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FRAME FIELDS ON MANIFOLDS

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1. Introduction. Consider the following stable secondary cohomology operations associated with the relations in the mod 2 Steenrod algebra: $\mathfrak{A}$

$$\phi_4 : Sq^2(Sq^2Sq^1) = 0;$$

$$\phi_5 : (Sq^2Sq^1)(Sq^2Sq^1) + Sq^1(Sq^2Sq^3) = 0$$

such that

$$Sq^2\phi_4 = Sq^1\phi_5 = 0.$$

Let ψ_5 be a stable tertiary cohomology operation associated with the above relation. We assume that (ϕ_4, ϕ_5) and ψ_5 are chosen to be spin trivial in the sense of Theorem 3.7 of [**14**].

Let $\phi_{0,0}$, $\phi_{1,1}$ be the stable Adams basic secondary cohomology operations associated with the relations:

$$\phi_{0,0} : Sq^1Sq^1 = 0 \quad \text{and}$$

$$\phi_{1,1} : Sq^2Sq^2 + Sq^3Sq^1 = 0$$

respectively.

Let n be a positive integer with $n \equiv 7 \bmod 8 \geqq 15$. Suppose that M is a closed, connected and smooth manifold of dimension n which is 3-connected mod 2 and satisfies the condition $w_4(M) = 0$, where $w_i(M)$ is the ith-mod 2 Stiefel-Whitney class of the tangent bundle of M. Let the mod 2 semi-Kervaire characteristic be defined by

$$\chi_2(M) = \sum_{2i<n} \dim_{\mathbf{Z}_2}(H^i(M)) \bmod 2.$$

All cohomology will be ordinary singular cohomology with $\mathbf{Z}_2$ coefficients unless otherwise specified. Let

$$\delta : H^*(-, \mathbf{Z}_2) \rightarrow H^{*+1}(-, \mathbf{Z})$$

be the Bockstein operator associated with the exact sequence

$$0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}_2 \rightarrow 0.$$

We shall prove the following theorems:

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THEOREM 1.1. *Suppose*

$$\mathrm{Indet}^{n-4}(\psi_5, M) = Sq^2H^{n-6}(M) \quad \textit{and}$$
$$Sq^2H^{n-7}(M; \mathbf{Z}) = Sq^2H^{n-7}(M).$$

(i) *If* $n \equiv 15 \bmod 16 \geqq 15$, *then* $\mathrm{Span}(M) \geqq 7$.
(ii) *Suppose* $n \equiv 7 \bmod 16 > 7$. *Then* $\mathrm{span}(M) \geqq 7$ *if and only if*

$$0 \in \phi_4(w_{n-9}(M)) \quad \textit{and} \quad 0 \in \psi_5(w_{n-9}(M)).$$

THEOREM 1.2. *Suppose*

$$\mathrm{Indet}^{n-4}(\psi_5, M) = Sq^2H^{n-6}(M) \quad \textit{and}$$
$$Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M).$$

(i) *If* $n \equiv 15 \bmod 16 > 15$, $\mathrm{span}(M) \geqq 8$.
(ii) *If* $n \equiv 7 \bmod 16 > 7$, $\mathrm{span}(M) \geqq 8$ *if and only if* $w_{n-7}(M) = 0$, $0 \in \phi_4(w_{n-9}(M))$, $0 \in \psi_5(w_{n-9}(M))$ *and* $\chi_2(M) = 0$.

We have the following immediate corollaries.

COROLLARY 1.3. *Suppose* $n \equiv 15 \bmod 16$.
(i) *If* M *is* 4-*connected* mod 2 *and*

$$Sq^2H^{n-7}(M; \mathbf{Z}) = Sq^2H^{n-7}(M),$$

then $\mathrm{span}(M) \geqq 7$.
(ii) *If* M *is* 5-*connected* mod 2 *and* $n > 15$, *then* $\mathrm{span}(M) \geqq 8$.

COROLLARY 1.4. *If* M *is* 5-*connected* mod 2 *and* $n \equiv 7 \bmod 16$ *with* $n > 7$, *then*
(a) $\mathrm{Span}(M) \geqq 7$
(b) $\mathrm{Span}(M) \geqq 8$ *if and only if* $w_{n-7}(M) = 0$ *and* $\chi_2(M) = 0$.

Throughout the rest of the paper M is assumed to be 3-connected mod 2.

2. The modified Postnikov tower. We shall consider the problem of finding a k-field as a lifting problem. Let $B\hat{S}O_n\langle 8\rangle$ be the classifying space of orientable n-plane bundles ξ satisfying

$$w_2(\xi) = w_4(\xi) = 0$$

where $w_i(\xi)$ is the i-th mod 2 Stiefel-Whitney class of the bundle ξ. Let

$$g: M \to B\hat{S}O_n\langle 8\rangle$$

classify an n-plane bundle η over M. Then the problem of finding k-linearly independent sections of η is equivalent to lifting g to $B\hat{S}O_{n-k}\langle 8\rangle$. Hence we shall consider a Postnikov tower for the fibration

$$V_{n,k} \to B\hat{S}O_{n-k}\langle 8\rangle \xrightarrow{\pi} B\hat{S}O_n\langle 8\rangle,$$

and inspect the obstructions to lifting g to $B\hat{S}O_{n-k}\langle 8\rangle$. Following [**3**] we shall consider the n-MPT for π for $k = 7$ or 8. The computation is done in [**8**]. We list the results in the following tables:

TABLE 1

The n-Postnikov tower for $\pi: B\hat{S}O_{n-7}\langle 8\rangle \to B\hat{S}O_n\langle 8\rangle$.

	k-invariant	Dimension	Defining relation
Stage 1	k_1^1	$n-6$	δw_{n-7}
	k_2^1	$n-5$	w_{n-5}
	k_3^1	$n-3$	w_{n-3}
Stage 2	k_1^2	$n-5$	$Sq^2k_1^1 = 0$
	k_2^2	$n-4$	$Sq^2k_2^1 + Sq^3k_1^1 = 0$
	k_3^2	$n-3$	$Sq^4k_1^1 = 0$
	k_4^2	$n-3$	$Sq^2Sq^1k_2^1 + Sq^1k_3^1 = 0$
	k_5^2	$n-2$	$Sq^4k_2^1 = 0$
	k_6^2	n	$Sq^4k_3^1 = 0$
Stage 3	k_1^3	$n-4$	$Sq^2k_1^2 = 0$
	k_2^3	$n-3$	$Sq^2Sq^1k_1^2 + Sq^1k_3^2 = 0$
	k_3^3	$n-3$	$Sq^1k_3^2 + Sq^2k_2^2 + Sq^1k_4^2 = 0$
	k_4^3	n	$\chi Sq^4k_3^2 + Sq^2Sq^4k_1^2 = 0$
Stage 4	k^4	$n-3$	$Sq^2k_1^3 + Sq^1k_2^3 = 0$

TABLE 2

The n-MPT for $\pi: B\hat{S}O_{n-8}\langle 8\rangle \to B\hat{S}O_n\langle 8\rangle$.

	k-invariant	Dimension	Defining relation
Stage 1	k^1	$n-7$	$k^1 = w_{n-7}$
Stage 2	k_1^2	$n-5$	$Sq^2Sq^1k^1 = 0$
	k_2^2	$n-3$	$Sq^4Sq^1k^1 = 0$
	k_3^2	$n-2$	$Sq^4Sq^2k^1 = 0$
	k_4^2	n ($n > 15$)	$(Sq^8 + w_8)k^1 = 0$
Stage 3	k_1^3	$n-4$	$Sq^2k_1^2 = 0$
	k_2^3	$n-3$	$(Sq^2Sq^1)k_1^2 + Sq^1k_2^2 = 0$
	k_3^3	n	$(Sq^2Sq^4)k_1^2 + \chi Sq^4k_2^2 = 0$
Stage 4	k^4	$n-3$	$Sq^2k_1^3 + Sq^1k_2^3 = 0$

By the connectivity condition on M we only need to consider for the case of lifting g to $B\hat{S}O_{n-7}\langle 8\rangle$, $\delta w_{n-7}(\eta)$, $w_{n-5}(\eta)$, $k_1^2(\eta)$, $k_2^2(\eta)$, $k_6^2(\eta)$, $k_1^3(\eta)$ and $k_4^3(\eta)$ whenever these are defined.

According to [**14**, Proposition 4.2] we have the following technical result:

PROPOSITION 2.1. *Let w_{n-9} be the $(n-9)$-th* mod 2 *universal Stiefel Whitney class considered as in $H^{n-9}(B\hat{S}O_{n-7}\langle 8\rangle)$. Then*

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(a) $(0, 0) \in (\phi_4, \phi_5)(w_{n-9}) \subset H^{n-5}(B\hat{S}O_{n-7}\langle 8\rangle)$
$\oplus H^{n-4}(B\hat{S}O_{n-7}\langle 8\rangle)$.

(b) $0 \in \psi_5(w_{n-9}) \subset H^{n-4}(B\hat{S}O_{n-7}\langle 8\rangle)$.

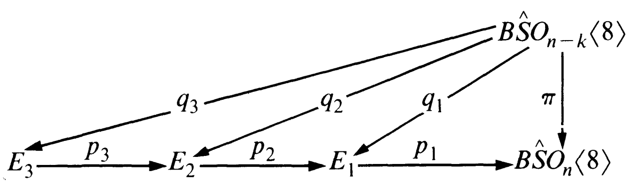
The proof is entirely analogous to that of Theorem 4.2 of [**14**]. We shall not present it here.

According to [**14**] $\phi_{1,1}$ is spin trivial and so we have

PROPOSITION 2.2. (E. Thomas)

$$0 \in \phi_{1,1}(w_{n-7}) \subset H^{n-4}(B\hat{S}O_{n-7}\langle 8\rangle).$$

Let the n-MPT for $\pi:B\hat{S}O_{n-k}\langle 8\rangle \to B\hat{S}O_n\langle 8\rangle$ for $k = 7$ or 8 be indicated by the following diagram:



By the connectivity condition on M, there is no obstruction to lifting any map from M into E_3 to $B\hat{S}O_{n-k}\langle 8\rangle$.

Recall the definition of a generating class in [**13**]. Then we have the following Proposition due to E. Thomas. The proof is identical to that of Proposition 4.1 in the case $k = 7$ and to Proposition 4.5 in the case $k = 8$ in [**14**].

PROPOSITION 2.3. (a) *The class w_{n-9} in $H^{n-9}(B\hat{S}O_n\langle 8\rangle)$ is a generating class for the pair $(k_1^2, 0)$ in $H^{n-5}(E_1) \oplus H^{n-4}(E_1)$, relative to the pair (ϕ_4, ϕ_5).*

(b) *The class $p_1{}^*w_{n-9}$ is a generating class for k_1^3, relative to the operation ψ_5.*

Similarly we have

PROPOSITION 2.4. *For $\pi:B\hat{S}O_{n-7}\langle 8\rangle \to B\hat{S}O_n\langle 8\rangle$, the class w_{n-7} in $H^{n-7}(B\hat{S}O_n\langle 8\rangle)$ is a generating class for k_2^2 in $H^{n-4}(E_1)$.*

Now by inspection of the k-invariants for the n-MPT for π and the connectivity condition on M, together with Proposition 2.1, 2.2, 2.3, 2.4 and the generating class theorem of Thomas [**13**] we have

THEOREM 2.5. (The case $k = 7$.) *Let η be an orientable n-plane bundle over M satisfying*

$$w_4(\eta) = 0,\ \delta w_{n-7}(\eta) = 0,\ w_{n-5}(\eta) = 0.$$

Suppose

$$\mathrm{Indet}^{n-4}(\phi_{1,1}, M) = Sq^2H^{n-6}(M),$$
$$\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M) \quad \textit{and}$$
$$Sq^2H^{n-7}(M; \mathbf{Z}) = Sq^2H^{n-7}(M).$$

Then

(i) $(0, 0) \in (k_1^2, k_2^2)(\eta)$ *if and only if*

$$0 \in \phi_4(w_{n-9}(\eta)) \quad \textit{and} \quad 0 \in \phi_{1,1}(w_{n-7}(\eta))$$

(ii) $0 \in k_1^3(\eta)$ *if and only if*

$$0 \in \phi_4(w_{n-9}(\eta)),\ 0 \in \phi_{1,1}(w_{n-7}(\eta))$$
$$0 \in k_6^2(\eta) \quad \textit{and} \quad 0 \in \psi_5(w_{n-9}(\eta)).$$

THEOREM 2.6. (The case $k = 8$.) *Let η be an orientable n-plane bundle over M satisfying $w_4(\eta) = w_{n-7}(\eta) = 0$. Suppose*

$$\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M).$$

If either $w_8(\eta) = V_8(M)$, the 8th Wu *class of M, and*

$$Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M) \textit{ or } Sq^2H^5(M) = 0,$$

then

(i) $0 \in k_1^2(\eta)$ *if and only if* $0 \in \phi_4(w_{n-9}(\eta))$

(ii) $0 \in k_1^3(\eta)$ *if and only if*

$$0 \in \phi_4(w_{n-9}(\eta)),\ 0 \in k_4^2(\eta) \textit{ and } 0 \in \psi_5(w_{n-9}(\eta)).$$

3. The top dimensional secondary obstructions. Let ζ_6 be a choice of stable cohomology operation of Hughes-Thomas type associated with the following relation in $\mathfrak{A}$:

$$\zeta_6{:}Sq^4Sq^{n-3} + Sq^2(Sq^{n-3}Sq^2) + Sq^1(Sq^{n-3}Sq^3 + Sq^{n-1}Sq^1) = 0$$

such that

$$Sq^4(b_{n-4}) \cup b_{n-4} \in \zeta_6(b_{n-4})$$

where b_{n-4} is the fundamental class of the space Y_{n-4} over K_{n-4} with classifying map $(Sq^2, Sq^1)\iota_{n-4}$.

Then the following is proved in [**8**].

THEOREM 3.1. *Consider the n-*MPT *for the fibration*

$$\pi{:}B\hat{S}O_{n-k}\langle 8\rangle \to B\hat{S}O_n\langle 8\rangle.$$

Let γ be the pull back of the universal orientable n-plane bundle over $B\hat{S}O_n\langle 8\rangle$. Using this bundle induce bundles over E_1, E_2 by p_1 and $p_2 \circ p_1$

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respectively. Denote the Thom class of the resulting bundles by $U(E_1)$ *and* $U(E_2)$ *respectively. Suppose* $k = 7$. *Then*

$$U(E_1) \cdot k_6^2 \in \zeta_6(U(E_2)).$$

Let η be an orientable n-plane bundle over M satisfying

$$w_4(\eta) = w_{n-5}(\eta) = 0, \quad \delta w_{n-7}(\eta) = 0.$$

Then by Theorem 3.1 together with the fact that

$$\mathrm{Indet}^{2n}(T(\eta)) = \psi\, \mathrm{Indet}^n(M, k_6^2)$$

(where ψ is the Thom isomorphism and $T(\eta)$ the Thom space of η), we have

THEOREM 3.2. $0 \in k_6^2(\eta)$ *if and only if* $0 \in \zeta_6(U(\eta))$ *where* $U(\eta)$ *is the Thom class of* η.

3.3. Consider now the case $k = 8$. Then Theorem 5.10 of [**8**] applies to give the existence of a secondary cohomology operation, ζ_8 (stable if $n \equiv 15(16)$ and non-stable if $n \equiv 7(16)$) associated with the relation

$$\begin{aligned}\zeta_8 : Sq^8 Sq^{n-7} &+ Sq^4(Sq^{n-7}Sq^4) \\ &+ Sq^2(Sq^{n-3}Sq^2 + Sq^{n-7}Sq^2Sq^4) \\ &+ Sq^1(Sq^{n-1}Sq^1 + Sq^{n-5}Sq^5 + Sq^{n-3}Sq^3 \\ &+ Sq^{n-7}Sq^7) = 0\end{aligned}$$

satisfying

$$d_{n-8} \cup Sq^8 d_{n-8} + Sq^6 d_{n-8} \cup Sq^2 d_{n-8} \in \zeta_8(d_{n-8}),$$

where d_{n-8} is the fundamental class of an universal example for $(n - 8)$ dimensional class x satisfying $Sq^4 x = 0$. Then for the n-MPT for π for the case $k = 8$, we have

(3.4) $\quad U(E_1) \cdot (k_4^2 + w_8 \cdot w_{n-8}) \in \zeta_8(U(E_1)).$

Since

$$Sq^1(U(E_1) \cdot (w_8 \cdot w_{n-9})) = U(E_1) \cdot (w_8 \cdot w_{n-8})$$

by (3.4) and the connectivity condition on M we have

THEOREM 3.5. (The case $k = 8$.) *Let* η *be an orientable* n*-plane bundle over* M *satisfying*

$$w_4(\eta) = w_{n-7}(\eta) = 0.$$

If $w_4(M) = 0$ *then* $0 \in k_4^2(\eta)$ *if and only if* $0 \in \zeta_8(U(\eta))$.

Of course if $w_8(\eta) \neq V_8(M)$ then

$$(Sq^8 + w_8(\eta)\cdot)H^{n-8}(M) = H^n(M)$$

and so trivially $0 \in k_4^2(\eta)$.

4. The top dimensional tertiary obstructions.

4.1. Let $\phi_{0,2}$, $\phi_{2,2}$ be the basic stable Adams secondary cohomology operations associated with the relation:

$$\phi_{0,2}: Sq^1Sq^4 + (Sq^2Sq^1)Sq^2 + Sq^4Sq^1 = 0 \quad \text{and}$$
$$\phi_{2,2}: Sq^4Sq^4 + Sq^6Sq^2 + Sq^7Sq^1 = 0$$

respectively.

Then Lemma 4.7, 4.17 of [**8**] says there exist stable secondary cohomology operations ζ_1, ζ_3, η_1 and η_2 associated with the following relations (denoted by the same symbols)

$$(4.2)\quad \begin{cases} \zeta_1: Sq^2(Sq^{n-6} + Sq^{n-7}Sq^1) = 0 \\ \zeta_3: Sq^4(Sq^{n-6} + Sq^{n-7}Sq^1) + Sq^4(Sq^{n-9}Sq^2Sq^1 + Sq^{n-8}Sq^2) \\ \qquad + (Sq^5Sq^1)(Sq^{n-11}Sq^2Sq^1) = 0 \\ \eta_1: (Sq^4Sq^2)(Sq^{n-9}Sq^2Sq^1 + Sq^{n-8}Sq^2) \\ \qquad + Sq^2(Sq^{n-7}Sq^2Sq^3) = 0 \\ \eta_2: (Sq^2Sq^1)(Sq^{n-9}Sq^2Sq^3 + Sq^{n-7}Sq^2Sq^1) \\ \qquad + Sq^1(Sq^{n-7}Sq^2Sq^3) + (Sq^4Sq^2Sq^1 \\ \qquad + Sq^7)(Sq^{n-11}Sq^2Sq^1) = 0 \end{cases}$$

satisfying

$$(4.3)\quad \Omega: (Sq^2Sq^4)\zeta_1 + \chi Sq^4\zeta_3 + Sq^2\eta_1 + Sq^3\eta_2 = 0$$

such that on b_{n-7} the fundamental class of Y_{n-7} over K_{n-7} with k-invariant $(Sq^1, Sq^2)\iota_{n-7}$,

$$(4.4)\quad \begin{cases} Sq^4b_{n-7} \cup b_{n-7} + (Sq^{n-7}Sq^4 + Sq^{n-9}Sq^6 \\ \qquad + Sq^{n-10}Sq^7)b_{n-7} + Sq^{n-6}\phi_{1,1}(b_{n-7}) \\ \qquad + Sq^{n-7}Sq^3\phi_{0,0}(b_{n-7}) \in \zeta_3(b_{n-7}); \\ Sq^{n-6}\phi_{0,0}(b_{n-7}) \in \zeta_1(b_{n-7}); \\ (Sq^{n-4} + Sq^{n-6}Sq^2)\phi_{1,1}(b_{n-7}) \in \eta_1(b_{n-7}) \text{ and} \\ Sq^{n-7}Sq^2\phi_{1,1}(b_{n-7}) \in \eta_2(b_{n-7}) \end{cases}$$

Let D_k be the universal example space for k-dimensional mod 2 cohomology class x satisfying $Sq^1x = Sq^2x = Sq^4x = 0$, $\phi_{0,0}(x) = 0$ and $\phi_{1,1}(x) = 0$. Let d_k be the fundamental class of D_k. Let $\tilde{\zeta}_1$, $\tilde{\zeta}_3$ be the relations obtained from ζ_1, ζ_3 of (4.2) respectively by replacing

$$Sq^2(Sq^{n-6} + Sq^{n-7}Sq^1) \quad \text{and} \quad Sq^4(Sq^{n-6} + Sq^{n-7}Sq^1)$$

by

$$(Sq^2Sq^1)Sq^{n-7} + Sq^2(Sq^{n-7}Sq^1) \quad \text{and}$$
$$(Sq^4Sq^1)Sq^{n-7} + Sq^4(Sq^{n-7}Sq^1)$$

respectively. Then there exist stable secondary cohomology operations associated with $\widetilde{\zeta}_1$, $\widetilde{\zeta}_3$ also denoted by the same symbols such that

(4.5) $\quad \widetilde{\zeta}_1 \subset \zeta_1, \widetilde{\zeta}_3 \subset \zeta_1 \quad$ and

$$\widetilde{\Omega}:(Sq^2Sq^4)\widetilde{\zeta}_1 + \chi Sq^4\widetilde{\zeta}_3 + Sq^2\eta_1 + Sq^3\eta_2 = 0.$$

Then Theorem 4.19 of [**8**] gives us

THEOREM 4.6. *There exist stable tertiary cohomology operations*, Ω *and* $\widetilde{\Omega}$ *associated with the relations* (4.3) *and* (4.5) *respectively such that*

$$d_{n-7} \cup (\phi_{2,2}(d_{n-7}) + Sq^3\phi_{0,2}(d_{n-7})) \in \Omega(d_{n-7})$$
$$\widetilde{\Omega} \subset \Omega \quad \textit{and} \quad 0 \in \widetilde{\Omega}(d_{n-8}).$$

Let $\nu_4 \in H^4(B\hat{S}O_n\langle 8\rangle) \approx \mathbf{Z}_2$ be a generator. Then by the admissible class theorem of [**8**], and Theorem 4.6 we have

THEOREM 4.7. (1) (The case $k = 7$.)

$$U(E_2) \cdot (k_4^3 + (p_2 \circ p_1)^* w_{n-7} \cdot Sq^3\nu_4) \in \Omega(U(E_2)).$$

(2) (The case $k = 8$.)

$$U(E_2) \cdot k_3^3 \in \widetilde{\Omega}(U(E_2)).$$

This is Theorem 5.8 of [**8**].

5. The case of sectioning orientable bundle η over M with $w_4(\eta) \neq w_4(M)$. The n-MPT for the fibration

$$\widetilde{\pi}:B \text{ Spin}_{n-k} \rightarrow B \text{ Spin}_n$$

is similar to that given by Table 1 or Table 2 depending on whether $k = 7$ or 8. We will retain the same notation. Note that for $k = 7$, k_6^2 and k_4^3 will be defined by

$$(Sq^4 + w_4)k_3^1 = 0 \text{ and } (\chi Sq^4 + w_4 \cdot)k_3^2 + Sq^2Sq^4k_1^2 = 0$$

respectively and for $k = 8$, k_4^2 and k_3^3 will be defined by

$$(Sq^8 + w_8 \cdot)k^1 = 0 \quad \text{and}$$
$$(\chi Sq^4 + w_4 \cdot)k_2^2 + Sq^2Sq^4k_1^2 = 0.$$

Thus if $w_4(\eta) \neq w_4(M)$, for $k = 7$,

$$(0, \mu) \in \mathrm{Indet}^{n-4,n}((k_1^3, k_4^3), M)$$

where $\mu \in H^n(M)$ is a generator. Also for $k = 8$,

$$(0, \mu) \in \mathrm{Indet}^{n-4,n}((k_1^3, k_3^3), M).$$

This means that once we have a lifting of an n-plane bundle η satisfying $w_4(\eta) \neq w_4(M)$ to E_2 we can ignore the top dimensional tertiary obstruction.

5.1. Note that the analogue of Theorem 2.5 for an orientable n-plane bundle η over M satisfying $w_{n-5}(\eta) = 0$ and $\delta w_{n-6}(\eta) = 0$ holds. The proof is exactly the same. Hence we have by the above remarks and the analogue of Theorem 2.5:

THEOREM 5.1. *Suppose η is an orientable n-plane bundle over M satisfying $w_4(\eta) \neq w_4(M)$. Suppose*

$$\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M),$$

$$Sq^2H^{n-7}(M; \mathbf{Z}) = Sq^2H^{n-7}(M) \quad \textit{and}$$

$$\mathrm{Indet}^{n-4}(\phi_{1,1}, M) = Sq^2H^{n-6}(M).$$

Then η has 7-linearly independent cross sections if and only if

$$\delta w_{n-7}(\eta) = 0,\ w_{n-5}(\eta) = 0,\ 0 \in \phi_4(w_{n-9}(\eta)),$$

$$0 \in \phi_{1,1}(w_{n-7}(\eta)) \textit{ and } 0 \in \psi_5(w_{n-9}(\eta)).$$

5.2. Similarly the analogue of Theorem 2.6 holds for an orientable n-plane bundle satisfying $w_{n-7}(\eta) = 0$. Therefore by the discussion at the beginning of this section and the analogue of Theorem 2.6 we have the following existence theorem.

THEOREM. *Suppose*

$$w_4(\eta) \neq w_4(M), \quad Sq^2H^5(M) = 0,$$

$$\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M) \quad \textit{and}$$

$$w_8(\eta) \neq V_8(M),$$

the 8-th Wu *class of M. Then η has 8 linearly independent cross-sections if and only if*

$$w_{n-7}(\eta) = 0,\ \phi_4(w_{n-9}(\eta)) = 0 \quad \textit{and} \quad 0 \in \psi_5(w_{n-9}(\eta)).$$

6. Indeterminacy of Ω. In addition to all the cohomology operations we have used so far we need to consider the following stable secondary cohomology operations associated with the following relations

$$(6.1)\quad \begin{cases} \Gamma_1:(Sq^2Sq^4)Sq^2 + \chi Sq^4Sq^4 = 0 \\ \Gamma_2:Sq^2(Sq^4Sq^2) + \chi Sq^4Sq^4 = 0 \\ \Gamma_3:\chi Sq^4(Sq^5Sq^1) + Sq^3(Sq^7 + \chi Sq^7) = 0 \\ \Gamma_5:Sq^3(Sq^2Sq^1) = 0 \end{cases}$$

By virtue of the last section we shall now assume for an orientable n-plane bundle η over M that $w_4(\eta) = w_4(M) = 0$. According to Atiyah [**2**], the S-dual of $T(\eta)$ is the Thom space of the stable bundle $\alpha = -\eta - \tau$ where τ is the tangent bundle of M. Primary piece of

$$\mathrm{Indet}^{n,n-4}(k_4^3, k_1^3) = \{0\} \times Sq^2H^{n-6}(M)$$

for the case $k = 7$.

$$\mathrm{Indet}^{n,n-4}(k_4^3, k_1^3) = (\Gamma_1, \phi_{1,1}^*)D^{n-7},$$

where

$$D^{n-7} = \{x \in H^{n-7}(M; \mathbf{Z}):Sq^2x = 0\}$$

and ϕ_{11}^* is the stable secondary cohomology operation of degree 3 defined on integral class and associated with the relation

$$Sq^2Sq^2 = 0.$$

Now by inspection, if $w_4(\eta) = w_4(M)$,

$$(6.2)\quad \mathrm{Indet}^{2n}(\Omega, T\eta) = \Gamma_1D^{2n-7}(T\eta) + \Gamma_2H^{2n-7}(T\eta) + \Gamma_3H^{2n-9}(T\eta) + \Gamma_5H^{2n-5}(T\eta)$$

where $D^{2n-7} \subset H^{2n-7}(T\eta)$ is defined by

$$D^{2n-7} = \{x \in H^{2n-7}(T\eta):Sq^2x = 0\}.$$

Notice that

$$\Gamma_1D^{2n-7}(T\eta) \subset \Gamma_2H^{2n-7}(T\eta).$$

Apply the S-duality pairing and by Maunder [**6**], we have for any $x \in H^{2n-7}(T\eta)$

$$\begin{aligned} &\langle \Gamma_2x, U(-\eta - \tau) \rangle \\ &= \langle x, \chi\Gamma_2U(-\eta - \tau) \rangle \\ &= \langle x, Sq^3\phi_{0,2}U(-\eta_1 - \tau) \rangle \end{aligned}$$

where $U(-\eta - \tau)$ is the Thom class of $-\eta - \tau$.

This is because

$$\chi\Gamma_2 = Sq^3\phi_{0,2} + \phi_{2,2}.$$

Since $w_4(-\eta - \tau) = 0$ and M is 3-connected mod 2, $\alpha = -\eta - \tau$ is classified by a map

$$g:M \to B\hat{S}O_N\langle 8\rangle,$$

for some large N. Then

$$\phi_{2,2}(U(B\hat{S}O_N\langle 8\rangle)) = 0,$$

where $U(B\hat{S}O_N\langle 8\rangle)$ is the Thom class of the universal N-plane bundle over $B\hat{S}O_N\langle 8\rangle$. Now let

$$\nu_4 \in H^4(B\hat{S}O_N\langle 8\rangle) \approx \mathbf{Z}_2$$

be a generator. Then

$$\phi_{0,2}U(B\hat{S}O_N\langle 8\rangle) = U(B\hat{S}O_N\langle 8\rangle) \cdot \nu_4.$$

Now for any bundle ξ over M classified by a map

$$h:M \to B\hat{S}O_N\langle 8\rangle.$$

Define $\nu_4(\xi)$ to be $h^*(\nu_4)$. Hence we have by the above remarks,

(6.3) $$\begin{aligned}\langle \Gamma_2 x, U(-\eta - \tau)\rangle &= \langle x, Sq^3(U(-\eta - \tau) \cdot \nu_4(\alpha))\rangle \\ &= \langle x \cdot U(\alpha) \cdot Sq^3\nu_4(\alpha)\rangle \\ &= 0 \quad \text{if } Sq^3\nu_4(\alpha) = 0.\end{aligned}$$

Similarly since

$$\chi(\Gamma_5) = Sq^1\phi_{1,1} \circ Sq^1$$

is trivial on integral classes, for any $x \in H^{2n-5}(T\eta)$, $\Gamma_5(x) = 0$ modulo zero indeterminacy because

$$\begin{aligned}\langle \Gamma_5 x, U(\alpha)\rangle &= \langle x, \chi\Gamma_5 U(\alpha)\rangle = \langle x, Sq^1\phi_{1,1}(Sq^1 U(\alpha))\rangle \\ &= 0 \quad \forall x \in H^{2n-5}(T\eta).\end{aligned}$$

Now the S-dual of Γ_3, $\chi\Gamma_3$, is associated with the relation

(6.4) $$(Sq^5Sq^1)Sq^4 + (Sq^7 + \chi Sq^7)(Sq^2Sq^1) = 0.$$

Therefore on $U(\alpha)$,

$$\chi(\widetilde{\Gamma}_3) = Sq^2Sq^3\phi_{0,2} + Sq^6Sq^2\phi_{0,0}.$$

Thus for any $x \in H^{2n-9}(T\eta)$

$$\begin{aligned}\langle \Gamma_3(x), U(\alpha)\rangle &= \langle x, Sq^2Sq^3\phi_{0,2}U(\alpha)\rangle \\ &= \langle x, U(\alpha) \cdot Sq^2Sq^3\nu_4(\alpha)\rangle \\ &= 0\end{aligned}$$

since $Sq^1\nu_4 = 0$ in $H^5(B\hat{S}O_N\langle 8\rangle)$.

Hence we have the following

THEOREM 6.5. *Suppose* $w_4(\eta) = w_4(M)$. *Then*

$$\mathrm{Indet}^{2n}(\Omega, T\eta) = \Gamma_2(H^{2n-7}(T\eta))$$

and is trivial if $Sq^3 \nu_4(\alpha) = 0$.

Similarly we have

THEOREM 6.6. *Suppose* $w_4(M) = 0$. *Then*

$$\mathrm{Indet}^{2n}(\Omega, M \times M) = \Gamma_2 H^{2n-7}(M \times M) \quad \textit{and}$$

$$\mathrm{Indet}^{2n}(\Omega, M \times M) = 0$$

if $Sq^3 \nu_4((-\tau) \times (-\tau)) = 0$ *or if* $Sq^3 \nu_4(-\tau) = 0$.

7. The case when the top dimensional tertiary obstruction has non-trivial indeterminacy. Let η be an orientable n-plane bundle over M. Suppose that

$$\mathrm{Indet}^{n-4}(\psi_5, M) = Sq^2 H^{n-6}(M) \quad \text{and} \quad w_4(\eta) = w_4(M).$$

7.1. The case $k = 7$. If

$$\mathrm{Indet}^n(k_4^3, M) \neq 0,$$

since the primary piece of $\mathrm{Indet}^n(k_4^3, M)$ is trivial, we see that

$$(0, 0) \in (k_4^3, k_1^3)(\eta) \quad \text{if } 0 \in k_1^3(\eta).$$

Thus we have

THEOREM. *Suppose*

$$\mathrm{Indet}^{n-4}(\phi_{1,1}, M) = Sq^2 H^{n-6}(M),$$

$$Sq^2 H^{n-7}(M; \mathbf{Z}) = Sq^2 H^{n-7}(M) \quad \textit{and}$$

$$\mathrm{Indet}^n(k_4^3, M) \neq 0.$$

Then η *has* 7 *linearly independent sections if and only if*

$$\delta w_{n-7}(\eta) = 0, \ w_{n-5}(\eta) = 0,$$

$$0 \in \phi_4(w_{n-9}(\eta)), \ 0 \in \phi_{1,1}(w_{n-7}(\eta)),$$

$$\zeta_6(U(\eta)) = 0 \textit{ and } 0 \in \psi_5(w_{n-9}(\eta)).$$

This follows from a theorem similar to 2.5 where the condition $w_4(\eta) = 0$ is dropped.

7.2. The case $k = 8$. Suppose $w_4(\eta) = 0$.

If $\mathrm{Indet}^n(k_3^3, M) \neq 0$, $(0, 0) \in (k_3^3, k_1^3)(\eta)$ if $0 \in k_1^3(\eta)$. Then similar to the case $k = 7$, we have

THEOREM. *Suppose either*

$$w_8(\eta) = V_8(M) \quad \textit{and} \quad Sq^2 H^{n-7}(M) = Sq^2 Sq^1 H^{n-8}(M)$$

or

$$Sq^2 H^5(M) = 0.$$

If $\mathrm{Indet}^n(k_3^3, M) \neq 0$, *then* η *admits* 8 *linearly independent sections if and only if*

$$w_{n-7}(\eta) = 0, 0 \in \phi_4(w_{n-9}(\eta)),$$

$$0 \in \zeta_8(U(\eta)) \quad \textit{and} \quad 0 \in \psi_5(w_{n-9}(\eta)).$$

8. The case when the top dimensional tertiary obstruction has trivial indeterminacy. Let η be an orientable n-plane bundle over M with

$$w_4(\eta) = w_4(M) = 0.$$

8.1. The case $k = 7$. Recall from Section 6 that

$$\mathrm{Indet}^n(k_4^3, M) = \Gamma_1 D^{n-7}.$$

By S-duality $\Gamma_1 D^{n-7} = 0$ modulo zero indeterminacy if $0 \in \chi\Gamma_1(U(-\tau))$ or if

$$Sq^3 \nu_4(-\tau) \in Sq^2 H^5(M).$$

Theorem 2.5, Theorem 4.7 (1), 6.5 and the admissible class theorem of [**8**], give the following:

THEOREM. *Suppose*

$$Sq^2 H^{n-7}(M; \mathbf{Z}) = Sq^2 H^{n-7}(M),$$

$$Sq^3(\nu_4(-\eta) + \nu_4(-\tau)) = 0,$$

$$\mathrm{Indet}^n(k_4^3, M) = 0,$$

$$Sq^2 H^{n-6}(M) = \mathrm{Indet}^{n-4}(\phi_{1,1}, M) \textit{ and}$$

$$\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M).$$

Then η *admits* 7 *linearly independent cross sections if and only if*

$$\delta w_{n-7}(\eta) = 0, w_{n-5}(\eta) = 0, 0 \in \phi_4(w_{n-9}(\eta)),$$

$$0 \in \phi_{1,1}(w_{n-7}(\eta)), \zeta_6(U(\eta)) = 0, 0 \in \psi_5(w_{n-9}(\eta)) \quad \textit{and}$$

$$\Omega(U(\eta)) = 0.$$

8.2. The case $k = 8$. As for the case $k = 7$, we have a similar theorem for the existence of 8 linearly independent cross sections of η.

THEOREM. *Suppose*

$$Sq^3(\nu_4(-\eta) + \nu_4(-\tau)) = 0,$$

$$\text{Indet}^n(k_3^3, M) = 0 \quad \textit{and}$$

$$\text{Indet}^{n-4}(\psi_5, M) = \text{Indet}^{n-4}(k_1^3, M).$$

Suppose either

$$w_8(\eta) = V_8(M) \textit{ and } Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M) \quad \textit{or}$$

$$Sq^2H^5(M) = 0.$$

Then η *admits* 8 *linearly independent cross sections if and only if*

$$w_{n-7}(\eta) = 0, 0 \in \phi_4(w_{n-9}(\eta)), 0 \in \zeta_8(U(\eta)),$$

$$0 \in \psi_5(w_{n-9}(\eta)) \quad \textit{and} \quad \Omega(U(\eta)) = 0.$$

This is a consequence of Theorem 2.6, Theorem 3.5, Theorem 4.7 (2), 6.5, and the admissible class theorem of [**8**] applied to 6.5 and the fact that

$$\text{Indet}^{2n}(\tilde{\Omega}, T\eta) = \text{Indet}^{2n}(\Omega, T\eta) = 0.$$

9. Evaluation on Thom complex of the tangent bundle of M. We now specialise to the case when η is the tangent bundle over M. We shall be considering the stable cohomology operation ζ_6 and the secondary operation ζ_8 and the tertiary cohomology operation Ω.

Suppose M' is a closed, connected and smooth manifold of dimension q and q is odd. Let

$$g:M' \times M' \to T(\tau)$$

be the map that collapses the complement of a tubular neighbourhood of the diagonal in $M' \times M'$ to a point. Let $U = g^*(U(\tau))$, where $U(\tau)$ is the Thom class of the tangent bundle of M'. Then we have the decomposition of Milnor and Wu:

$$(9.1) \qquad U \bmod 2 = \sum_{2i<q} \sum_k \alpha_i^k \otimes \beta_{q-i}^k + \sum_{2i<q} \sum_k \beta_{q-i}^k \otimes \alpha_i^k$$

where $\alpha_i^k \in H^i(M')$, $\beta_{q-i}^k \in H^{q-i}(M)$ and $\alpha_i^k \cup \beta_{q-i}^j = \delta_{k \cdot j}\mu$, $\mu \in H^q(M)$ is a generator and δ_{kj} is the Kronecker function. Then we have

LEMMA 9.2. ([**15**, Section 4]). *Let*

$$A = \sum_{2i<q} \alpha_i^k \otimes \beta_{q-i}^k \in H^q(M' \times M')$$

be as given by 9.1. *Then*

(i) $U \bmod 2 = A + t^*A$, *where*

$$t^*:H^*(M' \times M') \to H^*(M' \times M')$$

is the homomorphism induced by the map that interchanges the factors.

(ii) $A \cup t^*A = \chi_2(M')\mu \otimes \mu$.

Then according to Mahowald and Randall ([**12**]), we have the following

THEOREM 9.3. *Suppose M' is a spin manifold of dimension $n \equiv 7 \bmod 8$ with $n > 7$. Let A be as given by Lemma 9.2. Then*

(i) $Sq^{n-3}A = Sq^{n-3}Sq^2A = (Sq^{n-3}Sq^3 + Sq^{n-1}Sq^1)A = 0$.

(ii) *ζ_6 is defined on A and so on t^*A. In particular $\zeta_6(U(\tau)) = 0$ modulo zero indeterminacy.*

Since n is congruent to 7 mod 8, and M' is a spin manifold it follows from Wu's formula, 6.6 of [**8**], that $w_{n-3}(M') = 0$. Thus

$$Sq^{n-3}(U(\tau)) = Sq^{n-3}(A + t^*A) = Sq^{n-3}A + t^*Sq^{n-3}A = 0.$$

But $Sq^{n-3}A$ is of bidegree $(n - 1, n - 2)$ and so

$$Sq^{n-3}A = 0.$$

Similarly, it is shown that

$$Sq^{n-3}Sq^2A = 0.$$

Now

$$Sq^{n-3}Sq^3 = Sq^2(Sq^{n-4}Sq^2) + Sq^1(Sq^{n-3}Sq^2) \quad \text{and}$$
$$Sq^2Sq^{n-2} = Sq^{n-1}Sq^1.$$

Therefore since M' is a spin manifold, by Wu's duality,

$$(Sq^{n-3}Sq^3 + Sq^{n-1}Sq^1)A = 0.$$

This proves (i). Therefore ζ_6 is defined on A and so on t^*A. The last assertion is proved in [**12**, Section 2].

Now we return to our manifold M. Recall that M is 3-connected mod 2. For the rest of this section we shall assume that $w_4(M) = 0$. Recall that ζ_8 is a stable cohomology operation if $n \equiv 15\ (16) > 15$ and is non-stable if $n \equiv 7\ (16) \geqq 23$. We shall exploit the technique of Mahowald [**5**] to evaluate $\zeta_8(U(\tau))$. Note that $\text{Indet}^{2n}(\zeta_8, T(\tau))$ is trivial since $w_8(M) = V_8(M)$, the 8-th Wu class of M.

Let $A \in H^n(M \times M)$ be the class given by the decomposition (9.1). Suppose $w_{n-7}(M) = 0$. Then

$$Sq^{n-7}(A + t^*A) = 0.$$

But it can be shown that $Sq^{n-7}A$ is of bidegree $(n - 7, n)$. Hence

$$Sq^{n-7}A = 0.$$

Since

$$Sq^{n-5}Sq^5 + Sq^{n-7}Sq^7 = Sq^2Sq^{n-7}Sq^5 + Sq^2Sq^{n-8}Sq^6 + Sq^1Sq^{n-7}Sq^6,$$

$$(Sq^{n-5}Sq^5 + Sq^{n-7}Sq^7)A = 0.$$

Hence we have

PROPOSITION 9.4. *Suppose* $w_{n-7}(M) = 0$. *Then*
(i) $Sq^{n-7}A = 0$,
(ii) ζ_8 *is defined on* A, *hence on* t^*A.

THEOREM 9.5. *Suppose* $w_{n-7}(M) = 0$. *Then* ζ_8 *is defined on* $U(\tau)$ *and modulo zero indeterminacy,*

$$\zeta_8(U(\tau)) = \begin{cases} 0 & \text{if } n \equiv 15 \bmod 16. \\ \chi_2(M) \cdot U(\tau) \cdot \mu & \text{if } n \equiv 7 \bmod 16. \end{cases}$$

To prove 9.5 we shall exploit the technique of Mahowald.

Let $p:P \to K_n$ be the universal example space for ζ_8 on n-dimensional mod 2 cohomology classes. Consider $A \in H^n(M \times M)$ as a map

$$A:M \times M \to K_n.$$

Then 9.4 says that A has a lifting $\bar{A}:M \times M \to P$ to P. Let $\zeta \in H^{2n}(P)$ be a representative for ζ_8. Note that $\bar{A} \circ t$ is a lifting of t^*A represented by $A \circ t$.

Now P is a H-space and so we have a multiplication map

$$m:P \times P \to P.$$

Then the map $h = m \circ (\bar{A}, \bar{A} \circ t)$ is a lifting of $A + t^*A$ regarded as a map $m \circ (A, A \circ t)$. Let $\zeta \in H^{2n}(P)$ be a representative for ζ_8. Then if ζ_8 is stable

$$m^*\zeta = 1 \otimes \zeta + \zeta \otimes 1 \quad \text{and}$$

$$m^*\zeta = 1 \otimes \zeta + \zeta \otimes 1 + p^* \iota_n \otimes p^*\iota_n$$

if ζ_8 is non-stable. Thus

$$h^*\zeta = \bar{A}^*\zeta + t^*\bar{A}\zeta \quad \text{for } n \equiv 15 \bmod 16$$

But $t^*:H^{2n}(M \times M) \to H^{2n}(M \times M)$ is the identity homomorphism. Therefore

$$h^*\zeta = 0 \text{ if } n \equiv 15 \bmod 16.$$

Similarly if $n \equiv 7 \bmod 16$,

$$h^*\zeta = \bar{A}^*\zeta + t^*\bar{A}^*\zeta + A \cup t^*A = \chi_2(M)(\mu \otimes \mu).$$

Let $U:T(\tau) \rightarrow K_n$ represent the Thom class of the tangent bundle of M reduced mod 2. Let

$$\bar{U}:T(\tau) \rightarrow P$$

be any lifting of U to P. Then $f = \bar{U} \circ g$ is a lifting of $A + t^*A$. Since g^* is a monomorphism in dimension $2n$, $\zeta_8(U(\tau))$ vanishes if and only if

$$g^*\zeta_8(U(\tau)) = f^*(\zeta) = 0.$$

Since f and h are both liftings of $g^*(U(\tau)$ mod 2), there is a map

$$l:M \times M \rightarrow \Omega C,$$

where

$$C = K_{2n-7} \times K_{2n-3} \times K_{2n-1} \times K_{2n},$$

unique up to homotopy such that f and $m \circ (i \circ l, h)$ are homotopic, where $i:\Omega C \rightarrow P$ is the inclusion of the fibre. We can identify l with the quadruple (a, b, c, d) where a, b, c, d represent some classes in $H^{2n-8}(M \times M)$, $H^{2n-4}(M \times M)$, $H^{2n-2}(M \times M)$ and $H^{2n-1}(M \times M)$ respectively.

The class $i \circ l$ is invariant under t since both f and h are obviously invariant under t. Thus the homotopy class $[l] + [l \circ t]$ lies in the image of the homomorphism,

$$[M \times M, K_{n-1}] \rightarrow [M \times M, \Omega C].$$

I.e., there exists $x \in H^{n-1}(M \times M)$ such that

$$\begin{aligned}(9.6) \quad [l] + [l \circ t] &= (Sq^{n-7}x, \\ &Sq^{n-7}Sq^4x, (Sq^{n-3}Sq^2 + Sq^{n-7}Sq^2Sq^4)x, \\ &(Sq^{n-1}Sq^1 + Sq^{n-3}Sq^3 + Sq^{n-5}Sq^5 + Sq^{n-7}Sq^7)x) \\ &= (Sq^{n-7}x, Sq^{n-7}Sq^4x, 0, 0).\end{aligned}$$

By the connectivity condition on M we may assume that c and d are trivial. Therefore, since $Sq^4H^{2n-4}(M \times M) = 0$,

$$\begin{aligned} f^*\zeta &= h^*\zeta + Sq^8a + Sq^4b \\ (9.7) \quad &= \begin{cases} Sq^8a & \text{if } n \equiv 15(16) \\ \chi_2(M)\mu \otimes \mu + Sq^8a & \text{if } n \equiv 7(16) \end{cases}\end{aligned}$$

From (9.6) we have that

$$(9.8) \quad a + t^*a \in Sq^{n-7}H^{n-1}(M \times M).$$

Note that Sq^8 is trivial on any class in $H^i(M) \otimes H^{2n-8-i}(M)$ with bidegree $(i, 2n - 8 - i)$ different from $(n - 8, n)$ and $(n, n - 8)$. We shall show that $Sq^8a = 0$. This would prove 9.5. For this we need the following.

LEMMA 9.9. *Let M' be an orientable closed, connected and smooth manifold of dimension $n \equiv 7 \bmod 8$. Suppose $w_2(M') = 0$. Let*

$$p:H^{2n-8}(M' \times M') \rightarrow H^{n-8}(M') \otimes H^n(M')$$

be the projection corresponding to the Künneth formula. Then

$$Sq^{n-7}H^{n-1}(M' \times M') \subset \operatorname{Ker} P.$$

The proof is easy. Let

$$\alpha \otimes \beta \in H^{n-1}(M' \times M').$$

Then by the Cartan formula and Wu-duality we see that $Sq^{n-7}(\alpha \otimes \beta)$ does not have any non-trivial element with bidegree $(n - 8, n)$ and $(n, n - 8)$.

Therefore, since

$$a + t^*a \in Sq^{n-7}H^{n-1}(M \times M),$$

by 9.9 a is symmetric in the classes with bidegree $(n - 8, n)$ and $(n, n - 8)$. Therefore $Sq^8a = 0$. And this completes the proof of 9.5.

Following 6.9 of [**8**] we can derive the following.

THEOREM 9.10. *Let $A \in H^n(M \times M)$ be as given by the decomposition of* 9.1. *Suppose $w_4(M) = 0$. Then*

(i) $Sq^{n-8}Sq^2A = Sq^{n-9}Sq^2Sq^1A = Sq^{n-7}Sq^1A = Sq^{n-6}A = 0$;
$Sq^{n-11}Sq^2Sq^1A = 0$; $(Sq^{n-9}Sq^2Sq^3 + Sq^{n-7}Sq^1)A = 0$;

(ii) *Suppose* $0 \in \phi_4(w_{n-9}(M))$. *Then Ω is defined on A. Hence Ω is defined on t^*A. In particular $\Omega(U(\tau)) = 0$ modulo zero indeterminacy.*

(iii) *Suppose $w_{n-7}(M) = 0$ and $0 \in \phi_4(w_{n-9}(M))$, then $\widetilde{\Omega}$ is defined on A and $\widetilde{\Omega}(U(\tau)) = 0$.*

Proof. The proof of (i) is similar to that of 6.9 in [**8**]. If $n = 7 + 8s$, then for any $x \in H^{3+4s}(M)$, $y \in H^{4+4s}(M)$,

$$Sq^{4s-3}Sq^2Sq^1x = Sq^{4s-1}Sq^1x,$$

$$Sq^{4s-1}y = Sq^{4s-3}Sq^2y \text{ if } s \text{ is odd, and}$$

$$Sq^{4s-3}Sq^2Sq^1x = Sq^{4s-3}Sq^2y = 0 \text{ if } s \text{ is even.}$$

Now it can be shown that

$$Sq^{n-11}Sq^2Sq^1A = \sum_k (Sq^{4s-3}Sq^2Sq^1\alpha^k_{3+4s} \otimes Sq^{4s-1}\beta^k_{4+4s} + Sq^{4s-1}Sq^1\alpha^k_{3+4s} \otimes Sq^{4s-3}Sq^2\beta^k_{4-4s}).$$

Thus by the above remark

$$Sq^{n-11}Sq^2Sq^1A = 0.$$

The other cases are similar.

Part (iii) follows from (ii) and naturally since $\widetilde{\Omega} \subset \Omega$ and that $w_{n-7}(M) = 0$ implies that

$$Sq^{n-7}A = Sq^{n-7}t^*A = 0.$$

Part (ii) is harder. First we check that ζ_1 is defined and trivial on A. It can be shown that if $n = 7 + 8s$, then

$$\begin{aligned}Sq^{n-9}A &= w_{n-9}(M) \otimes \mu \\ &+ \sum_k (\alpha^k_{1+4s})^2 \otimes Sq^{4s-3}\beta^k_{4s+6} \\ &+ \sum_k (Sq^{4s+1}\alpha^k_{4s+2} \otimes Sq^{4s-3}\beta^k_{4s+5} \\ &+ Sq^{4s}\alpha^k_{4s+2} \otimes Sq^{4s-2}\beta^k_{4s+5}) \\ &+ \sum_k (Sq^{4s}\alpha^k_{4s+3} \otimes Sq^{4s-2}\beta^k_{4s+4} \\ &+ Sq^{4s-1}\alpha^k_{4s+3} \otimes Sq^{4s-1}\beta^k_{4s+4}).\end{aligned}$$

ζ_1 can be chosen in such a way that

$$\zeta_1(U(\tau)) = \phi_4(Sq^{n-9}U(\tau)).$$

Hence

$$\begin{aligned}g^*\zeta_1(U(\tau)) &= \zeta_1(g^*U(\tau)) = \phi_4(Sq^{n-9}(A + t^*A)) \\ &= \phi_4(Sq^{n-9}A) + t^*(\phi_4(Sq^{n-9}A)).\end{aligned}$$

Since M is 3-connected mod 2 and $w_4(M) = 0$, by a Cartan formula for ϕ_4 and the above proceeding,

$$\begin{aligned}(9.11)\quad \phi_4(Sq^{n-9}A) &= \phi_4(w_{n-9}(M)) \otimes \mu \\ &+ \sum_k (\alpha^k_{4s+1})^2 \otimes \phi_4(Sq^{4s-3}\beta^k_{4s+6}) \\ &+ \sum_k \{\phi_4(Sq^{4s+1}\alpha^k_{4s+2}) \otimes Sq^{4s-3}\beta^k_{4s+5} \\ &+ Sq^{4s}\alpha^k_{4s+2} \otimes \phi_4(Sq^{4s-2}\beta^k_{4s+5})\} \\ &+ \sum_k \{\phi_4(Sq^{4s}\alpha^k_{4s+3}) \otimes Sq^{4s-2}\beta^k_{4s+4} \\ &+ Sq^{4s-1}\alpha^k_{4s+3} \otimes \phi_4(Sq^{4s-1}\beta^k_{4+4s})\}\end{aligned}$$

modulo $\mathrm{Indet}^{2n-5}(\zeta_1, M \times M)$.

But by S-duality

$$\chi\phi_4(U(-\tau)) = Sq^1\phi_{1,1}(U(-\tau)) = 0.$$

Therefore ϕ_4 which is defined on $H^{n-4}(M)$ is trivial on $H^{n-4}(M)$ modulo zero indeterminacy. It follows from (9.11) that

$$\phi_4(Sq^{n-9}A) = \phi_4(w_{n-9}(M)) \otimes \mu.$$

Thus

$$0 \in \phi_4(w_{n-9}(M)) \Rightarrow 0 \in \phi_4(Sq^{n-9}A).$$

Hence $0 \in \zeta_1(A)$. Thus Ω is defined on A, hence on t^*A.

Let $P_2 \rightarrow P_1 \rightarrow K_n$ be the universal example tower of space for the operation Ω. Let U be the Thom class of τ reduced mod 2 and represented by a map

$$U:T(\tau) \rightarrow K_n.$$

Let $\bar{U}$ be lifting of U to P_1 such that $\bar{U}$ also has a lifting $\bar{\bar{U}}$ to P_2. Let

$$m_1:P_1 \times P_1 \rightarrow P_1 \quad \text{and}$$

$$m_2:P_2 \times P_2 \rightarrow P_2$$

be the multiplication maps. Let $A \in H^n(M \times M)$ be represented by a map

$$A:M \times M \rightarrow K_n$$

also denoted by the same symbol. If $0 \in \phi_4(w_{n-9}(M))$, Ω is defined on A. Let $\bar{A}$ be a lifting of A to P_1 and $\bar{\bar{A}}$ a lifting of $\bar{A}$ to P_2. Then

$$h = m_1 \circ (\bar{A}, \bar{A} \circ t)$$

is a lifting of $U \circ g$ to P_1 and

$$\bar{h} = m_2 \circ (\bar{\bar{A}}, \bar{\bar{A}} \circ t)$$

is a lifting of h to P_2. Let $f = \bar{U} \circ g$. Then f is also a lifting of $U \circ g$ to P_1.

Since f and h are both liftings of $U \circ g$ there is a map

$$l:M \times M \rightarrow \Omega C_1,$$

where

$$C_1 = K_{2n-6} \times K_{2n-6} \times K_{2n-8} \times K_{2n-4} \times K_{2n-2}$$

such that f and $h_1 = m_1 \circ (i_1 \circ l, h)$ are homotopic where

$$i_1:\Omega C_1 \rightarrow P_1$$

is the inclusion of the fibre.

Consider the following fibre square.

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{i}_1} & P_2 \\ \bar{P}_2 \downarrow & & \downarrow P_2 \\ \Omega C_1 & \xrightarrow{i_1} & P_1 \end{array}$$

We can represent l as a vector $(y, z, c, d, 0)$, where

$$y, z \in H^{2n-7}(M \times M), \quad c \in H^{2n-9}(M \times M) \quad \text{and}$$
$$d \in H^{2n-5}(M \times M).$$

The class $i_1 \circ l$ is invariant under t since both f and h are obviously invariant under t. Thus the homotopy class $[l] + [l \circ t]$ lies in the image of the homomorphism

$$[M \times M, K_{n-1}] \to [M \times M, \Omega C_1].$$

Note that since both f and h lift to P_2, l must lift to G_1 with a lifting

$$\bar{l}: M \times M \to G_1.$$

There is a class $\theta \in H^{n-1}(M \times M)$ such that

$$[l] + [l \circ t] = (Sq^{n-6}\theta + Sq^{n-7}Sq^1\theta,$$
$$(Sq^{n-9}Sq^2Sq^1 + Sq^{n-8}Sq^2)\theta,$$
$$Sq^{n-11}Sq^2Sq^1\theta,$$
$$(Sq^{n-9}Sq^2Sq^3 + Sq^{n-7}Sq^2Sq^1)\theta, 0).$$

It can be easily checked that

$$Sq^{n-6}H^{n-1}(M \times M) = 0$$

and $(Sq^{n-7}Sq^1\theta, (Sq^{n-9}Sq^2Sq^1 + Sq^{n-8}Sq^2)\theta)$ is of the form

$$((Sq^1\alpha)^2 \otimes \mu + \mu \otimes (Sq^1\alpha)^2,$$
$$(Sq^{4s-1}Sq^2 + Sq^{4s-2}Sq^2Sq^1)\alpha \otimes \mu$$
$$+ \mu \otimes (Sq^{4s-1}Sq^2 + Sq^{4s-2}Sq^2Sq^1)\alpha),$$

where $\alpha \in H^{4s-1}(M)$.

Since

$$H^{2n-7}(M \times M) \approx H^{n-7}(M) \otimes H^n(M) \oplus H^n(M) \otimes H^{n-7}(M)$$

we can write

$$y = y' \otimes \mu + \mu \otimes y''$$

where $y', y'' \in H^{n-7}(M)$. Therefore

$$y + t^*y = (y' + y'') \otimes \mu + \mu \otimes (y' + y'').$$

Since Γ_1 is defined on y, Γ_1 is defined on y' and y''. Therefore modulo zero indeterminacy

$$\Gamma_1(y' + y'') = \Gamma_1(y') + \Gamma_1(y'').$$

Now

$$\Gamma_1(Sq^1\alpha)^2 = \Gamma_1(Sq^{4s}Sq^1\alpha) = \Gamma_1(Sq^2Sq^{4s-1}\alpha).$$

But by S-duality pairing,

$$\langle \Gamma_1(Sq^2Sq^{4s-1}\alpha), U(-\tau) \rangle = \langle Sq^2Sq^{4s-1}\alpha, \chi\Gamma_1 U(-\tau) \rangle$$
$$= \langle Sq^{4s-1}\alpha, Sq^2(U(-\tau) \cdot Sq^3 v_4(-\tau)) \rangle$$
$$= \langle Sq^{4s-1}\alpha, U(-\tau)Sq^2Sq^3v_4(-\tau) \rangle.$$

But $Sq^2Sq^3v_4(-\tau) = 0$. Thus $\Gamma_1(Sq^1\alpha)^2 = 0$. Hence $\Gamma_1(y' + y'') = 0$ and so

$$\Gamma_1(y') = \Gamma_1(y'').$$

Thus

$$\Gamma_1(y' \otimes \mu + \mu \otimes y'') = \Gamma_1(y') \otimes \mu + \mu \otimes \Gamma_1(y'') = 0.$$

Similarly we can show that $\Gamma_2(z) = 0$. The proof of Theorem 6.5 shows that $\Gamma_3(c) = 0$, $\Gamma_5(d) = 0$. Hence

$$\Gamma_1(y) + \Gamma_2(z) + \Gamma_3(c) + \Gamma_5(d) = 0.$$

Now $\bar{\bar{h}} = m_2 \circ (\bar{i}_1 \circ \bar{l}, \bar{h})$ is a lifting of $m_1 \circ (i_1 \circ l, h) \sim f$. Let w be a representative for the operation Ω. Then

$$\bar{\bar{h}}^*w = \bar{h}^*w + \bar{l}^*\bar{i}_1^*w.$$

Now

$$\bar{l}^*\bar{i}_1^*w \in \Gamma_1(y) + \Gamma_2(z) + \Gamma_3(c) + \Gamma_5(d) = 0.$$

Therefore

$$\bar{\bar{h}}^*w = \bar{h}^*w = \bar{\bar{A}}^*w + t^*\bar{\bar{A}}^*w = 0.$$

Now $\bar{f} = \bar{\bar{u}} \circ g$ is a lifting of

$$f \sim m_1 \circ (i_1 \circ l, h).$$

Since the primary piece of the indeterminacy of Ω is trivial,

$$\bar{f}^*w = \bar{\bar{h}}^*w = 0.$$

That is

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$$g^*\overline{\overline{U}}^*w = 0.$$

Since g^* is injective,

$$\overline{\overline{U}}^*w = 0.$$

Thus $\Omega(U(\tau)) = 0$ modulo zero indeterminacy.

10. Vector fields on manifolds. We shall now prove Theorem 1.1 and Theorem 1.2.

Suppose $w_4(M) = 0$. Recall that then

$$\text{Indet}^n(k_4^3, M) = \Gamma_1 D^{n-7}$$

for the case $k = 7$.

10.1. Proof of Theorem 1.1. $\text{Indet}^{n-4}(\psi_5, M) = \text{Indet}^{n-4}(k_1^3, M)$ implies that

$$\text{Indet}^{n-4}(\phi_{1,1}, M) = Sq^2H^{n-6}(M).$$

Furthermore if $n \equiv 7 \bmod 8$,

$$\delta w_{n-7}(M) = 0,\ w_{n-5}(M) = 0.$$

In particular if $n \equiv 15 \bmod 16$,

$$w_{n-7}(M) = w_{n-9}(M) = 0.$$

If $\text{Indet}^n(k_4^3, M) \neq 0$, the hypothesis of Theorem 7.1 is satisfied. Thus it follows from 7.1 and 9.3 (ii) that $\text{Span}(M) \geqq 7$ if and only if

$$0 \in \phi_4(w_{n-9}(M)),\ 0 \in \phi_{1,1}(w_{n-7}(M)) \quad \text{and}$$
$$0 \in \psi_5(w_{n-9}(M)).$$

Thus by the above remark if $n \equiv 15 \bmod 16$, $\text{Span}(M) \geqq 7$. If $n = 7 + 16s$ with $n > 7$, then $w_{n-7}(M) = V_{8s}^2$, where $V_{8s} \in H^{8s}(M)$ is the $8s$-th Wu class of M. It is easily seen that

$$Sq^1V_{8s} = Sq^2V_{8s} = 0.$$

Therefore by a Cartan formula for $\phi_{1,1}$,

$$\phi_{1,1}(w_{n,-7}(M)) = \phi_{1,1}(V_{8s}) \cdot V_{8s} + V_{8s} \cdot \phi_{1,1}(V_{8s}) = 0$$

modulo indeterminacy of $\phi_{1,1}$. Thus

$$0 \in \phi_{1,1}(w_{n-7}(M)).$$

This proves the assertion in (ii) when $n \equiv 7 \bmod 16$ and

$$\text{Indet}^n(k_4^3, M) \neq 0.$$

The case when $\text{Indet}^n(k_4^3, M) = 0$ follows from 8.1, 9.3 and 9.10. This completes the proof.

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Notice, if $Sq^3\nu_4(-\tau) \in Sq^2H^5(M)$, in applying 8.1 we only require that

$$\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$$

for the case $k = 7$. We have actually proved a stronger result.

THEOREM 10.2. (The case $k = 7$.) *Suppose*

$$w_4(M) = 0,$$
$$Sq^3\nu_4(-\tau) \in Sq^2H^5(M),$$
$$\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M) \quad \textit{and}$$
$$Sq^2H^{n-7}(M; \mathbf{Z}) = Sq^2H^{n-7}(M).$$

Then:

(i) *If* $n \equiv 15 \bmod 16$, $\mathrm{span}(M) \geqq 7$;
(ii) *If* $n \equiv 7 \bmod 16 > 7$, $\mathrm{span}(M) \geqq 7$ *if and only if*

$$0 \in \phi_4(w_{n-9}(M)) \quad \textit{and} \quad 0 \in \psi_5(w_{n-9}(M)).$$

The proof of 1.2 is similar to that of 1.1, using Theorem 8.2, 9.5 and 9.10. We have in fact a stronger result:

THEOREM 10.3. (The case $k = 8$). *Suppose*

$$w_4(M) = 0,$$
$$Sq^3\nu_4(-\tau) \in Sq^2H^5(M),$$
$$Sq^2H^{n-7}(M) = Sq^2Sq^1H^{n-8}(M) \quad \textit{and}$$
$$\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M).$$

(i) *If* $n \equiv 15 \bmod 16$ *with* $n > 15$, *then* $\mathrm{span}(M) \geqq 8$;
(ii) *If* $n \equiv 7 \bmod 16 > 7$, *then* $\mathrm{span}(M) \geqq 8$ *if and only if*

$$w_{n-7}(M) = 0,\ 0 \in \phi_4(w_{n-9}(M)),$$
$$0 \in \psi_5(w_{n-9}(M)) \quad \textit{and} \quad \chi_2(M) = 0.$$

11. Application. It is well known that $\mathrm{Span}(S^{8s+3}) = 3$. Let us consider

$$M = S^{3+8s} \times QP^{1+2k}, \quad s \geqq 1,\ k \geqq 0,$$

where QP^j is the quaternionic projective space of real dimension $4j$. Then

$$\mathrm{Indet}^{n-4}(\psi_5, M) = 0,$$
$$H^{8(s+k)}(M) = H^{8(s+k)+1}(M) = 0,$$

$$H^7(M) = H^{8(s+k)-2}(M) = 0 \text{ and}$$

$$\chi_2(M) = 0.$$

By 1.2 we have the following immediate result.

THEOREM 11.1.

$$\text{Span}(S^{3+8s} \times QP^{1+2k}) \geqq 8 \text{ for } s \geqq 1, k \geqq 0.$$

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