k} = x_{n,k}$. Since the Möbius transformation

$$T(z) = \frac{1-z}{1+z}$$

maps the upper half of the unit circle onto the negative y-axis it follows that

$$Z\big(GP_{B_n}(x)\big) \subset \Big(-\infty, -\frac{1}{3}\Big) \quad n = 1, 2, \ldots.$$

Moreover, since $T(z)$ is a homeomorphism of the Riemann sphere onto itself, and since the roots of unity are dense in the unit circle, it follows that $\cup_{n=1}^{\infty} Z\big(GP_{B_n}(x)\big)$ is in fact a dense subset of $(-\infty, -\frac{1}{3})$. It also follows from the interlacing of the $(n-1)$-st and n-th imaginary roots of unity on the unit circle, and from the bicontinuity of $T(z)$, that

$$B_{n-1}(x) <, \leq B_n(x).$$

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

We conclude from (18) and Lemmas 3.3 (2.) and 3.4 (2.) that

$$B_n(x) <, \leq A_n(x)$$

Finally, it follows from Example 5.1 and Proposition 5.2 that

$$GP_{C_n}(x) = A_n(x) + xB_n(x).$$

Consequently, by Lemmas 3.3 (2.) and 3.4 (2.) we have

$$A_n(x) <, \leq GP_{C_n}(x)$$

Thus, $Z\big(GP_{C_n}(x)\big)$ is a subset of $(-\infty, -\frac{1}{3})$ for each $n = 1, 2, 3, \ldots$. Moreover, since the transformation T is a homeomorphism, it follows from the fact that $\cup_{n=1}^{\infty} Z\big(B_n(x)\big)$ is a dense subset of $(-\infty, -\frac{1}{3})$ that so are first $\cup_{n=1}^{\infty} Z\big(A_n(x)\big)$ and next $\cup_{n=1}^{\infty} Z\big(GP_{C_n}(x)\big)$ dense subsets of $(-\infty, -\frac{1}{3})$. ∎

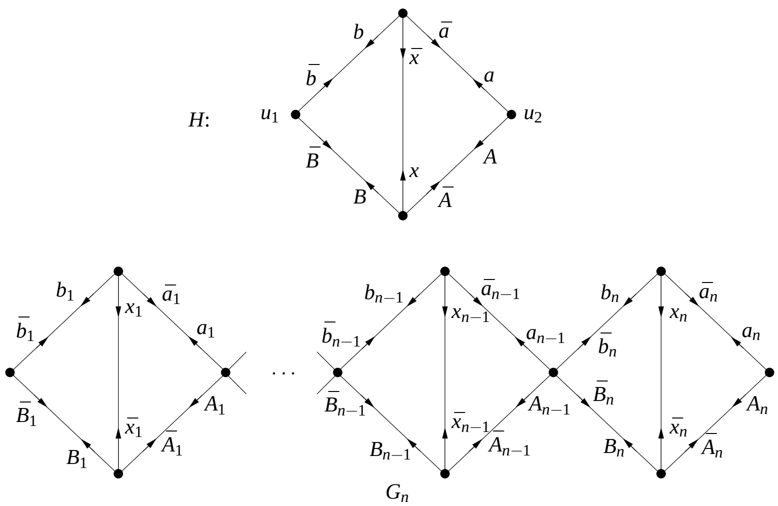


FIGURE 10

The procedure used in the above proposition can also be used to show that the zeros in the genus polynomials of the ladders of Example 5.3 are real, negative, and dense in the interval $(-\infty, -\frac{1}{8})$. The following proposition illustrates a different technique for proving that the genus polynomials of an H-linear family of graphs have real and negative zeros.

PROPOSITION 5.7. *Let $G = \{D_n\}_{n=1}^{\infty}$ denote the family of diamond bands. Then all the zeros of $GP_{D_n}(x)$ are real and negative for each $n = 1, 2, 3, \ldots$.*

PROOF. Let $M = 4\begin{pmatrix} 2+3x & 1 \\ 4x & 2x \end{pmatrix}$ be the generating matrix of the diamond bands $D = \{D_n\}_{n=1}^{\infty}$ (Example 5.5), and let

$$M^n = \begin{pmatrix} A_n(x) & B_n(x) \\ \cdot & \cdot \end{pmatrix}.$$

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

We will show that the polynomials $A_n(x)$ and $B_n(x)$ interlace. To do this we show that if $P(x)$, $Q(x)$, $P'(x)$, $Q'(x)$ are polynomials such that

$$\big(P(x), Q(x)\big)M = \big(P'(x), Q'(x)\big)$$

then

$$Q(x) < P(x) \text{ implies } Q'(x) < P'(x). \tag{19}$$

However, by Lemma 3.3 (2.) and 3.4 (2.) we have

$$\begin{aligned} P(x) \leq 2xQ(x) + P(x) &< 3xP(x) + 2[2xQ(x) + P(x)] \\ &= (2 + 3x)P(x) + 4xQ(x) \end{aligned}$$

Since $Q'(x) = P(x) + 2xQ(x)$ and $P'(x) = (2 + 3x)P(x) + 4xQ(x)$, we have a proof of (19). In as much as $\big(A_n(x)\ B_n(x)\big)M = \big(A_{n+1}(x)\ B_{n+1}(x)\big)$ it now follows by a straightforward induction argument that

$$B_n(x) < A_n(x) \quad n = 1, 2, 3, \ldots.$$

It follows from Proposition 5.2 and Example 5.5 that

$$GP_{D_n}(x) = 2(1 + x)A_n(x) + 4xB_n(x).$$

However, by Lemma 3.4 (1.)

$$A_n(x) + 2B_n(x) \leq A_n(x)$$

and so by Proposition 5.2, Example 5.5, and Lemma 3.4 (4.)

$$\begin{aligned} GP_{D_n}(x) &= 2(1 + x)A_n(x) + 4xB_n(x) = 2A_n(x) + 2x[A_n(x) + 2B_n(x)] \\ &\geq 2A_n(x) + 2[A_n(x) + 2B_n(x)] = 4[A_n(x) + B_n(x)]. \end{aligned}$$

Thus, the zeros of $GP_{D_n}(x)$ are real and negative for each positive integer n. ∎

6. **Conclusion.** It was conjectured in [9] that the genus distribution (*i.e.*, the sequence of the coefficients of the genus polynomial) of every graph is log concave. The evidence in favor of this conjecture is not overwhelming. Nothing is known above and beyond the facts proved or reproved in this paper and the theorem of [11] which states that the genus polynomial of a graph G with a bridge e is the product of the genus polynomials of the two connected components of $G-e$. The main theorem of [15] guarantees that in that case G inherits the log concavity of the components of $G - e$. We offer here some related questions and conjectures.

QUESTION 6.1. Is $\cup_{n=1}^{\infty} Z\big(GP_{B_n}(x)\big)$ a dense subset of $(-\infty, 0)$?

It is of course already known that the union in question is a subset of $(-\infty, 0)$ and that 0 is one of its limit points. The gist of this question therefore is whether this union is unbounded and/or dense. Since the genus polynomials of B_n, as well as those of all vertex-forest multijoins, are known to converge in some sense to the generating polynomial of the Stirling numbers [21], this question might lend itself to resolution.

QUESTION 6.2. Is $\cup_{n=1}^{\infty} Z\big(GP_{\mathrm{DP}_n}(x)\big)$ a dense subset of $(-\infty, 0)$?

As was noted in the proof of Proposition 4.7, the zeros of genus polynomials of the dipoles interlace with two polynomials whose coefficients are alternate Stirling numbers. This should give us even more of a grip on them.

The results of Section 5 suggest that unlike the zeros of vertex-forest multijoins, those of H-linear families of graphs are bounded away from zero. Since 0 is a zero of the genus polynomial of every nonplanar graph, some care must be exercised in phrasing the appropriate conjecture.

CONJECTURE 6.3. For every H-linear family $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ of graphs, $\cup_{n=1}^{\infty} Z\big(GP_{G_n}(x)\big)$ is disjoint from some punctured neighborhood of zero.

The log concavity conjecture has the following version in this context.

CONJECTURE 6.4. For every graph G, $Z\big(GP_G(x)\big)$ is a subset of $(-\infty, 0]$.

This might be the place to mention four more H-linear families of graphs for which the genus generating matrices are known, but the zeros of whose genus polynomials have still not been proven to be real.

EXAMPLE 6.5. The triple ladders of Figure 11 have genus generating matrix M and initial vector $\mathbf{v}_1$

$$M = \begin{pmatrix} 192x & 96 + 288x \\ 72x + 192x^2 & 24 + 288x \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 18 + 18x \\ 6 + 30x \end{pmatrix}.$$

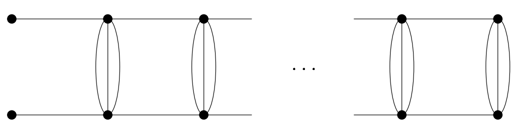


FIGURE 11

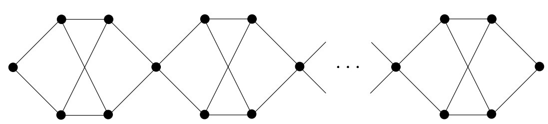


FIGURE 12

EXAMPLE 6.6. The K_4-linear graphs of Figure 12 has genus generating matrix M and initial vector $\mathbf{v}_1$

$$M = \begin{pmatrix} 8 + 68x & 4 + 16x \\ 32x + 48x^2 & 16x \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 + 14x \\ 8x + 8x^2 \end{pmatrix}.$$

EXAMPLE 6.7. The W_4-linear graphs of Figure 13 have genus generating matrix M and initial vector $\mathbf{v}_1$

$$M = \begin{pmatrix} 8 + 260x + 216x^2 & 4 + 88x \\ 64x + 416x^2 & 32x + 64x^2 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 2 + 58x + 36x^2 \\ 16 + 80x \end{pmatrix}.$$

https://doi.org/10.4153/CJM-1997-029-5 Published online by Cambridge University Press

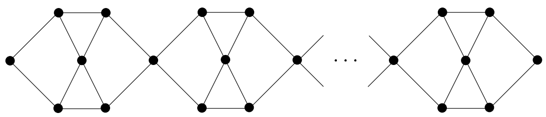


FIGURE 13

EXAMPLE 6.8. The triangular prisms of Figure 14 have genus generating matrix M and initial vector $\mathbf{v}_1$

$$M = \begin{pmatrix} 0 & 162x & 54 \\ 24x^2 & 72x & 12 + 108x \\ 11x^2 & 15x + 117x^2 & 1 + 72x \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 8 \\ 4 + 4x \\ 1 + 7x \end{pmatrix}.$$

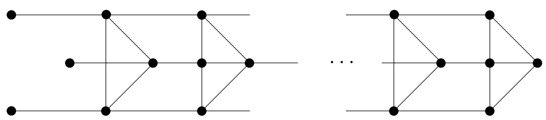


FIGURE 14

The following special case of Conjecture 6.4 is probably not hard.

CONJECTURE 6.9. The zeros of the genus polynomials of the graphs listed in Examples 6.5–6.8 are real and negative.

The techniques used in Section 5 suggest the following general question.

QUESTION 6.10. Let M be a matrix with entries in $\mathbb{R}(x)$. Under what conditions can it be guaranteed that if $\big(P(x)\ Q(x)\big)$ is a pair of polynomials whose zeros interlace, then so do the zeros of the two components of the vector $\big(P(x)\ Q(x)\big)M$ interlace?

Most of the proofs of Section 5 were facilitated by matrices that possess this property. On the other hand, the matrix

$$M_0 = \begin{pmatrix} 3x & 3 \\ 2x & 1000 + 3x \end{pmatrix}$$

fails to have this property, as is easily verified by taking $P(x) = (x + 2)(x + 4)$ and $Q(x) = (x+1)(x+3)(x+5)$. Matrices that do possess this property seem to have a related property that is mentioned in the next question.

QUESTION 6.11. Let M be a matrix with entries in $\mathbb{R}[x]$. Under what conditions can it be guaranteed that the zeros of each of the entries of $M^n n = 1, 2, 3, \ldots$ are all real?

The aforementioned matrix M_0 fails to have this property too, as is seen by examining its third power.

REFERENCES

1. D. Archdeacon, *Calculations on the average genus and genus distribution of graphs*, Congr. Numer. **67**(1988), 114–124.
2. ______, *The orientable genus is nonadditive*, J. Graph Theory **10**(1986), 385–402.
3. ______, *The nonorientable genus is additive*, J. Graph Theory **10**(1986), 363–384.
4. F. Brenti, G. F. Royle and D. G. Wagner, *Location of zeros of chromatic and related polynomials of graphs*, Canad. J. Math. **46**(1994), 55-80.
5. L. Comtet, *Advanced combinatorics*, Reidel Publ. Co. Boston, 1974.
6. R. Cori, A. Machi, J. G. Penaud and B. Vauquelin, *On the automorphism group of a planar hypermap*, European J. Combin. **2**(1981), 331–334.
7. R. A. Duke, *The genus, regional number, and Betti number of a graph*, Canad. J. Math. **134**(1966), 817–822.
8. M. L. Furst, J. L. Gross and R. Statman, *Genus distributions of two classes of graphs*, J. Combinatorial Theory (B) **46**(1989), 22–36.
9. J. L. Gross, D. P. Robbins and T. Tucker, *Genus distributions for bouquets of circles*, J. Combin. Theory Ser. B **47**(1989), 292–306.
10. J. L. Gross and T. Tucker, *Topological graph theory*, Wiley-Interscience, New York, 1987.
11. J. L. Gross and M. Furst, *Hierarchy for imbedding distribution invariants of a graph*, J. Graph Theory, **11**(1987), 205–220.
12. J. Harer and D. Zagier, *The Euler characteristics of the moduli space of curves*, Invent. Math. **85**(1986), 457–485.
13. O. J. Heilman and E. H. Lieb, *Theory of monomer-dimer systems*, Commun. Math. Phys. **25**(1972), 190–232.
14. D. M. Jackson, *Counting cycles in permutations by group characters, with applications to a topological problem*, Trans. Amer. Math. Soc. **299**(1987), 785–801.
15. J. Keilson and H. Gerber, *Some results for discrete unimodality*, J. Amer. Statist. Assoc. **66**(1971), 386–389.
16. L. A. McGeoch, *Genus distributions for circular and Möbius ladders*, manuscript.
17. R. G. Rieper, Doctoral dissertation, Western Mischigan University, August 1987.
18. S. Stahl, *Permutation-partition pairs: A combinatorial generalization of graph embeddings*, Trans. Amer. Math. Soc. **259**(1980), 129–145.
19. ______, *Permuttion-partition pairs III: Embedding distributions of linear families of graphs*, J. Combin. Theory Ser. B **52**(1991), 191–218.
20. ______, *On the number of maximum genus embeddings of almost all graphs*, European J. Combin. **13** (1992), 119–126.
21. ______, *Region distributions of some small diameter graphs*, Discrete Math. **89**(1991), 281–299.
22. ______, *A combinatorial analog of the Jordan Curve Theorem*, J. Combin. Theory Ser. B **35**(1983), 28–38.
23. ______, *A duality for permutations*, Discrete Math. **71**(1988), 257–271.
24. R. P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, In: Graph Theory and its Applications: East and West, Ann. New York Acad. Sci. **576**(1986), 500–535.
25. D. G. Wagner, *The partition polynomial of a finite set system*, J. Combin. Theory Ser. A **56**(1991), 138–159.
26. ______, *Total positivity of Hadamard products*, J. Math. Anal. Appl. **163**(1992), 459–483.
27. D. W. Walkup, *How many ways can a permutation be factored into two n-cycles*? Discrete Math. **28**(1979), 315–319.

Department of Mathematics
University of Kansas
Lawrence, KS
U.S.A. 66045
e-mail: stahl@kuhub.cc.ukans.edu