Canad. Math. Bull. Vol. 62 (3), 2019 pp. 583–601
http://dx.doi.org/10.4153/S0008439518000140

Tits Triangles

Bernhard Mühlherr and Richard M. Weiss

Abstract. A Tits polygon is a bipartite graph in which the neighborhood of every vertex is endowed with an "opposition relation" satisfying certain properties. Moufang polygons are precisely the Tits polygons in which these opposition relations are all trivial. There is a standard construction that produces a Tits polygon whose opposition relations are not all trivial from an arbitrary pair (Δ, T), where Δ is a building of type Π, Π is a spherical, irreducible Coxeter diagram of rank at least 3, and T is a Tits index of absolute type Π and relative rank 2. A Tits polygon is called k-plump if its opposition relations satisfy a mild condition that is satisfied by all Tits triangles coming from a pair (Δ, T) such that every panel of Δ has at least $k+1$ chambers. We show that a 5-plump Tits triangle is parametrized and uniquely determined by a ring R that is alternative and of stable rank 2. We use the connection between Tits triangles and the theory of Veldkamp planes as developed by Veldkamp and Faulkner to show existence.

1 Introduction

An irreducible spherical building of rank 2 is a generalized polygon. Generalized polygons are too numerous to classify, but Tits observed that the generalized polygons associated with absolutely simple algebraic groups of F-rank 2 satisfy a symmetry property he called the *Moufang condition*. Moufang polygons (that is to say, generalized polygons satisfying the Moufang condition) were classified in [14].

The notion of a Tits polygon was introduced in [11]. A Tits polygon is a bipartite graph in which the neighborhood of each vertex is endowed with an "opposition relation" satisfying certain axioms. Moufang polygons are precisely the Tits polygons in which these opposition relations are all trivial. Every pair (Δ, T), where Δ is an irreducible spherical building of rank at least 3 and T is a Tits index of relative rank 2 whose absolute type is the Coxeter system of Δ, gives rise, in a natural way, to a Tits polygon in which the opposition relations are not all trivial. We call these (together with those that are actually Moufang polygons) the Tits polygons of *index type*.

Almost all Tits polygons of index type share a natural property of their tori. We call a Tits polygon *dagger-sharp* if it has this property or simply *sharp* if it has a slightly weaker version of this property. All Moufang polygons are trivially dagger-sharp.

In [11] we developed the basic theory of Tits polygons. In particular, we showed that sharp Tits n-gons exist only for $n = 3, 4, 6$, and 8 (as is the case with Moufang n-gons). In the second part of [11] and in forthcoming work, we give characterizations of large classes of Tits quadrangles and hexagons (*i.e*, Tits n-gons for $n = 4$ and 6) of index type, all arising from buildings of exceptional type.

Received by the editors May 16, 2018; revised October 25, 2018.
Published online on Cambridge Core April 17, 2019.
The work of the first author was partially supported by a grant from the DFG and the work of the second author was partially supported by a collaboration grant from the Simons Foundation.
AMS subject classification: **17D05**, 20E42, 51E12, 51E24.
Keywords: Tits polygon, Veldkamp plane, building.

In this paper we consider the case $n = 3$. Under the assumption that a Tits triangle X is 5-plump (a mild condition satisfied, for example, by all Tits polygons of index type arising from a building whose panels all contain at least 6 chambers), we show in Section 5 that X is parametrized and uniquely determined by a ring R that is alternative and of stable rank 2. Rings of stable rank 2 were introduced in [1], and their connection to incidence geometry was first explored by Veldkamp in [15]. It turns out that a Tits triangle is essentially the same thing as a Veldkamp plane that is (C, l)-transitive for all incident pairs C, l as defined in [6, §2], and hence it follows from [6, Thm. 9] that every alternative ring of stable rank 2 arises as the parametrizing ring of a Tits triangle. In Section 7 we obtain as a corollary that [6, Thm. 9] continues to hold if we replace the configurational condition (QS) in the definition of a Moufang–Veldkamp plane on [6, pp. 190-191] by the much milder 5-plump condition. In Section 5 we also show that a 5-plump Tits triangle X is dagger-sharp if and only if the ring R is simple.

Conventions Let G be a group. As in [14], we set

$$a^b = b^{-1}ab \quad \text{and} \quad [a, b] = a^{-1}b^{-1}ab$$

for all $a, b \in G$. Thus,

$$[ab, c] = [a, c]^b \cdot [b, c] \quad \text{and} \quad [a, bc] = [a, c] \cdot [a, b]^c \tag{1.1}$$

for all a, b, c. We compose permutations from left to right. Other functions will be written on the left and composed from right to left. When i and j are subscripts, we sometimes use $[i, j]$ to indicate the set of integers $\{i, i+1, \ldots, j\}$ as, for example, in Definition 2.1(ii).

2 Tits Polygons

In this section, we define Tits polygons and summarize their most basic properties.

Definition 2.1 A *dewolla* is a triple

$$X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V}),$$

where the following hold:

(i) Γ is a bipartite graph with vertex set V and $|\Gamma_v| \geq 3$ for each $v \in V$, where Γ_v denotes the set of vertices adjacent to v.

(ii) For each $v \in V$, $\equiv_v$ is an anti-reflexive symmetric relation on Γ_v. We say that vertices $u, w \in V$ are *opposite at* v if $u, w \in \Gamma_v$ and $u \equiv_v w$, and a path $(w_0, w_1, \ldots, w_m)$ in Γ is called *straight* if w_{i-1} and w_{i+1} are opposite at w_i for all $i \in [1, m-1]$.

(iii) $\mathcal{A}$ is a set of $2n$-cycles for some $n \geq 3$ and every cycle in $\mathcal{A}$ contains only straight paths.

The parameter n is called the *level* of X. The automorphism group $\mathrm{Aut}(X)$ is the subgroup of $\mathrm{Aut}(\Gamma)$ consisting of all $g \in \mathrm{Aut}(\Gamma)$ such that $\mathcal{A}^g = \mathcal{A}$ and for all $v \in V$ and all $u, w \in \Gamma_v$, u and w are opposite at v if and only if u^g and w^g are opposite at v^g. A *root* of X is a straight path of length n.

Definition 2.2 A *Tits n-gon* is a dewolla

$$X = \left(\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V}\right)$$

of level n for some $n \geq 3$ such that Γ is connected and the following axioms hold:

(i) For all $v \in V$ and all $u, w \in \Gamma_v$, there exists $z \in \Gamma_v$ that is opposite both u and w at v.

(ii) For each straight path $\delta = (w_0, \dots, w_k)$ of length $k \leq n-1$, δ is the unique straight path of length at most k from w_0 to w_k.

(iii) For $G = \mathrm{Aut}(X)$ and for each root $\alpha = (w_0, \dots, w_n)$ of X, the group U_α acts transitively on the set of vertices opposite w_{n-1} at w_n, where U_α is the pointwise stabilizer of the set

$$\Gamma_{w_1} \cup \Gamma_{w_2} \cup \cdots \cup \Gamma_{w_{n-1}}$$

in $\mathrm{Aut}(X)$. The group U_α is called the *root group* associated with the root α.

A *Tits polygon* is a Tits n-gon for some $n \geq 3$. A Tits n-gon is called a *Tits triangle* if $n = 3$, a *Tits quadrangle* if $n = 4$, etc.

Notation 2.3 Let $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ be a Tits n-gon. A *coordinate system* for X is a pair $(\gamma, i \mapsto w_i)$ where γ is an element of $\mathcal{A}$ and $i \mapsto w_i$ is a map from $\mathbb{Z}$ to the vertex set of γ such that w_{i+1} is adjacent to w_i and distinct from w_{i-1} for each i. For each coordinate system $(\gamma, i \mapsto w_i)$, we denote by U_i the root group associated with the root $(w_i, w_{i+1}, \dots, w_{i+n})$ for each $i \in \mathbb{Z}$ and call the map $i \mapsto U_i$ the associated *root group labeling*. Thus, $w_i = w_j$ and $U_i = U_j$ whenever i and j have the same image in $\mathbb{Z}_{2n}$. If $(\gamma, i \mapsto w_i)$ is a coordinate system for X, then so is $(\gamma, i \mapsto w_{n+1-i})$.

For the rest of this section, we fix a Tits n-gon $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ and a coordinate system $(\gamma, i \mapsto w_i)$ of X and let $i \mapsto U_i$ be the corresponding root group labeling.

Proposition 2.4 *The following hold:*

(i) $[U_i, U_j] \subset U_{[i+1,j-1]}$ *for all* i, j *such that* $i < j < i+n$*, where*

$$U_{[k,m]} = \begin{cases} U_k U_{k+1} \cdots U_m & \text{if } k \leq m, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, $[U_i, U_{i+1}] = 1$ *for all* i.

(ii) *The product map* $U_1 \times U_2 \times \cdots \times U_n \to U_{[1,n]}$ *is bijective.*

Proof This holds by [11, 1.3.36]. ∎

Proposition 2.5 *For each* i*,* U_i *acts sharply transitively on the set of vertices that are opposite* w_{i+1} *at* w_i *and on the set of vertices that are opposite* w_{i+n-1} *at* w_{i+n}.

Proof This holds by [11, 1.3.25]. ∎

Notation 2.6 Let

$$U_i^\sharp = \{a \in U_i \mid w_{i+n+1}^a \text{ is opposite } w_{i+n+1} \text{ at } w_{i+n}\}$$

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

for each i. By [11, 1.4.3], $U_i^\sharp$ is non-empty and closed under inverses, and by [11, 1.4.8], we have

$$U_i^\sharp = \{a \in U_i \mid w_{i-1}^a \text{ is opposite } w_{i-1} \text{ at } w_i\}$$

for each i

Proposition 2.7 *For each $i \in \mathbb{Z}$, there exist unique maps κ_γ and λ_γ from $U_i^\sharp$ to $U_{i+n}^\sharp$ such that for each $a \in U_i^\sharp$, the product*

$$\mu_\gamma(a) := \kappa_\gamma(a) \cdot a \cdot \lambda_\gamma(a)$$

interchanges the vertices w_{i+n-1} and w_{i+n+1}. For each $a \in U_i^\sharp$, the element $\mu_\gamma(a)$ fixes the vertices w_i and w_{i+n} and interchanges the vertices w_{i+j} and w_{i-j} for all $j \in \mathbb{Z}$ and

$$U_k^{\mu_\gamma(a)} = U_{2i+n-k} \tag{2.1}$$

for all $k \in \mathbb{Z}$.

Proof This holds by [11, 1.4.4]. ∎

Remark 2.8 Let $m_i \in \mu_\gamma(U_i^\sharp)$ for $i = 1$ and n. By Proposition 2.7, the subgroup $\langle m_1, m_n\rangle$ acts transitively on the set of edges of γ. If n is odd, $\langle m_1, m_n\rangle$ acts transitively on the set $\{U_i \mid i \in \mathbb{Z}\}$, but if n is even, U_i and U_j are in the same $\langle m_1, m_n\rangle$-orbit if and only if i and j have the same parity.

Proposition 2.9 *Let $a \in U_i^\sharp$ for some i. Then $\mu_\gamma(a)$ is the unique element of $U_{i+n}aU_{i+n}$ that interchanges w_{i-1} and w_{i+1}.*

Proof Let $b, c \in U_{i+n}$ and suppose that $m := bac$ interchanges w_{i-1} and w_{i+1}. Then $w_{i+1}^b = w_{i-1}^{a^{-1}}$ and $w_{i-1}^a = w_{i+1}^{c^{-1}}$. By Proposition 2.5, therefore, b and c are unique. Hence $b = \kappa_\gamma(a)$, $c = \lambda_\gamma(a)$ and $m = \mu_\gamma(a)$. ∎

Proposition 2.10 *Let $u \in U_i^\sharp$ for some i. Then $\mu_\gamma(u^{-1}) = \mu_\gamma(u)^{-1}$.*

Proof By Notation 2.6, $u^{-1} \in U_i^\sharp$. The claim holds, therefore, by [11, 1.4.13(i)]. ∎

Proposition 2.11 *Suppose that $[a_1, a_n^{-1}] = a_2 \cdots a_{n-1}$ with $a_i \in U_i$ for each $i \in [1, n]$. Then the following hold:*

(i) *If $a_1 \in U_1^\sharp$, then $a_2 = a_n^{\mu_\gamma(a_1)}$ and $[a_2, \lambda_\gamma(a_1)^{-1}] = a_3 \cdots a_{n-1}a_n$.*

(ii) *If $a_n \in U_n^\sharp$, then $a_1 = a_{n-1}^{\mu_\gamma(a_n)}$ and $[\kappa_\gamma(a_n), a_{n-1}^{-1}] = a_1a_2 \cdots a_{n-2}$.*

Proof This holds by [11, 1.4.16]. ∎

Proposition 2.12 *$w_{i-1}^{U_iU_{i+n}} = \Gamma_{w_i} = w_{i+1}^{U_{i+n}U_i}$ for each i.*

Proof This holds by [11, 1.3.4]. ∎

Proposition 2.13 *The following hold:*

(i) *If $a \in U_1$ and $U_2^{ba} = U_n$ for some $b \in U_{n+1}$, then $a \in U_1^\sharp$.*

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

(ii) *If $a \in U_n$ and $U_{n-1}^{ba} = U_1$ for some $b \in U_0$, then $a \in U_n^\sharp$.*

Proof This holds by [11, 1.4.27]. ∎

Definition 2.14 Let $k \geq 3$. As in [11, 1.4.21], we call X *k-plump* if for all $v \in V$, and for every subset N of Γ_v of cardinality at most k, there exists a vertex that is opposite u at v for all $u \in N$. Thus, k-plump implies $(k-1)$-plump, where "2-plump" is simply Definition 2.2(i). See Remark 8.4.

Notation 2.15 Let $G = \mathrm{Aut}(X)$, let $G^\dagger$ denote the subgroup of G generated by all the root groups of X and let G_γ (resp. $G_\gamma^\dagger$) denote the pointwise stabilizer of γ in G (resp. $G^\dagger$). The subgroup G_γ normalizes U_i for each i. We say that X is *sharp* if for each i, every nontrivial G_γ-invariant normal subgroup of U_i contains elements of $U_i^\sharp$, where $U_i^\sharp$ is as in Notation 2.6. We say that X is *dagger-sharp* if for each i, every nontrivial $G_\gamma^\dagger$-invariant normal subgroup of U_i contains elements of $U_i^\sharp$. Note that dagger-sharp implies sharp.

Remark 2.16 Let G_γ and $G_\gamma^\dagger$ be as in Notation 2.15. It follows from Remark 2.8 that to show that X is sharp (or dagger-sharp), it suffices to show that every nontrivial G_γ-invariant (or $G_\gamma^\dagger$-invariant) normal subgroup of U_i contains elements of $U_i^\sharp$ for any one value of i if n is odd, respectively, for $i = 1$ and n if n is even.

Proposition 2.17 *If X is $(n+1)$-plump, then $G_\gamma^\dagger = Y_1 Y_n$, where $G_\gamma^\dagger$ is as in Notation 2.15 and*

$$Y_i = \left\langle \mu_\gamma(a)\mu_\gamma(b) \mid a, b \in U_i^\sharp \right\rangle$$

for $i = 1$ and n.

Proof This holds by [11, 1.5.28]. ∎

Proposition 2.18 *Suppose that X is $(n+1)$-plump and let $m_i \in \mu_\gamma(U_i^\sharp)$ for $i = 1$ and n. Then X is uniquely determined by $U_{[1,n]}$, the action of m_1 on $U_{[2,n]}$, the action of m_n on $U_{[1,n-1]}$ and the action of the product $m_i\mu_\gamma(u)$ on $U_{[1,n]}$ for $i = 1$ and n and for all $u \in U_i^\sharp$.*

Proof This holds by [11, 1.5.29]. ∎

3 Tits Triangles

The main result of this section is Theorem 3.2. The proof is derived from results in [14, Chapter 19].

Notation 3.1 Let A be a unital ring (*i.e., A* contains a two-sided identity 1) that is not necessarily commutative or associative. We will say that an element a of A is *invertible* if there exists $b \in A$ such that $xa \cdot b = x$ and $b \cdot ax = x$ for all $x \in A$. The element b, if it exists, is unique and is denoted by a^{-1}. We denote by $A^\times$ the set of invertible elements of A.

Theorem 3.2 *Let $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$ be a Tits triangle, let $(\gamma, i \mapsto w_i)$ be a coordinate system of X, and let $i \mapsto U_i$ be the corresponding root group labeling. Then there exists a unital ring R and isomorphisms x_i from the additive group of R to U_i for all $i \in [1,3]$ such that the following hold:*

(i) $[x_1(s), x_3(t)] = x_2(st)$ *for all* $s, t \in R$.
(ii) $x_i(t) \in U_i^\sharp$ *if and only if* $t \in R^\times$.
(iii) *The set* $R^\times$ *of invertible elements of* R *is closed under multiplication and if* $u, v \in R^\times$, *then* $(uv)^{-1} = v^{-1}u^{-1}$.
(iv) *The set* $R^\times$ *is closed under inverses and* $(t^{-1})^{-1} = t$ *for all* $t \in R^\times$.
(v) *The Moufang identities* $(tu \cdot v)u = t(u \cdot vu)$, $u(t \cdot uv) = (ut \cdot u)v$ *and* $uv \cdot tu = u(vt \cdot u) = (u \cdot vt)u$ *hold for all* $u \in R^\times$ *and all* $t, v \in R$.

We prove Theorem 3.2 in a series of steps.

Remark 3.3 Suppose that $[a_i, a_{i+2}] = a_{i+1}$ for some i with $a_{i+j} \in U_{i+j}$ for $j \in [0,2]$. By Proposition 2.4(i), we have $[U_j, U_{j+2}] \subset U_{j+1}$ and $[U_j, U_{j+1}] = 1$ for all j. Applying (1.1) to $[a_i \cdot a_i^{-1},\ a_{i+2}]$ and to $[a_i,\ a_{i+2} \cdot a_{i+2}^{-1}]$, we conclude that $[a_i^{-1}, a_{i+2}] = a_{i+1}^{-1}$ and $[a_i, a_{i+2}^{-1}] = a_{i+1}^{-1}$.

Proposition 3.4 *The following hold:*

(i) *Let* $a_1 \in U_1^\sharp$. *Then* $a_2^{\mu_\gamma(a_1)^2} = a_2^{-1}$ *and* $a_3^{\mu_\gamma(a_1)^2} = a_3^{-1}$ *for all* $a_2 \in U_2$ *and all* $a_3 \in U_3$.
(ii) *Let* $a_3 \in U_3^\sharp$. *Then* $a_1^{\mu_\gamma(a_3)^2} = a_1^{-1}$ *and* $a_2^{\mu_\gamma(a_3)^2} = a_2^{-1}$ *for all* $a_1 \in U_1$ *and* $a_2 \in U_2$.

Proof Choose $a_1 \in U_1^\sharp$ and $a_3 \in U_3$, and let $a_2 = [a_1, a_3^{-1}]$. By Proposition 2.10, $a_1^{-1} \in U_1^\sharp$ and $\mu_\gamma(a_1^{-1}) = \mu_\gamma(a_1)^{-1}$, and by Remark 3.3, $[a_1^{-1}, a_3^{-1}] = a_2^{-1}$. By Proposition 2.11(ii), therefore, $a_2 = a_3^{\mu_\gamma(a_1)}$ follows from $a_2 = [a_1, a_3^{-1}]$, and $a_2^{-1} = a_3^{\mu_\gamma(a_1^{-1})} = a_3^{\mu_\gamma(a_1)^{-1}}$ follows from $a_2^{-1} = [a_1^{-1}, a_3^{-1}]$. Hence, $a_i^{\mu_\gamma(a_1)^2} = a_i^{-1}$ for $i = 2$ and 3. Thus, (i) holds; (ii) holds by a similar argument. ∎

Proposition 3.5 *U_i is abelian for all i.*

Proof By Proposition 3.4, the map $x \mapsto x^{-1}$ is an automorphism of U_3. It follows that U_3 is abelian. By Remark 2.8, therefore, U_i is abelian for all i. ∎

Proposition 3.6 *Let $a_i \in U_i$ for some i. Then the maps $u_{i+2} \mapsto [a_i, u_{i+2}]$ from U_{i+2} to U_{i+1} and the map $u_{i-2} \mapsto [u_{i-2}, a_i]$ from U_{i-2} to U_{i-1} are both homomorphisms.*

Proof As was observed in Remark 3.3, we have $[U_j, U_{j+2}] \subset U_{j+1}$ and $[U_j, U_{j+1}] = 1$ for all j. The claim holds, therefore, by (1.1) and Proposition 3.5. ∎

Notation 3.7 We now choose an additive group R isomorphic to U_1 and an isomorphism $t \mapsto x_1(t)$ from R to U_1. Thus,

$$x_1(t+u) = x_1(t)x_1(u) \quad \text{and} \quad x_1(t)^{-1} = x_1(-t)$$

for all $t, u \in R$. We then choose $e_1 \in U_1^\sharp$ and $e_3 \in U_3^\sharp$, and let

$$ {}_2(t) = x_1(t)^{\mu_\gamma(e_3)} \quad \text{and} \quad x_3(t) = x_2(t)^{\mu_\gamma(e_1)} $$

for all $t \in R$.

By (2.1) and Notation 3.7, $t \mapsto x_i(t)$ is an isomorphism from R to U_i for $i = 2$ and 3. Next, we define a multiplication on R.

Notation 3.8 Let $(u, v) \in R \times R$. Then uv (or $u \cdot v$) is the unique element of R such that

$$ \left[x_1(u), x_3(v)\right] = x_2(uv). $$

Proposition 3.9 *The left and right distributive laws hold in R.*

Proof Let $u, u', v \in R$. By Proposition 3.6, we have

$$ \begin{aligned} x_2((u+u')v) &= \left[x_1(u+u'), x_3(v)\right] = \left[x_1(u), x_3(v)\right] \cdot \left[x_1(u'), x_3(v)\right] \\ &= x_2(uv)x_2(u'v), \end{aligned} $$

and thus $(u + u')v = uv + u'v$. By a similar argument, $v(u + u') = vu + vu'$. ∎

Proposition 3.10 *The following hold: $x_1(t)^{\mu_\gamma(e_3)} = x_2(t)$, $x_2(t)^{\mu_\gamma(e_1)} = x_3(t)$, $x_2(t)^{\mu_\gamma(e_3)} = x_1(-t)$ and $x_3(t)^{\mu_\gamma(e_1)} = x_2(-t)$ for all $t \in R$.*

Proof The first two assertions hold by definition. The second two follow by Proposition 3.4. ∎

Notation 3.11 Now let $1 \in R$ denote the pre-image of e_1 under the isomorphism $t \mapsto x_1(t)$, and let $e_2 = [e_1, e_3]$. Then $[e_1, e_3^{-1}] = e_2^{-1}$ by Proposition 3.6. By Proposition 2.11, therefore, $e_1^{-1} = e_2^{\mu_\gamma(e_3)}$ and $e_2^{-1} = e_3^{\mu_\gamma(e_1)}$. By Proposition 3.10, it follows that $e_2 = x_2(1)$ and then $e_3 = x_3(1)$.

Proposition 3.12 *The element* 1 *is a* 2*-sided identity,* i.e. $1 \cdot t = t \cdot 1 = t$ *for all* $t \in R$.

Proof Choose $t \in R$. Since $e_1 = x_1(1)$, we have

$$ x_2(1 \cdot t) = [e_1, x_3(t)] = [e_1, x_3(-t)^{-1}] = x_3(-t)^{\mu_\gamma(e_1)} = x_2(t), $$

and hence $1 \cdot t = t$ by Proposition 2.11, Notation 3.8, and Proposition 3.10. By Proposition 2.10, we have $e_3^{-1} \in U_3^\sharp$ and $\mu_\gamma(e_3^{-1})^{-1} = \mu_\gamma(e_3)$. Since $e_3 = x_3(1)$, it follows that

$$ x_2(t \cdot 1) = [x_1(t), e_3] = [x_1(t), (e_3^{-1})^{-1}] = x_1(t)^{\mu_\gamma(e_3)} = x_2(t), $$

and hence $t \cdot 1 = t$, again by Proposition 2.11, Notation 3.8, and Proposition 3.10. ∎

Proposition 3.13 *R is a unital ring, and Theorem* 3.2(i) *holds.*

Proof By Propositions 3.9 and 3.12, R is a unital ring, and Theorem 3.2(i) is simply a restatement of Notation 3.8. ∎

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

Now let $x_0(t) = x_3(t)^{\mu_\gamma(e_3)}$ and $x_4(t) = x_1(t)^{\mu_\gamma(e_1)}$ for all t. By (2.1), $t \mapsto x_i(t)$ is an isomorphism from R to U_i for $i = 0$ and 4.

Proposition 3.14 *The following hold:*

$$[x_0(u), x_2(v)] = x_1(vu) \quad \textit{and} \quad [x_2(u), x_4(v)] = x_3(vu)$$

for all $u, v \in R$.

Proof Choose $u, v \in R$. Then $[x_1(u), x_3(v)] = x_2(uv)$. Conjugating by $\mu_\gamma(e_3)$, we obtain $[x_2(u), x_0(v)] = x_1(-uv)$ by Proposition 3.10, so

$$[x_0(v), x_2(u)] = x_1(uv).$$

Conjugating instead by $\mu_\gamma(e_1)$, we obtain $[x_4(u), x_2(-v)] = x_3(uv)$. By Proposition 3.6,

$$[x_4(u), x_2(-v)] = [x_4(u), x_2(v)]^{-1}.$$

Thus, $[x_2(v), x_4(u)] = x_3(uv)$. ∎

Proposition 3.15 *Theorem* 3.2(ii) *holds.*

Proof Suppose first that $u \in R^\times$. By Notation 3.8 and Proposition 3.14,

$$x_2(t)^{x_0(u^{-1})x_3(u)} = \left(x_1(-tu^{-1})x_2(t)\right)^{x_3(u)} = x_1(-tu^{-1})$$

for all $t \in R$. By Proposition 2.13(ii), therefore, $x_3(u) \in U_3^\sharp$.

Suppose, conversely, that $x_3(u) \in U_3^\sharp$. Thus,

$$x_1(u) = x_3(u)^{\mu_\gamma(e_1)\mu_\gamma(e_3)} \in U_1^\sharp$$

by Proposition 3.10. We have $\kappa_\gamma(x_3(u)) = x_0(v)$ and $\lambda_\gamma(x_1(u)) = x_4(w)$ for some $w, v \in R$, where κ_γ and λ_γ are as in Proposition 2.7. Let $t \in R$ be arbitrary. By Proposition 3.6 and Notation 3.8, we have

$$[x_1(t), x_3(u)^{-1}] = x_2(tu)^{-1}.$$

By Proposition 2.11(ii), therefore,

$$[x_0(v), x_2(tu)] = x_1(t).$$

By Proposition 3.14, on the other hand, $[x_0(v), x_2(tu)] = x_1(tu \cdot v)$. Thus, $t = tu \cdot v$ for all $t \in R$.

Again, let $t \in R$ be arbitrary. By Proposition 3.6 and Notation 3.8, we have

$$[x_1(u), x_3(t)^{-1}] = x_2(ut)^{-1}.$$

By Proposition 2.11(i), therefore,

$$[x_2(ut)^{-1}, x_4(w)^{-1}] = x_3(t).$$

By Propositions 3.6 and 3.14, on the other hand

$$\left[x_2(ut)^{-1}, x_4(w)^{-1}\right] = \left[x_2(ut), x_4(w)\right] = x_3(w \cdot ut).$$

Thus, $t = w \cdot ut$ for all $t \in R$.

Setting $t = 1$ in $t = tu \cdot v$, we have $uv = 1$, since $1 \cdot u = u$ by Proposition 3.12. Setting $t = v$ in $t = w \cdot ut$, we then obtain $v = w \cdot uv = w$, since $w \cdot 1 = w$ by Proposition 3.12.

Thus, Theorem 3.2(ii) holds for $i = 3$. Thus, by (2.1) and Proposition 3.10, it also holds for $i = 1$ and 2. ∎

Lemma 3.16 *Let x be an element of Γ_{w_i} that is opposite w_{i+1} at w_i for some i and let $g \in U_i$. Then $g \in U_i^\sharp$ if and only if x^g is opposite x at w_i.*

Proof Let g_1 be the unique element of U_i mapping x to w_{i-1}. By Proposition 3.5, U_i is abelian. Hence, $x^{g_1 g g_1^{-1}} = x^g$. Therefore, x^g is opposite x at w_i if and only if $w_{i-1}^g = x^{g_1 g}$ is opposite $w_{i-1} = x^{g_1}$ at w_i. ∎

Proposition 3.17 *Theorem* 3.2(iii) *holds.*

Proof Let $u, v \in R^\times$. By Proposition 3.13, $[x_1(u), x_3(v)] = x_2(uv)$. We claim that $x_2(uv) \in U_2^\sharp$. By Proposition 3.15, we have $x_1(u) \in U_1^\sharp$ and $x_3(v) \in U_3^\sharp$ and hence $x_3(v)^{-1} \in U_3^\sharp$ by Notation 2.6. Let $d = w_1^{x_3(v)^{-1}}$ and $f = w_2^{x_3(v)^{-1}}$. Since $x_1(u)$ fixes w_1, we have

$$w_1^{x_2(uv)} = w_1^{[x_1(u), x_3(v)]} = d^{x_1(u) x_3(v)}. \tag{3.1}$$

Since $x_3(v)^{-1} \in U_3^\sharp$, the vertex f is opposite w_2 at w_3. Hence, there exists $g \in U_0$ mapping f to w_4. By Lemma 3.16 with $i = 3$, $d^{g x_1(u)}$ is opposite d^g at $w_4 = f^g$. Since $[g, x_1(u)] \in [U_0, U_1] = 1$, it follows that $d^{x_1(u)}$ is opposite d at f. Hence, $d^{x_1(u) x_3(v)}$ is opposite $d^{x_3(v)} = w_1$ at $f^{x_3(v)} = w_2$. By (3.1), therefore, $x_2(uv) \in U_2^\sharp$ as claimed. Hence, $uv \in R^\times$ by Proposition 3.15. It follows from $u = uv \cdot v^{-1}$ that $(uv)^{-1} \cdot u = v^{-1}$, and thus $(uv)^{-1} = v^{-1} u^{-1}$. ∎

Proposition 3.18 *Theorem* 3.2(iv) *holds.*

Proof Let $t \in R^\times$ and $u \in R$. By Notation 3.8 and Proposition 3.14, we have

$$x_2(u)^{x_0(t) x_3(t^{-1})} = \left(x_1(-ut) x_2(u)\right)^{x_3(t^{-1})} = x_1(-ut) x_2(u - ut \cdot t^{-1}) = x_1(-ut).$$

By Proposition 2.13(ii), it follows that $x_3(t^{-1}) \in U_3^\sharp$. By Proposition 3.15, therefore, $t^{-1} \in R^\times$. It follows from $1 = t^{-1}(t \cdot 1) = t^{-1} t$ that $t = (t^{-1})^{-1} \cdot t^{-1} t = (t^{-1})^{-1} \cdot 1 = (t^{-1})^{-1}$. ∎

Notation 3.19 Let

$$\alpha_u = \mu_\gamma\left(x_3(1)\right)^{-1} \mu_\gamma\left(x_3(u)\right) \quad \text{and} \quad \beta_u = \mu_\gamma\left(x_1(1)\right)^{-1} \mu_\gamma\left(x_1(u)\right)$$

for each $u \in R^\times$.

Proposition 3.20 *The following hold for all $u \in R^\times$ and all $t \in R$:*

$$x_1(t)^{\alpha_u} = x_1(tu^{-1}), \qquad x_2(t)^{\alpha_u} = x_2(tu), \qquad x_3(t)^{\alpha_u} = x_3(u \cdot tu)$$

and

$$x_1(t)^{\beta_u} = x_1(ut \cdot u), \qquad x_2(t)^{\beta_u} = x_2(ut), \qquad x_3(t)^{\beta_u} = x_3(u^{-1} t).$$

Proof Choose $t \in R$ and $u \in R^\times$. By Proposition 3.6 and Notation 3.8,

$$[x_1(t), x_3(u)^{-1}] = x_2(tu)^{-1}.$$

By Proposition 2.11, therefore,

$$x_2(tu)^{\mu_\gamma(x_3(u))} = x_1(t)^{-1}.$$

Thus, $x_2(t)^{\mu_\gamma(x_3(u))} = x_1(-tu^{-1})$ by Theorem 3.2(iv) and $x_1(t)^{\mu_\gamma(x_3(u))} = x_2(tu)$ by Proposition 3.4(ii). It follows that $x_1(t)^{\alpha_u} = x_1(tu^{-1})$ and $x_2(t)^{\alpha_u} = x_2(tu)$.

By Proposition 3.4(ii) and Notation 3.8, $[x_1(u), x_3(t)^{-1}] = x_2(ut)^{-1}$. By Proposition 2.11, therefore,

$$x_3(t)^{\mu_\gamma(x_1(u))} = x_2(-ut).$$

By Proposition 3.4(i), it follows that $x_2(ut)^{\mu_\gamma(x_1(u))} = x_3(t)$, and thus $x_2(t)^{\mu_\gamma(x_1(u))} = x_3(u^{-1}t)$ by Theorem 3.2(iv). Hence, $x_2(t)^{\beta_u} = x_2(ut)$ and $x_3(t)^{\beta_u} = x_3(u^{-1}t)$.

There exist $w, v \in R^\times$ such that $x_3(t)^{\alpha_u} = x_3(v)$ and $x_1(t)^{\beta_u} = x_1(w)$. Conjugating $[x_1(1), x_3(t)] = x_2(t)$ by α_u, we obtain

$$[x_1(u^{-1}), x_3(v)] = x_2(tu),$$

and therefore $u^{-1}v = tu$. It follows that $v = u \cdot u^{-1}v = u \cdot tu$. Conjugating $[x_1(t), x_3(1)] = x_2(t)$ by β_u, we obtain $[x_1(w), x_3(u^{-1})] = x_2(ut)$ and therefore $wu^{-1} = ut$. Thus, $w = wu^{-1} \cdot u = ut \cdot u$. ∎

Proposition 3.21 *Theorem* 3.2(v) *holds.*

Proof Choose $u \in R^\times$ and $t, v \in R$. Conjugating $[x_1(tu), x_3(v)] = x_2(tu \cdot v)$ by α_u yields $[x_1(t), x_3(u \cdot vu)] = x_2((tu \cdot v)u)$ by Proposition 3.20. Thus, $t(u \cdot vu) = (tu \cdot v)u$. Similarly, conjugating $[x_1(t), x_3(uv)] = x_2(t \cdot uv)$ by β_u yields $(ut \cdot u)v = u(t \cdot uv)$. Setting $v = 1$, we conclude that

$$uz \cdot u = u \cdot zu \tag{3.2}$$

for all $z \in R$.

There exists $w \in R$ such that $x_4(v)^{\alpha_u} = x_4(w)$. By Proposition 3.14, $[x_2(1), x_4(v)] = x_3(v)$. Conjugating by α_u, we obtain $[x_2(u), x_4(w)] = x_3(u \cdot vu)$. By Proposition 3.14 and (3.2), therefore, $wu = u \cdot vu = uv \cdot u$. Hence, $w = (wu)u^{-1} = (uv \cdot u)u^{-1} = uv$. Conjugating $[x_2(t), x_4(v)] = x_3(vt)$ by α_u now yields $[x_2(tu), x_4(uv)] = x_3(u(vt \cdot u))$, and therefore $uv \cdot tu = u(vt \cdot u)$. Finally, we observe that $u(vt \cdot u) = (u \cdot vt)u$ by (3.2). ∎

This completes the proof of Theorem 3.2. ∎

4 Stable Rank 2

The main result of this section is Proposition 4.4.

Definition 4.1 Let A be a unital ring and let $[x, y, z] = xy \cdot z - x \cdot yz$ for all $x, y, z \in R$. The map $(x, y, z) \mapsto [x, y, z]$ is called the *associator* of A. It is tri-additive. The ring A is called *alternative* if the associator is alternating, *i.e.*, if $[x, y, z] = 0$ whenever two of the terms x, y, z are equal.

Remark 4.2 The notion of the *stable rank* (or stable range) of a ring was introduced in [1]. For an associative ring A, *stable rank* 2 is equivalent to the condition that for all $s, b, c \in A$ such that $sb + c$ is left-invertible, there exists $t \in A$ such that $b + tc$ is left-invertible. At the bottom of [5, p. 179], Faulkner adapted this as the definition of stable rank 2 for alternative rings and in [5, Thm. 2] he showed (for alternative rings) that "left-invertible" can be replaced by "invertible" in the definition. The relevance of this notion to incidence geometry was first examined by Veldkamp in [15].

Definition 4.3 Let A be a unital ring. We call A *Veldkamp* if for all $s, b, c \in A$ such that $sb + c \in A^\times$, there exists t such that $b + tc \in A^\times$. Thus, if A is alternative, then A is Veldkamp if and only if A has stable rank 2.

Proposition 4.4 *The ring R in Theorem 3.2 is Veldkamp as defined in Definition 4.3.*

Proof Let $(\gamma, i \mapsto w_i)$ and $i \mapsto U_i$ be as in Theorem 3.2. Suppose that $sb + c \in R^\times$ for some $s, b, c \in R$ and let $f = sb + c$. By Theorem 3.2(i) and Proposition 3.14, we have

$$x_3(f)^{x_1(p)x_4(s)} = \big(x_2(-pf)x_3(f)\big)^{x_4(s)} = x_2(-pf)x_3(f - s \cdot pf) = x_2(b)x_3(c)$$

for $p = bf^{-1}$. Let $u = w_5^{x_1(p)x_4(s)}$. By Proposition 2.12, there exist $a, t \in R$ such that $w_3^{x_4(a)x_1(t)} = u$. Thus, the product $g := x_1(p)x_4(s)x_1(t)^{-1}x_4(a)^{-1}$ maps w_5 to w_3. Since U_1 and U_4 both fix w_1 and w_4, so does g. By Definition 2.2(ii), therefore,

$$(w_4, w_5, w_6, w_1)^g = (w_4, w_3, w_2, w_1). \tag{4.1}$$

By Definition 2.2(iii), $(w_4, w_5)^g = (w_4, w_3)$ implies that $U_3^g = U_2$. Thus, $x_3(f)^g = x_2(e)$ for some $e \in R$. Using Theorem 3.2(i) and Proposition 3.14 again, we calculate that

$$\begin{aligned} x_2(b)x_3(c) &= x_3(f)^{x_1(p)x_4(s)} = x_3(f)^{gx_4(a)x_1(t)} \\ &= x_2(e)^{x_4(a)x_1(t)} = (x_2(e)x_3(ae))^{x_1(t)} = x_2(e - t \cdot ae)x_3(ae), \end{aligned}$$

so $ae = c$ and $e - t \cdot ae = b$ by Proposition 2.4(ii). Hence $b + tc = e$. Since $f \in R^\times$, we have $x_3(f) \in U_3^\sharp$ by Theorem 3.2(ii). Thus, $w_1^{x_3(f)}$ is opposite w_1 at w_6, so $w_1^{x_3(f)g} = w_1^{x_2(e)}$ is opposite $w_1^g = w_1$ at w_6^g, and by (4.1), $w_6^g = w_2$. Thus, $x_2(e) \in U_2^\sharp$. By another application of Theorem 3.2(ii), therefore, $b + tc = e \in R^\times$. ∎

5 The Moufang Identities

We assume that X, $(\gamma, i \mapsto w_i)$, $i \mapsto U_i$, x_i for all $i \in [1, 3]$, and R are as in Theorem 3.2. The main result of this section is Theorem 5.8.

Notation 5.1 Let $u, v \in R^\times$. We denote by S the subset of R consisting of all products of terms equal to u, v, u^{-1}, or v^{-1}. Then $1 \in S$, and by Theorem 3.2(iii) and (iv), $S \subset R^\times$ and S is closed under inverses.

Proposition 5.2 *Let u, v, and S be as in Notation 5.1. Then S is a group.*

Proof By Theorem 3.2(iii)–(v), $R^\times$ is a Moufang loop. The set S is the intersection of all the subloops containing u and v. Thus, S is the subloop generated by $\{1, u, v\}$. The claim holds, therefore, by Moufang's Theorem; see, for example, [3]. ∎

Corollary 5.3 *Let $u, v \in R^\times$ and let $\langle u, v\rangle$ denote the subring generated by $\{u, v, u^{-1}, v^{-1}\}$. Then $\langle u, v\rangle$ is associative.*

Proof The additive group of the subring $\langle u, v\rangle$ is generated by the set S defined in Notation 5.1. The claim holds, therefore, by Proposition 5.2. ∎

Proposition 5.4 *Let A be a unital ring and let*

$$I_u = \{v \in A^\times \mid u - v \in A^\times\}$$

for all $u \in A$. Suppose that the following hold:

(i) *For all $w, v \in A^\times$, the subring generated by $\{w, v\}$ is associative.*

(ii) *For all $u \in A$, A is generated as an additive group by I_u.*

Then A is alternative.

Proof Let $u \in A$. By (i), we have $u, v \in \langle u - v, v\rangle$, and hence

$$[u, u, v] = [u, v, u] = [v, u, u] = 0 \tag{5.1}$$

for all $v \in I_u$. Since the associator is tri-additive, it follows from (ii) that (5.1) holds for all $v \in A$. Thus, R is alternative. ∎

Proposition 5.5 *Let $g \in U_3$, let $z = w_2^g$, and let*

$$U_g = \{g_1 \in U_3^\sharp \mid w_2^{g_1} \text{ is opposite } z\}.$$

Then for each $g_1 \in U_g$, there exists $g_2 \in U_3^\sharp$ such that $g = g_2 g_1$.

Proof Let

$$g_1 \in U_g, \quad y = w_2^{g_1}, \quad \text{and} \quad z = w_2^g.$$

Then $z^{g_1^{-1}}$ is opposite both $y^{g_1^{-1}} = w_2$ and $w_4^{g_1^{-1}} = w_4$ at w_3, so by Definition 2.2(iii) and Notation 2.6, there exists $g_2 \in U_3^\sharp$ mapping w_2 to $z^{g_1^{-1}}$. Therefore, $w_2^{g_2 g_1} = z$. Thus, $g = g_2 g_1$ by Proposition 2.5. ∎

Proposition 5.6 *Let U_g be as in Proposition 5.5 for some $g \in U_3$. Suppose that X is 5-plump. Then $U_3 = U_g \cdot U_g^{-1}$.*

Proof Let $g_0 \in U_3$, let $y = w_2^{g_0}$, and let $z = w_2^g$. Since X is 5-plump, we can choose $x \in \Gamma_{w_3}$ such that x is opposite w_4, w_2, y, z and z^{g_0} at w_3. By Definition 2.1(iii), there exist elements g_1 and g_2 in U_3 such that $w_2^{g_1} = x$ and $y^{g_2} = x$. Then $g_1 \in U_g$ and by Lemma 3.16, $g_2 \in U_3^\sharp$. Since U_3 is abelian (by Proposition 3.5), we have

$$w_2^{g_2} = w_2^{g_0 g_2 g_0^{-1}} = y^{g_2 g_0^{-1}} = x^{g_0^{-1}}.$$

Since x is opposite z^{g_0} at w_3, it follows that $w_2^{g_2}$ is opposite z at w_3. Hence $g_2 \in U_g$. Since $w_2^{g_1 g_2^{-1}} = y = w_2^{g_0}$, we have $g_0 = g_1 g_2^{-1} \in U_g \cdot U_g^{-1}$ by Proposition 2.5. ∎

Proposition 5.7 *If X is* 5*-plump, then R satisfies the conditions Proposition* 5.4(i) *and* (ii).

Proof By Corollary 5.3, Proposition 5.4(i) holds. Let $g = x_3(u)$ for some $u \in R$. By Theorem 3.2(ii) and Proposition 5.5, we have $U_g \subset x_3(I_u)$. Thus, by Proposition 5.6, Proposition 5.4(ii) holds. ■

Theorem 5.8 *If X is* 5*-plump, then R is alternative and has stable rank* 2*.*

Proof By Propositions 5.4 and 5.7, R is alternative. By Definition 4.3 and Proposition 4.4, it follows that R has stable rank 2. ■

Proposition 5.9 *Suppose that X is* 4*-plump. Then X is dagger-sharp as defined in Notation* 2.15 *if and only if R is simple.*

Proof Suppose that X is dagger-sharp and let I be an ideal of R. By Propositions 2.17 and 3.20, $x_2(I)$ is $G_\gamma^\dagger$-invariant and hence contains an element of $U_2^\sharp$. By Theorem 3.2(ii), therefore, $I = R$. Thus, R is simple. Suppose, conversely, that R is simple and let I be an additive subgroup of R such that $x_2(I)$ is $G_\gamma^\dagger$-invariant. By [11, 1.4.23], U_2 is generated by $U_2^\sharp$, so by Theorem 3.2(ii), R is generated as an additive group by $R^\times$. By Proposition 3.20, it follows that I is an ideal of R. Hence, $x_2(I) = U_2$. By Remark 2.16, therefore, X is dagger-sharp. ■

Theorem 5.10 *If X is* 5*-plump and dagger-sharp, then either R is an octonion algebra or R is a simple associative ring.*

Proof The octonions are the only simple non-associative alternative rings, as shown in [8]. The claim holds, therefore, by Theorem 5.8 and Proposition 5.9. ■

Proposition 5.11 *Suppose that X is* 4*-plump. Then X is uniquely determined by the ring R.*

Proof By Proposition 2.4 and Theorem 3.2(i), the sequence Ω is uniquely determined by R. By Proposition 3.10, the action of $m_1 := \mu_\gamma(e_1)$ on $U_{[2,3]}$ and the action of $m_3 := \mu_\gamma(e_3)$ on $U_{[1,2]}$ are uniquely determined by R. By Propositions 2.18 and 3.20, therefore, X is uniquely determined by R. ■

6 Faulkner Graphs

The main result of this section is Theorem 6.10.

Definition 6.1 Let $\Gamma = (V, E)$ be a bipartite graph endowed with a symmetric binary relation $\equiv$ on V. We say that vertices u and v *have the same type* if $\mathrm{dist}(u, v)$ is even; we say that vertices u and v are *far* from each other if $u \equiv v$ and u and v have the same type, and we say that vertices u and v are *distant* if $u \equiv v$ and u and v do not have the same type. Then $(\Gamma, \equiv)$ is a *Veldkamp graph* if the following axioms hold:

(i) If $u \equiv v$, then $\mathrm{dist}(u, v) \geq 2$.

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

(ii) Vertices u and v are far from each other if and only if there exists $w \in \Gamma_v$ such that u and w are distant.
(iii) If u and v are far from each other, then there exists a unique vertex in $\Gamma_u \cap \Gamma_v$ denoted by $u \wedge v$.
(iv) If u and v are far from each other and $w \in \Gamma_v$ is far from $u \wedge v$, then w and u are distant.
(v) For all vertices u and v that have the same type, there exists a vertex w that is distant from both u and v.

Remark 6.2 Let $(\Gamma, \equiv)$ be a Veldkamp graph. By Definition 6.1(i)–(iii), $\text{dist}(u, v) = 3$ if u and v are distant and $\text{dist}(u, v) = 2$ if u and v are far from each other. By Definition 6.1(v), it follows that Γ is connected.

Remark 6.3 In [15, Def. 3.1] Veldkamp introduced the notion of a *projective Barbilian plane*. A projective Barbilian plane is an incidence geometry with point set P and line set L endowed with a relation $\approx$ on $P \cup L$ satisfying six axioms. By [15, Prop. 3.2], the dual statements to all six axioms hold. With this in mind, it is easy to check that a bipartite graph Γ endowed with a relation $\equiv$ on its vertices is a Veldkamp graph if and only if it is the incidence graph of a projective Barbilian plane in which $u \approx v$ exactly when $u \not\equiv v$.

Proposition 6.4 *Let $(\Gamma, \equiv)$ be a Veldkamp graph. Then for all vertices u, v, w such that w has the same type as neither u nor v, there exists a vertex $z \in \Gamma_w$ that is far from both u and v.*

Proof This holds by Remark 6.3 and [15, Prop. 3.2(iii)]. ∎

Corollary 6.5 *Let $(\Gamma, \equiv)$ be a Veldkamp graph. If $u, v \in \Gamma_w$, then there exists $z \in \Gamma_w$ that is far from both u and v*

Proof If $u, v \in \Gamma_w$, then w has the same type as neither u nor v, and hence the claim holds by Proposition 6.4 ∎

Corollary 6.6 *If $(\Gamma, \equiv)$ is a Veldkamp graph, then $|\Gamma_z| \geq 3$ for all vertices z.*

Proof Let z be a vertex of Γ. Since Γ is connected, $\Gamma_z \neq \varnothing$. By two applications of Corollary 6.5, the first with $u = v$ and the second with $u \neq v$, it follows that $|\Gamma_z| \geq 3$. ∎

Corollary 6.7 *Let $(\Gamma, \equiv)$ be a Veldkamp graph. Then there exist 6-circuits γ such that for each vertex u of γ, the two vertices in Γ_u contained in γ are far from each other.*

Proof By Definition 6.1(ii) and (v), there exist vertices u_0, u_2 far from each other. Let $u_1 = u_0 \wedge u_2$. By Corollary 6.5, there exists $u_3 \in \Gamma_{u_2}$ far from u_1. The vertex u_3 has the same type as neither u_0 nor u_2. By Proposition 6.4, therefore, there exists a vertex $u_4 \in \Gamma_{u_3}$ that is far from both u_0 and u_2. Let $u_0 = u_6$ and let $u_5 = u_4 \wedge u_6$. Let γ be the subgraph with vertex set $\{u_i \mid i \in [1, 6]\}$ and edge set $\{\{u_{i-1}, u_i\} \mid i \in [1, 6]\}$. By construction, γ is a 6-circuit, u_0, u_2, and u_4 are pairwise far from each other and u_1

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

is far from u_3. By Definition 6.1(iv), u_i, and u_{i+3} are distant for $i = 0$ and 1. Since u_0 and u_3 are distant and $u_0 \in \Gamma_{u_5}$, Definition 6.1(i) implies that u_3 and u_5 are far from each other. Since u_4 is distant from u_1 and $u_4 \in \Gamma_{u_5}$, Definition 6.1(i) implies that u_5 and u_1 are far from each other. ∎

Remark 6.8 Let γ and $u_1, \ldots, u_6$ be as in Corollary 6.7. By Definition 6.1(iv), u_i is distant from u_{i+3} for $i = 0, 1$, and 2. By Definition 6.1(i), it follows that $u_i \notin \Gamma_{u_{i+3}}$ for $i = 0, 1$ and 2. Thus, the only edges of Γ containing two vertices of γ are already edges of γ. In other words, γ is the subgraph of Γ spanned by the vertices $\{u_i \mid i \in [1, 6]\}$.

Definition 6.9 Let $Y = (\Gamma, \equiv)$ be a Veldkamp graph. An automorphism of Y is an automorphism of Γ that preserves $\equiv$. A *root* is a path (u_0, u_1, u_2, u_3) of length 3 such that u_i is far from u_{i-2} for $i = 2$ and 3. The *root group* U_α associated with a root $\alpha = (u_0, u_1, u_2, u_3)$ is the pointwise stabilizer of the set $\Gamma_{u_1} \cup \Gamma_{u_2}$ in $\mathrm{Aut}(Y)$. We say that Y is a *Moufang* if for each root $\alpha = (u_0, u_1, u_2, u_3)$, U_α acts transitively on the set of vertices that are far from u_2 and adjacent to u_3. Requiring Y to be Moufang is the same as requiring that the corresponding Veldkamp plane be (C, l)-transitive for all incident pairs C, l, as defined at the top of [6, p. 190]. We define a *Faulkner graph* to be a Veldkamp graph that is Moufang.

Theorem 6.10 *The following hold.*

(i) *Let $Y = (\Gamma, \equiv)$ be a Faulkner graph, let V be the vertex set of Γ, let $u \equiv_v w$ for $u, v, w \in V$ whenever u and w are far from each other and $v = u \wedge w$ and let $\mathcal{A}$ be the set of 6-circuits in Γ such that for each vertex v of Γ, the two vertices of γ in Γ_v are far from each other. Then $X := (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$ is a Tits triangle.*

(ii) *Let $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$ be a 4-plump Tits triangle. Let $u \equiv v$ for $u, v \in V$ if and only if there exists a straight path from u to v of length 2 or 3. Then $Y := (\Gamma, \equiv)$ is a Faulkner graph.*

Proof Let X be as in (i). By Corollary 6.7, $\mathcal{A} \neq \varnothing$; by Corollary 6.6, X is a dewolla of level 3; by Remark 6.2, Γ is connected; by Corollary 6.5, X satisfies Definition 2.2(i), and by Definition 6.1(iii), X satisfies Definition 2.2(i). It follows from Definition 6.9, that X satisfies Definition 2.2(iii). Thus, (i) holds.

Suppose now that Y is as in (ii). Every 1-path is straight. By Definition 2.2(ii), therefore, Y satisfies Definition 6.1(i). Let $u, v \in V$. If (u, z, v) is a straight 2-path from u to v, then by Definition 2.2(i), there exists $w \in \Gamma_v$ such that (u, z, v, w) is a straight 3-path. If there exists a straight 3-path from u to a vertex w in Γ_v, then by [11, 1.3.18], there exists a straight 2-path from u to v. Thus, Y satisfies Definition 6.1(ii). Now suppose that w, z are vertices such that (u, w, v) and (u, z, v) are both 2-paths from u to v and that (u, w, v) is straight. By Definition 2.2(i), we can choose $y \in \Gamma_v$ that is opposite both w and z at v. Thus, (u, w, v, y) is a straight 3-path, so by [11, 1.3.18], there exists a vertex z' such that (u, z, z', y) is a straight 3-path. Since (z, v, y) is also straight, it follows by Definition 2.2(ii) that $z' = v$. Hence, (u, z, v) is straight. By another application of Definition 2.2(ii), therefore, $z = w$. Thus, Y satisfies Definition 6.1(iii). If (u, z, v) is a straight 2-path from u to v, and (z, y, w) is a straight 2-path from z to a vertex w in Γ_v, then by Definition 6.1(iii), $v = y$ and hence

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

the 3-path (u, z, v, w) is straight. Thus, Y satisfies Definition 6.1(iv). By [11, 1.3.16 and 1.5.1(iii)] and the assumption that X is 4-plump, Y satisfies Definition 6.1(v). By Definition 2.2(iii), Y is Moufang. ∎

7 Existence

The main results of this section are Theorems 7.2 and 7.3. The first is a corollary of a result of Faulkner on the existence of Moufang–Veldkamp planes.

Notation 7.1 Let R be an alternative ring of stable rank 2 and let $\mathcal{P}(R)$ be the plane constructed in [6, Lemma 8] and the paragraph preceding [6, Thm. 9]. By [6, Thm. 9], $\mathcal{P}(R)$ is a Moufang–Veldkamp plane as defined at the bottom of [6, p. 190]. Thus, in particular, the Veldkamp graph corresponding to $\mathcal{P}(R)$ is a Faulkner graph as defined in Definition 6.9. Let $\mathcal{T}(R)$ be the Tits triangle obtained by applying Theorem 6.10(i) to this Faulkner graph.

Theorem 7.2 *For every alternative ring R of stable rank 2, there exists a Tits triangle X such that R is isomorphic to the ring obtained by applying Theorem 3.2 to X.*

Proof Let $X = \mathcal{T}(R)$ as described in Notation 7.1. By [6, Lemma 4 and eq. (3)], R is isomorphic to the ring obtained by applying Theorem 3.2 to X. ∎

Theorem 7.3 *If X is a 5-plump Tits triangle, then X is isomorphic to $\mathcal{T}(R)$, as described in Notation 7.1, for some alternative ring R of stable rank 2.*

Proof This holds by Theorem 5.8, Proposition 5.11, and Theorem 7.2. ∎

Notation 7.4 Let $Y = (\Gamma, \equiv)$ be a Veldkamp graph. With Corollary 6.5 in mind, we define Y to be k-plump for some $k \geq 3$ if for every vertex v and for every subset N of Γ_v of cardinality at most k, there exists a vertex that is far from u for all $u \in N$. Thus, Y is k-plump if and only if the Tits triangle in Theorem 6.10(i) is k-plump, as defined in Definition 2.14. We say that a Veldkamp plane is k-plump if its corresponding Veldkamp graph is k-plump.

Remark 7.5 Let $Y = (\Gamma, \equiv)$ be a Faulkner graph as defined in Definition 6.9 and let $\mathcal{V}$ denote the corresponding Veldkamp plane; see Remark 6.3. Suppose that $\mathcal{V}$ is 5-plump as defined in Notation 7.4. Let X be the Tits triangle in Theorem 6.10(i) corresponding to Y. By Theorem 7.3, X is isomorphic to $\mathcal{T}(R)$ for some alternative ring R of stable rank 2, so by Notation 7.1, $\mathcal{V}$ is isomorphic to $\mathcal{P}(R)$. By [6, Thm. 9], therefore, the Veldkamp plane $\mathcal{V}$ satisfies the configurational condition (QS) on [6, p. 191]. In other words, a Veldkamp plane that is (C, l)-transitive for all incident pairs C, l and 5-plump automatically satisfies Faulkner's condition (QS).

Suppose that R is alternative and simple and that R satisfies the descending chain condition on right ideals if R is associative. Then R is either a matrix ring over a skew field or an octonion algebra. As was observed in the last paragraph of [5, §2], these rings are all of stable rank 2, and hence $\mathcal{T}(R)$ exists.

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

8 Tits Indices

We close with a few comments about Tits polygons of index type (defined in Example 8.3). The notion of a Tits polygon arose from a study of these structures. We assume some familiarity with the theory of buildings in this section.

Notation 8.1 Let Π be a Coxeter diagram with vertex set S and let (W, S) denote the corresponding Coxeter system. For each subset $J \subset S$, we denote by W_J the subgroup of W generated by J. A subset J is *spherical* if W_J is finite. If J is a spherical subset of S, we denote by w_J the longest element in the Coxeter system (W_J, J). For each spherical subset J of S, $s \mapsto w_J s w_J$ is an automorphism of the subdiagram Π_J spanned by J; see, for example, [9, 19.6].

Definition 8.2 A *Tits index* is a triple

$$T = (\Pi, \Theta, A),$$

where Π is a Coxeter diagram with vertex set S, Θ is a subgroup of $\mathrm{Aut}(\Pi)$, and A is a Θ-invariant subset of S such that for each $s \in S\backslash A$, the subset $J_s := A \cup s^\Theta$ is spherical and A is stabilized by the automorphism $s \mapsto w_{J_s} s w_{J_s}$ of Π_{J_s}, where s^Θ is the Θ-orbit containing s. The Coxeter diagram Π is called the *absolute type* of T, and $|S|$ is called the *absolute rank* of T. For each $s \in S\backslash A$, we denote by $\tilde{s}$ the product $w_A w_{J_s}$. There is one element $\tilde{s}$ for each Θ-orbit disjoint from A. Let $\tilde{S}$ denote the set of all these elements and let $\tilde{W} = \langle \tilde{S} \rangle$. By [9, 20.4, 20.32 and 20.34], $(\tilde{W}, \tilde{S})$ is also a Coxeter system. It is called the *relative Coxeter system* of T and $|\widetilde{S}|$ is called the *relative rank* of T. The *relative type* $\widetilde{\Pi}$ of T is the Coxeter diagram corresponding to its relative Coxeter system.

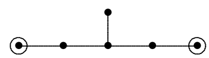
Examples 8.3 Let $T = (\Pi, \Theta, A)$ be a Tits index as in Definition 8.2 whose absolute type Π is an irreducible spherical Coxeter diagram and whose relative rank is 2. Thus, $\widetilde{W}$ is a dihedral group, where $\tilde{W}$ is as in Definition 8.2. Let $n = |\tilde{W}|/2$ and let $J_i = A \cup s_i^\Theta$ for $i = 1$ and 2, where s_1 and s_2 are representatives of the two Θ-orbits disjoint from A. By [9, 20.39 and 20.40], $n \geq 3$. Let Δ be a building of type Π. We assume that Δ satisfies the Moufang condition; this is automatic if the rank of T is greater than 2. Let V_i be the set of J_i-residues of Δ for $i = 1$ and 2 and let Γ be the bipartite graph whose vertex set is $V := V_1 \cup V_2$, where a residue in V_1 is adjacent to a residue in V_2 whenever their intersection is an A-residue of Δ. By [11, 1.2.12 and 1.2.28(i)], there is a natural way to endow Γ with sets $\mathcal{A}$ and $\{\equiv_v\}_{v \in V}$ so that $X_{\Delta,T} := (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$ is a Tits n-gon. We call the Tits polygons that arise in this way for some pair (Δ, T) the Tits polygons *of index type*.

Remark 8.4 Suppose that $X = X_{\Delta,T}$ for some pair (Δ, T) as in Example 8.3. If every panel of X contains at least $k + 1$ chambers for some k, then by [11, 1.2.7], X is k-plump as defined in Definition 2.14. It follows that X is k-plump whenever the field of definition of Δ (or one of its fields of definition) as defined in [9, 28.8] contains at least k elements. In particular, if Δ is the spherical building associated with the group

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

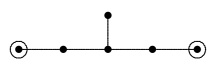
of F-rational points of an absolutely simple algebraic group and $|F| \geq k$, then X is k-plump.

Remark 8.5 By [9, 20.31], the relative type of a Tits index is spherical if and only if the absolute type is spherical. With this in mind, it can be checked that the only Tits indices whose absolute type is irreducible and whose relative type is A_2 are



(drawn following the conventions described in [9, 34.2]) and $T_m := (\Pi, \Theta, A)$, where $\Pi = A_{3m-1}$ for some $m \geq 1$, Θ is the trivial subgroup of $\mathrm{Aut}(\Pi)$ and $A = S\backslash\{v_m, v_{2m}\}$, where S is the vertex set of Π and $v_1, \ldots, v_{3m-1}$ are the elements of S going from left to right.

Remark 8.6 Let Δ be the building $E_6(K)$ for some field K, let T be the Tits index



and let X be the Tits triangle $X_{\Delta,T}$ as defined in Remark 8.4. Suppose that $|K| \geq 5$, so that X is 5-plump by Remark 8.4 and let R be as in Theorem 3.2. It can be shown, in this case, that R is an 8-dimensional non-division algebra over K and that the Veldkamp plane corresponding to X does not satisfy axiom 8 in [15, 4.16], so by [15, 5.12], R is not associative. Using the connection to a polar space of type D_5, it can be shown that X is dagger-sharp. Hence, R is simple by Proposition 5.9. By Theorem 5.10, it follows that R is the split octonion algebra over K. If $|K| \leq 4$, we can reach the same conclusion by embedding Δ as the set of fixed points of a Galois group acting on the building $E_6(L)$, where L is a suitable separable extension of K.

Remark 8.7 Let $\Delta = A_{3m-1}(E)$ for some $m \geq 1$ and some skew field E, let T_m be the Tits index described in Remark 8.5, let $X =_{\Delta,T_n}$, let R be as in Theorem 3.2, and let R_0 be the ring of $m \times m$ matrices over E. We identify Δ with the projective space associated with R^3 regarded a left-vector space over E. With the description of the root groups of Δ in [10, §7] in mind, it is an elementary calculation to show that R is isomorphic to R_0.

Remark 8.8 By [14, 40.22(vi)] (in light of [14, 40.15 and 40.17]) every building of type E_6 is isomorphic to $E_6(K)$ for some field K and by [14, 17.2, 17.3 and 40.22(i)], every building of type A_{3m-1} is isomorphic to $A_{3m-1}(E)$, where either $m > 1$ and E is a skew field or $m = 1$ and E is either a skew field or an octonion division ring. If $\Delta = A_2(E)$ for E either a skew field or an octonion division ring, then X_{Δ,T_1} is the Moufang polygon described in [14, 16.1], the ring R in Theorem 3.2 is isomorphic to E and $\mathcal{T}(R)$ is the projective plane associated with E.

Remark 8.9 By Remarks 8.5–8.8, the Tits triangles of index type as defined in Example 8.3 are precisely the Tits triangles of the form $\mathcal{T}(R)$ where R is either an octonion algebra or a matrix ring over a skew field.

https://doi.org/10.4153/S0008439518000140 Published online by Cambridge University Press

Remark 8.10 We mention that there is another interesting family of Tits triangles that can be described in terms of spheres of finite or infinite radius in a suitable affine building. The rings that arise in this context are quotients of the ring of integers of an alternative division ring with valuation.

Remark 8.11 Freudenthal observed in 1951 (in a paper re-published as [7]) that the automorphism group of the projective plane over the octonions is a real form of type E_6 of relative rank 2. The idea that the split group of type E_6 acts on a "projective plane over the split octonions" is suggested by Tits in the footnote on [13, p. 25]. In their paper [12], Springer and Veldkamp give a concrete construction of the projective planes over the split Cayley-Dickson algebras (in characteristic different from 2 and 3). They remark without proof that these correspond precisely to the geometries considered by Tits in [13]. Later, Faulkner provided a characteristic free approach to the projective planes over the split Cayley–Dickson algebras in [4].

References

[1] H. Bass, *K-theory and stable algebra*. Inst. Hautes Études Sci. Publ. Math. 22(1964), 5–60.

[2] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*. Springer-Verlag, Berlin, 2002. https://doi.org/10.1007/978-3-540-89394-3.

[3] A. Drápal, *A simplified proof of Moufang's theorem*. Proc. Amer. Math. Soc. 139(2011), 93–98. https://doi.org/10.1090/S0002-9939-2010-10501-4.

[4] J. R. Faulkner, *Octonion planes defined by quadratic Jordan algebras*, Mem. Amer. Math. Soc., 104, American Mathematical Society, Providence, RI, 1970.

[5] J. R. Faulkner, *Stable range and linear groups for alternative rings*. Geom. Dedicata 14(1983), 177–188. https://doi.org/10.1007/BF00181623.

[6] J. R. Faulkner, *Coordinatization of Moufang-Veldkamp planes*. Geom. Dedicata 14(1983), 189–201. https://doi.org/10.1007/BF00181624.

[7] H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*. Geom. Dedicata 19(1985), 7–63. https://doi.org/10.1007/BF00233101.

[8] E. Kleinfeld, *Simple alternative rings*. Ann. Math. 58(1953), 544–547. https://doi.org/10.2307/1969753.

[9] B. Mühlherr, H. P. Petersson, and R. M. Weiss, *Descent in buildings*, Annals of Mathematics Studies, 190, Princeton University Press, Princeton, NJ, 2015. https://doi.org/10.1515/9781400874019.

[10] B. Mühlherr and R. M. Weiss, *Galois involutions and exceptional buildings*. Enseign. Math. 62(2016), 207–260. https://doi.org/10.4171/LEM/62-1/2-13.

[11] B. Mühlherr and R. M. Weiss, *Tits polygons*. Mem. Amer. Math. Soc., to appear.

[12] T. A. Springer and F. D. Veldkamp, *On Hjelmslev-Moufang planes*. Math. Z. 107(1968), 249–263. https://doi.org/10.1007/BF01110014.

[13] J. Tits, *Sur la géométrie des R-espaces*. J. Math. Pure Appl. 36(1957), 17–38.

[14] J. Tits and R. M. Weiss, *Moufang polygons*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. https://doi.org/10.1007/978-3-662-04689-0.

[15] F. D. Veldkamp, *Projective planes over rings of stable rank 2*. Geom. Dedicata 11(1981), 285–308. https://doi.org/10.1007/BF00149352.

Mathematisches Institut, Universität Giessen, 35392 Giessen, Germany
e-mail: bernhard.m.muehlherr@math.uni-giessen.de

Department of Mathematics, Tufts University, Medford, MA 02155, USA
e-mail: rweiss@tufts.edu