Canad. Math. Bull. Vol. **27** (4), 1984

COMMUTATIVE SYSTEMS OF COVARIANCE AND A GENERALIZATION OF MACKEY'S IMPRIMITIVITY THEOREM

BY

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ABSTRACT. Two results are obtained in this paper. The first is a generalization of the imprimitivity theorem of Mackey, when the associated projection-valued measure is replaced by a commutative positive operator valued measure. The second is a necessary and sufficient condition for such a system of covariance to possess an overcomplete, covariant family of coherent states.

1. **Introduction.** In the recent mathematical and physical literature there have appeared several papers [1, 2, 3, 4, 5] on systems of covariance (also termed generalized systems of imprimitivity), and an associated generalization of the imprimitivity theorem of Mackey [6]. The technique used for such a generalization has so far been to embed the positive operator valued (POV)-measure into a projection valued (PV)-measure in a minimally extended Hilbert space, in the sense of Naimark [7], and then to extend the corresponding group representation also to this enlarged space. One ends up in this way once more with a Mackey system of imprimitivity on the enlarged space. It is then possible to characterize the original group representation as necessarily being a subrepresentation of an induced representation. Conversely, it can be shown that every subrepresentation of an induced representation gives rise to a system of covariance.

In this paper we look at a somewhat different type of a generalization of the Mackey imprimitivity theorem, in the special case where the POV-measure associated to the system of covariance is commutative. We have proved elsewhere [5] that every regular commutative POV-measure can be written uniquely, using Choquet's integral representation theorems, (cf. for example, [8]) as an integral over PV-measures. We use this result in the present note to characterize the associated system of covariance. Specifically, we prove that the corresponding group representation for each such system is one which is induced from a subgroup, and that the system both determines and is determined by a unique probability measure which is invariant under the action of

Received by the editors October 19, 1982 and, in revised form February 1, 1984.

AMS Subject Classification (1980); 28B15, 52A99, 22D30

https://doi.org/10.4153/CMB-1984-060-6 Published online by Cambridge University Press

the inducing subgroup. This result is contained in Proposition 1. We also prove an allied result on the possible existence of densities for such a POV-measure and show that a commutative system of covariance does not admit a set of generalized coherent states in the sense of Scutaru [2], Proposition 2, unless the representing measure for the associated POV-measure in the system of covariance, is absolutely continuous with respect to the invariant group measure.

2. **Commutative systems of covariance; representing measures.** Let X be a metrizable, locally compact topological space, $\mathscr{B}(X)$ the set of all Borel sets of X, G a metrizable locally compact topological group, $\mathscr{H}$ a separable Hilbert space and $\mathscr{L}(\mathscr{H})$ the set of all bounded operators on $\mathscr{H}$. Let $E \to a(E)$ be a normalized POV-measure [2, 5] defined on $\mathscr{B}(X)$ and taking values in $\mathscr{L}(\mathscr{H})$. Throughout this paper we shall assume that a is *commutative*, i.e., for all $E, F \in \mathscr{B}(X)$, $a(E)$ and $a(F)$ commute. Let $g \to U_g$, $g \in G$, be a strongly continuous unitary representation of G on $\mathscr{H}$. Let X be a transitive G-space [6] and let us denote the action of G on X by $x \mapsto g[x]$, for $g \in G$ and $x \in X$.

DEFINITION 1. The pair $\{a, U\}$ is said to form a commutative system of covariance if, for all $g \in G$ and $E \in \mathscr{B}(X)$

$$U_g a(E) U_g^* = a(g[E]). \tag{2.1}$$

REMARK 1. As is well known, in the special case where a is a PV-measure, the pair $\{a, U\}$ reduces to the usual Mackey system of imprimitivity. It has been common in the literature to impose a further continuity condition on the POV-measure a. This condition has then been used to continuously extend the representation U to a canonically enlarged Hilbert space. However, as shown in [3] this condition is always satisfied, and hence is redundant.

Let $\mathscr{A}(a)$ be the commutative von Neumann algebra generated by the operators $a(E)$, for all $E \in \mathscr{B}(X)$, and denote by $\mathscr{M}_I(X; \mathscr{A}(a))$ the set of all POV-measures b defined on $\mathscr{B}(X)$, such that $b(E) \in \mathscr{A}(a)$ for all $E \in \mathscr{B}(X)$, and which satisfy the normalization condition, $b(x) = I$ (= identity operator on $\mathscr{H}$). It has been shown in [5] that $\mathscr{M}_I(X; \mathscr{A}(a))$ has a natural topology under which it is compact and convex. Furthermore, the set of its extreme points $\mathfrak{G}$ is a G_δ, and consists of all the PV-measures in it. The following theorem, which has been proved in [5], will be used in the sequel.

THEOREM. *If a is a normalized, regular, commutative POV-measure defined on the Borel sets of the metrizable, locally compact space X, there exists a unique probability Borel measure ν carried by $\mathfrak{G}$, such that, for all $E \in \mathscr{B}(X)$,*

$$a(E) = \int_{\mathfrak{G}} P(E)\, d\nu(P) \tag{2.2}$$

https://doi.org/10.4153/CMB-1984-060-6 Published online by Cambridge University Press

the integral converging weakly. The measure ν is unaltered if we replace $\mathfrak{G}$ by any larger set $\mathfrak{G}'$ ($\supset\mathfrak{G}$) of PV-measures.

Let $C_\infty(X)$ denote the commutative C^*-algebra (under the sup norm) of all bdd. complex continuous functions which vanish at infinity. For any $f \in C_\infty(X)$, let

$$a(f) = \int_X f(x)\, da(x) \tag{2.3}$$

and let $\mathcal{A}_c(a)$ be the commutative C^*-algebra of operators in $\mathcal{L}(\mathcal{H})$ generated by the $a(f)$'s, for all $f \in C_\infty(X)$. Then it follows from (2.1) and the continuity of the action $x \mapsto g[x]$ of G on X that both the algebras $\mathcal{A}_c(a)$ and $\mathcal{A}(a)$ are left invariant by G, i.e., for all $g \in G$,

$$\begin{cases} U_g \mathcal{A}_c(a) U_g^* = \mathcal{A}_c(a) \\ U_g \mathcal{A}(a) U_g^* = \mathcal{A}(a) \end{cases} \tag{2.4}$$

It can then be proved [5] that $\mathcal{A}_c(a)$ is contained densely, in the weak operator topology, in $\mathcal{A}(a)$. If Y denotes the spectrum of $\mathcal{A}_c(a)$, the following statements hold [9, 10]:

1) There exists an algebraic isometry

$$i : \mathcal{A}_c(a) \to C_\infty(Y)$$

which is bijective, and which can be extended to

$$i : \mathcal{A}(a) \to L^\infty(Y, \lambda),$$

where λ is a basic measure on Y. This latter algebraic isometry is also bijective and if $u \in L^\infty(Y, \lambda)$ is the image of $A \in \mathcal{A}(a)$ under i, then

$$\|A\| = \operatorname*{Ess\,Sup}_{y \in Y} |u(y)|$$

2) The action (2.1) of the group G induces an action on Y, making it into a homogeneous G-space. Let us denote this action by $y \mapsto g[y]$, $(g, y) \in G \times Y$, and assume that it is *transitive*. Then there exists a closed subgroup M of G for which the topological homeomorphism

$$Y \cong G/M \tag{2.5}$$

holds, and the measure λ is quasi-invariant under the action of G. Hence

$$\int_Y u(g[y])\, d\lambda(y) = \int_Y u(y)\xi(g, y)\, d\lambda(y), \tag{2.6}$$

where u is an integrable Borel function on Y and ξ is the usual Radon-Nikodym derivative [9] for the quasi-invariant measure λ.

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3) The representation $g \mapsto U_g$ is induced [6] from a unitary representation $m \mapsto V(m)$ of M.

DEFINITION 2. The commutative system of covariance $\{a, U\}$ is said to be transitive if the action $y \mapsto g[y]$ of G on $Y = \text{Spectrum}\,[\mathcal{A}_c(a)]$ is transitive.

REMARK 2. Note that our definition of transitivity is different from that of [2–4], where transitivity of the action $x \mapsto g[x]$ of G on X is required.

3. **An extended imprimitivity theorem**. The objective of this section is to prove in Proposition 1 our main result, which is an extended version of Mackey's imprimitivity theorem.

PROPOSITION 1. *Let $g \mapsto U_g$ be a strongly continuous unitary representation of the metrizable, locally compact group G, on the separable Hilbert space $\mathcal{H}$, and let X be a metrizable, locally compact, homogeneous G-space. Then there exists a normalized POV-measure a on $\mathcal{B}(X)$ for which $\{a, U\}$ is a transitive commutative system of covariance if and only if U_g is a representation which is induced from some subgroup M of G and there exists a probability measure ν on $\mathcal{B}(X)$ which is invariant under M. Furthermore, given a, ν is uniquely fixed and vice versa.*

Proof. Let $g \mapsto U_g$ be induced from the unitary representation $m \mapsto V(m)$ of M, acting on the Hilbert space $\mathcal{H}_0$, and let Y be as in (2.5). Corresponding to M, let

$$g = k_g m_g \tag{3.1}$$

be the Mackey decomposition for any element $g \in G$, such that $k_g \in G$, $m_g \in M$. The coset representative $k_g \in G/M$ is to be chosen in such a way that $k_e = e$, the neutral element of G. Let

$$\beta : Y \to G$$

be the Borel section, for which, for all $y \in Y$,

$$\beta(y) = k_{\beta(y)}. \tag{3.2}$$

Then, following [6] we write $\mathcal{H}$ in the form

$$\mathcal{H} = \mathcal{H}_0 \otimes L^2(Y, \lambda) \tag{3.3}$$

and U_g as,

$$(U_g\phi)(y) = B(g, y)\phi(g^{-1}[y]), \tag{3.4}$$

for all $\phi \in \mathcal{H}$, where the multiplier $B(g, y)$ is given by

$$B(g, y) = [\xi(g, y)]^{1/2} V(m_{g^{-1}\beta(y)})^{-1}, \tag{3.5}$$

ξ as in (2.6)

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Suppose now that ν is a probability measure on $\mathscr{B}(X)$ which is invariant under M, i.e.,

$$\nu(m[E]) = \nu(E), \tag{3.6}$$

for all $E \in \mathscr{B}(X)$ and $m \in M$. For each $x \in X$, and $E \in \mathscr{B}(X)$, consider the operator $P_x(E)$ on $\mathscr{H}$,

$$(P_x(E)\phi)(y) = \chi_E(\beta(y)[x])\phi(y), \tag{3.7}$$

where χ_E is the characteristic function of the set E. It is then straightforward to check that for fixed x, $E \mapsto P_x(E)$ is a PV-measure on $\mathscr{H}$. Moreover, for fixed E, the function $x \mapsto \chi_E(\beta(y)[x])$ is measurable. Hence consider

$$a(E) = \int_X P_x(E)\, d\nu(x), \tag{3.8}$$

the integral being defined strongly. Once again, straightforward manipulations show that $E \mapsto a(E)$ is a normalized commutative POV-measure. Furthermore, $\{a, U\}$ is a system of covariance. Indeed, by (3.4) and (3.7), for all $\phi \in \mathscr{H}$,

$$\begin{aligned}(U_g a(E) U_g^* \phi)(y) &= \int_X d\nu(x)\chi_E(\beta(g^{-1}[y])[x])\phi(y)\\ &= \int_X d\nu(x)\chi_E(g^{-1}\beta(y)m^{-1}_{g^{-1}\beta(y)}[x])\phi(y)\\ &= \int_X d\nu(x)\chi_{(g^{-1}\beta(y))^{-1}[E]}(m^{-1}_{g^{-1}\beta(y)}[x])\phi(y)\end{aligned}$$

Hence, since ν is invariant under $m \in M$,

$$(U_g a(E) U_g^* \phi)(y) = \int_X d\nu(x)\chi_{g[E]}(\beta(y)[x])\phi(y)$$

so that

$$U_g a(E) U_g^* = a(g[E]).$$

From the Theorem stated in Sec. 2 above, it follows that the POV measure a in (3.8) is uniquely determined by ν.

Next suppose that $\{a, U\}$ is a commutative transitive system of covariance. Then, since the commutative von Neumann algebra $\mathscr{A}(a)$ is invariant under the action of G (cf. (2.4)) and the spectrum Y of $\mathscr{A}(a)$ is a transitive G-space, it follows from Takesaki's extension of Mackey's Imprimitivity Theorem to invariant von Neumann algebras [9], that $g \mapsto U_g$ is induced from a unitary representation of M (where Y and M are related by (2.5)). The existence of the measure ν, satisfying (3.8), is then established by Theorem 6 in [5], while its uniqueness follows again from the uniqueness of the representation (2.2). Q.E.D.

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4. **A result on coherent states.** For the discussion in this section, we shall assume that X is a homogeneous space under the action of G, so that

(4.1) $$X \cong G/H,$$

for some closed subgroup $H \subset G$. Also let

(4.2) $$\gamma : X \to G$$

be the Borel section, defined analogously to β in (3.2). We shall assume that X admits the *invariant* measure σ. Let $\mathcal{T}(\mathcal{H})_1^+$ denote the set of all positive trace-class operators on $\mathcal{H}$ with unit trace. Furthermore, let $x_0 \in X$ be the element for which

(4.3) $$\gamma(x_0) = e,$$

so that for arbitrary $x \in X$,

(4.4) $$x = \gamma(x)[x_0],$$

DEFINITION 3. A weakly measurable map $x \mapsto \rho_x$ from X into $\mathcal{T}(\mathcal{H})_1^+$ is called an overcomplete, covariant family of coherent states for the system of covariance $\{a, U\}$ if

(4.5) 1) $$\int_X (\phi, \rho_x \psi)\, d\sigma(x) = (\phi, \psi)$$

for all $\phi, \psi \in \mathcal{H}$

(4.6) 2) $$U_g \rho_x U_g^* = \rho_{g[x]}$$

for all $x \in X$ and $g \in G$

(4.7) 3) $$\int_E \rho_x\, d\sigma(x) = a(E),$$

for all $E \in \mathcal{B}(X)$, the integral converging weakly (hence strongly).

We find in Proposition 2 below the necessary and sufficient conditions for a commutative system of covariance to possess an over-complete, covariant family of coherent states.

In view of (4.4) and (4.6), we have,

(4.8) $$\rho_x = U_{\gamma(x)} \rho U_{\gamma(x)}^*,$$

where we have written ρ for ρ_{x_0}. Also, since a is commutative, it follows from (4.7) that ρ_x commutes with $\rho_{x'}$ for all $x, x' \in X$, while (4.8) implies that the operators ρ_x all have the same spectrum, namely the spectrum of ρ. Additionally, ρ being a positive trace class operator, its spectrum consists of a discrete positive sequence of points $\{\alpha_n\}_{n=0}^{\infty}$, converging to zero, and such that

(4.9) $$\operatorname{tr} \rho = \sum_{n=0}^{\infty} \alpha_n = 1.$$

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On the other hand, from the general theory of commutative von Neumann algebras [10] it follows that the spectrum Y of $\mathcal{A}_c(a)$ is homeomorphic to $\{\alpha_n\}_{n=0}^{\infty}$ (the latter being considered as a topological space with the obvious topology). Hence, Y consists of a convergent sequence $\{y_n\}_{n=0}^{\infty}$ of real numbers. Also,

$$(\rho\phi)(y_n) = \rho(y_n)\phi(y_n) \tag{4.10}$$

for all $\phi \in \mathcal{H}$, and where $\rho(y_n) = \alpha_n$ for all n. Next, for any $x \in X$, there exists a positive valued function $y_n \mapsto \rho_x(y_n)$ satisfying

$$(\rho_x\phi)(y_n) = \rho_x(y_n)\phi(y_n),$$

and (in virtue of (3.4), (4.8) and (4.10)),

$$\rho_x(y_n) = \rho(\gamma(x)^{-1}[y_n]). \tag{4.11}$$

We note, in addition, that (in virtue of (4.11) and (4.3)), for all $h \in H$,

$$U_h \rho U_h^* = \rho \tag{4.12}$$

Let ν be the measure associated with a through (3.8). We then have:

PROPOSITION 2. *The commutative system of covariance $\{a, U\}$ possesses an overcomplete, covariant family of coherent states $x \mapsto \rho_x$ if and only if the measure ν has a continuous density f with respect to the invariant measure σ on X, and the spectrum of $\mathcal{A}_c(a)$ is discrete.*

Proof. Suppose that for all $E \in \mathcal{B}(X)$

$$\nu(E) = \int_E f(x)\, d\sigma(x), \tag{4.13}$$

where f is a positive, real, continuous function on X. Since $\nu(X) = 1$,

$$\int_X f(x)\, d\sigma(x) = 1 \tag{4.14}$$

and, since ν is invariant under M,

$$f(m[x]) = f(x) \tag{4.15}$$

for all $m \in M$ and $x \in X$. Now, from (3.7) and (3.8), for all $\phi \in \mathcal{H}$,

$$(a(E)\phi)(y) = \int_X \chi_E(x) f(\beta(y)^{-1}[x])\phi(y)\, d\sigma(x) \tag{4.16}$$

Hence, defining an operator ρ on $\mathcal{H}$ via

$$(\rho\phi)(y) = f(\beta(y)^{-1}[x_0])\phi(y), \tag{4.17}$$

it is easily checked (using (4.15), and the fact that x_0 is stable under H) that ρ satisfies (4.12) and leads to a family of trace class operators ρ_x, satisfying (4.11).

Conversely, suppose a possesses such a coherent family of states $x \mapsto \rho_x$. Then (4.7) and (4.11) imply that, for all $\phi \in \mathcal{H}$,

$$(a(E)\phi)(y) = \int_X \chi_E(x)\rho(\gamma(x)^{-1}[y])\phi(y)\, d\sigma(x), \tag{4.18}$$

so that defining

$$f(x) = \rho(\gamma(x)^{-1}[y_0]), \tag{4.19}$$

the result follows. Q.E.D.

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