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SOLUTION OF A GENERAL HOMOGENEOUS LINEAR DIFFERENCE EQUATION

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Abstract

Solutions of a homogeneous $(r+1)$-term linear difference equation are given in two different forms. One involves the elements of a certain matrix, while the other is in terms of certain lower Hessenberg determinants. The results generalize some earlier results of Brown [1] for the solution of a 3-term linear difference equation.

1. Introduction

In a recent paper, Brown [1] has given the solution of the three-term linear difference equation

$$a_0(n)\,u_n + a_1(n)\,u_{n-1} + a_2(n)\,u_{n-2} = 0, \quad n \geqslant 2, \tag{1}$$

with $a_0(n) \neq 0$ for all $n \geqslant 2$, in terms of certain tri-diagonal determinants. In this paper, we consider the general $(r+1)$-term homogeneous linear difference equation

$$a_0(n)\,u_n + a_1(n)\,u_{n-1} + \ldots + a_r(n)\,u_{n-r} = 0, \quad n \geqslant r, \tag{2}$$

with the condition that $a_0(n) \neq 0$ for all $n \geqslant r$. We obtain two forms of the general solution for this difference equation, namely a matrix form, given in Section 2, and a determinantal form, given in Section 3. An interesting determinantal relation is derived in Section 4.

We introduce the following notations:

$$a_t(n) = 0 \quad \text{if } t \text{ is a negative integer or a positive integer} > r.$$

$$p_k = \prod_{l=r}^{k} a_0(l), \quad q_k = \prod_{l=r}^{k} a_r(l); \quad \text{empty products will be taken to be 1.}$$

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$$A_k = [a_{i-j}(kr+i-1)],\ i,j = 1,\dots,r,\ k = 1,2,\dots .$$

$$B_k = [a_{r-(j-i)}(kr+i-1)],\ i,j = 1,\dots,r,\ k = 1,2,\dots .$$

$N = [n/r]$, where $[x]$ denotes the integral part of x; $N' = n - Nr + 1$.

$$A_{(n)} = [a_{i-j}(Nr+i-1)], \quad i,j = 1,\dots,N'.$$

$$B_{(n)} = [a_{r-(j-i)}(Nr+i-1)], \quad i = 1,\dots,N', \quad j = 1,\dots,r.$$

$$U_{(r,n)} = [u_r\, u_{r+1} \dots u_n]^T; \quad U_{(kr,(k+1)r-1)} \equiv U_k.$$

$$D_m^n(r,s) = \begin{vmatrix} a_s(m) & a_0(m) & 0 & \dots & 0 & \dots & 0 \\ a_{s+1}(m+1) & a_1(m+1) & a_0(m+1) & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_r(m+r-s) & a_{r-s}(m+r-s) & . & \dots & a_0(m+r-s) & \dots & 0 \\ 0 & . & . & \dots & . & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ . & a_r(m+r) & a_{r-1}(m+r) & \dots & . & \dots & 0 \\ . & 0 & a_r(m+r+1) & \dots & . & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ . & . & . & \dots & . & \dots & a_0(n-1) \\ . & . & . & \dots & . & \dots & a_1(n) \end{vmatrix}$$

for $n \geqslant m+1$; $D_m^m(r,s) = a_s(m)$ and $D_r^n(r,s) \equiv D_s^n$.

2. Solution in terms of matrix elements

We shall first obtain a matrix solution of the linear difference equation (2). If we put $n = kr, kr+1, \dots, (k+1)r-1$ in (2), we get the matrix reduction formula

$$A_k U_k = -B_k U_{k-1}, \quad k \geqslant 1. \tag{3}$$

Therefore,

$$U_k = (-1)^k \left\{ \prod_{s=1}^{k} (A_s^{-1} B_s) \right\} U_0, \quad k \geqslant 1, \tag{4}$$

as the matrices A_s, $s = 1,\dots,k$, being lower triangular, are non-singular because of the condition that $a_0(n) \neq 0$ for all $n \geqslant r$. Thus, if $n \equiv t \pmod{r}$, then $n = kr+t$, $0 \leqslant t \leqslant r-1$, and so, from (4), we obtain the following general solution for the difference equation (2):

$$u_{kr+t} = \text{the } (t+1)\text{th element of the column matrix } U_k,\ 0 \leqslant t \leqslant r-1. \tag{5}$$

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3. Solution in terms of determinants

In this section, we obtain a solution of the linear difference equation (2) in terms of the determinants D_s^n, $s = 1, \ldots, r$.

Let $n = r, r+1, \ldots, n$ in (2). Then

$$A_{(r,n)} U_{(r,n)} = -\begin{bmatrix} B_1 \\ 0_{n-2r+1,r} \end{bmatrix} U_0, \tag{6}$$

where

$$A_{(r,n)} = \begin{bmatrix} A_1 & 0_r & 0_r & \cdots & 0_r & 0_r & 0_{r,N'} \\ B_2 & A_2 & 0_r & \cdots & 0_r & 0_r & 0_{r,N'} \\ 0_r & B_3 & A_3 & \cdots & 0_r & 0_r & 0_{r,N'} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_r & 0_r & 0_r & \cdots & B_{N-1} & A_{N-1} & 0_{r,N'} \\ 0_{N',r} & 0_{N',r} & 0_{N',r} & \cdots & 0_{N',r} & B_{(n)} & A_{(n)} \end{bmatrix},$$

with $0_{r,s}$ denoting the null matrix of dimension r by s, and with $0_{r,r} \equiv 0_r$.

Applying Cramer's rule to the linear non-homogeneous system (6), we get, in particular,

$$|A_{(r,n)}| u_n = C_{(r,n)}, \tag{7}$$

where

$$C_{(r,n)} = \begin{bmatrix} A_1 & 0_r & 0_r & \cdots & 0_r & 0_r & 0_{r,N'-1} & -B_1 U_0 \\ B_2 & A_2 & 0_r & \cdots & 0_r & 0_r & 0_{r,N'-1} & 0_{r,1} \\ 0_r & B_3 & A_3 & \cdots & 0_r & 0_r & 0_{r,N'-1} & 0_{r,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_r & 0_r & 0_r & \cdots & B_{N-1} & A_{N-1} & 0_{r,N'-1} & 0_{r,1} \\ 0_{N',r} & 0_{N',r} & 0_{N',r} & \cdots & 0_{N',r} & B_{(n)} & A^*_{(n)} & 0_{r,1} \end{bmatrix}, \tag{8}$$

the asterisk in the last row denoting the omission of the last column of the starred matrix. On expanding the determinant $C_{(r,n)}$ along the last column, we find that

$$\begin{aligned} C_{(r,n)} &= (-1)^{n-r+1} \sum_{t=0}^{r-1} (-1)^t \left\{ \sum_{s=1}^{r-t} a_{s+t}(r+t) u_{r-s} \right\} p_{r+t-1} D^n_{r+t+1}(r,1) \\ &= (-1)^{n-r+1} \sum_{s=1}^{r} u_{r-s} \left\{ \sum_{t=0}^{r-s} (-1)^t a_{s+t}(r+t) p_{r+t-1} D^n_{r+t+1}(r,1) \right\}. \end{aligned}$$

The last inner sum is seen to be the expansion, along the first column, of the

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determinant D_s^n. Hence, by (7), we obtain the following determinantal solution of the linear difference equation (2):

$$u_n = (-1)^{n-r+1} p_n^{-1} \sum_{s=1}^{r} D_s^n u_{r-s}, \quad n \geqslant r. \tag{9}$$

It is easy to see that, when $r = 2$, this reduces to the solution given by Brown [1], equation (3.10):

$$u_n = (-1)^{n-1} L_{n-1}^{-1} \{u_1 C_1^{n-1} + u_0 F(1) C_2^{n-1}\}, \quad n \geqslant 3,$$

if one observes that

$$C_m^n = D_{m+1}^{n+1}(2,1) = D_m^{n+1}(2,2)/\{a_2(m)\},$$

so that

$$D_1^n = C_1^{n-1} \quad \text{and} \quad D_2^n = a_2(2) C_2^{n-1}. \tag{10}$$

4. A relation between determinants

We can easily obtain a determinantal relation connecting the determinants D_s^{n+t}, $t = 0, 1, \dots, r-1$, $s = 1, \dots, r$.

Let

$$E_r^n \equiv |D_j^{n+i-1}|, \quad i,j = 1, \dots, r,$$

denote the determinant formed from these r^2 determinants. On expanding the determinant D_s^n along the last column, we have

$$D_s^n = \sum_{t=1}^{r} (-1)^{t-1} \left\{\prod_{l=1}^{t-1} a_0(n-l)\right\} a_t(n) D_s^{n-t}, \quad n \geqslant 2r, \tag{11}$$

for $s = 1, \dots, r$. Making use of relation (11) in the last row of determinant E_r^n, we find that

$$E_r^n = \left\{\prod_{l=n}^{n+r-2} a_0(l)\right\} a_r(n+r-1) E_r^{n-1}, \quad n \geqslant r+1. \tag{12}$$

We now need the following

Lemma.

$$E_r^r = \left\{\prod_{n=r}^{2r-2} p_n\right\} q_{2r-1}. \tag{13}$$

Proof. Let

$$D_r^* = [D_{r-j+1}^{r+i-1}] \quad \text{and} \quad P_r = [(-1)^{i-1} p_{r+i-1} \delta_j^i],$$

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where $i,j = 1,\ldots,r$ and δ^i_j is the Kronecker delta. Then the set of solutions (9) for $n = r, r+1, \ldots, 2r-1$, can be written as

$$D_r^* U_0 = -P_r U_1. \tag{14}$$

Since, by (4),

$$U_1 = -(A_1^{-1} B_1) U_0,$$

we have

$$(P_r^{-1} D_r^*) U_0 = (A_1^{-1} B_1) U_0. \tag{15}$$

Denoting by E_k the r-dimensional unit vector having unity in the kth place and zeros everywhere else, and taking $U_0 = E_k$, $k = 1, 2, \ldots, r$, successively in (15), we get

$$P_r^{-1} D_r^* = A_1^{-1} B_1,$$

so that

$$(-1)^{r(r-1)/2} E_r^r = \left\{ \prod_{n=r}^{2r-1} (-1)^{n-r} p_n \right\} p_{2r-1}^{-1} q_{2r-1},$$

whence (13) follows.

Using the reduction formula (12) and the value (13), after some rearrangement we finally get

$$E_r^n = \left\{ \prod_{k=n}^{n+r-2} p_k \right\} q_{n+r-1}, \quad n \geqslant r. \tag{16}$$

Thus, E_r^n vanishes if $a_r(l) = 0$ for some l, $r \leqslant l \leqslant n+r-1$.

When $r = 2$, (16) reduces to the corresponding relation given by Brown [1], equation (3.14), because of the relations (10).

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Reference

[1] A. Brown, "Solution of a linear difference equation", *Bull. Austral. Math. Soc.* 11 (1974), 325–331.

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