SOME THEOREMS ABOUT $p_r(n)$

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Introduction. If n is a non-negative integer, define $p_r(n)$ as the coefficient of x^n in

$$\prod_{n=1}^{\infty} (1 - x^n)^r;$$

otherwise define $p_r(n)$ as 0. In a recent paper **(1)** the author has proved that if r has any of the values 2, 4, 6, 8, 10, 14, 26 and p is a prime >3 such that $r(p + 1) \equiv 0 \pmod{24}$, then

$$(1) \qquad p_r(np + \Delta) = (-p)^{\frac{1}{2}r-1} p_r\left(\frac{n}{p}\right), \qquad \Delta = r(p^2 - 1)/24,$$

where n is an arbitrary integer.

In this note we wish to point out one or two additional facts implied by identity (1). The first remark is that (1) furnishes a simple, uniform proof of the Ramanujan congruences for partitions modulo 5,7,11, and a general congruence will be proved. The second is that for the values of r indicated, $p_r(n)$ is zero for arbitrarily long strings of consecutive values of n. Finally, some additional theorems not covered by (1) will be given without proof.

In what follows all products will be extended from 1 to ∞ and all sums from 0 to ∞, unless otherwise indicated.

THEOREM 1. *Let r = 4, 6, 8, 10, 14, 26. Let p be a prime greater than 3 such that $r(p + 1) \equiv 0 \pmod{24}$, and set $\Delta = r(p^2 - 1)/24$. Then if $R \equiv r \pmod{p}$ and $n \equiv \Delta \pmod{p}$,*

$$(2) \qquad p_R(n) \equiv 0 \pmod{p}.$$

Proof. Set $R = Qp + r$. Then

$$\sum p_r(n)x^n = \prod(1 - x^n)^R = \prod(1 - x^n)^{Qp+r}$$
$$\equiv \prod(1 - x^{np})^Q(1 - x^n)^r \pmod{p}.$$

Thus

$$\sum p_R(n)x^n \equiv \sum p_Q\left(\frac{n}{p}\right)x^n \sum p_r(n)x^n \pmod{p},$$

and so

$$p_R(n) \equiv \sum_{j=0}^{n} p_Q\left(\frac{j}{p}\right)p_r(n - j) \pmod{p},$$

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or

$$p_R(n) \equiv \sum_{0 \leqslant j \leqslant \frac{n}{p}} p_Q(j) p_r(n - pj) \qquad (\text{mod } p).$$

Now (1) implies that for $r > 2$ and $n \equiv \Delta \pmod p$, $p_r(n - pj) \equiv 0 \pmod p$. Thus $p_R(n) \equiv 0 \pmod p$, and so (2) is proved.

If we now note that for $R = -1$ the choices $r = 4$, $p = 5$; $r = 6$, $p = 7$; and $r = 10$, $p = 11$ (all with $Q = -1$) are permissible, and that for these values $\Delta = 4,12,50$ respectively, then the Ramanujan congruences $p(5n + 4) \equiv 0 \pmod 5$, $p(7n + 5) \equiv 0 \pmod 7$, $p(11n + 6) \equiv 0 \pmod{11}$ follow as a corollary, since $12 \equiv 5 \pmod 7$ and $50 \equiv 6 \pmod{11}$.

We go on now to the second remark. We first prove the following lemma.

LEMMA 1. *Let $a_1, a_2, \ldots, a_{n+1}$ be non-zero pairwise relatively prime integers, and let $c_1, c_2, \ldots, c_n$ be arbitrary integers. Then the simultaneous diophantine equations*

$$\begin{aligned} a_1x_1 - a_2x_2 &= c_1, \\ a_2x_2 - a_3x_3 &= c_2, \\ &\cdots \\ a_nx_n - a_{n+1}x_{n+1} &= c_n, \end{aligned} \tag{3}$$

always have infinitely many solutions.

Proof. Put $T = a_{n+1}x_{n+1}$, $C_i = c_i + c_{i+1} + \ldots + c_n$, $1 \leqslant i \leqslant n$. Then by summing the rows of (3) 1, 2, ... at a time beginning with the last, we find that the system (3) is equivalent to the system

$$\begin{aligned} a_ix_i &= C_i + T, \qquad 1 \leqslant i \leqslant n, \\ a_{n+1}x_{n+1} &= T. \end{aligned}$$

Since the a's are pairwise relatively prime, the Chinese remainder theorem assures us of the existence of an integer C such that

$$C \equiv -C_i \pmod{a_i}, \qquad 1 \leqslant i \leqslant n,$$

and

$$C \equiv 0 \qquad \pmod{a_{n+1}}.$$

Put $T = C + Ax$, where $A = a_1a_2 \ldots a_{n+1}$. Then (3) has the solution

$$\begin{aligned} x_i &= \frac{C + C_i}{a_i} + \frac{Ax}{a_i}, \qquad 1 \leqslant i \leqslant n, \\ x_{n+1} &= \frac{C}{a_{n+1}} + \frac{A}{a_{n+1}}x, \end{aligned}$$

where x is arbitrary. Thus Lemma 1 is proved.

If we now notice, for example, that $p_r(np^2 + p + \Delta) = 0$ (obtained by replacing n by $np + 1$ in (1)) and that any two distinct primes p are relatively prime, we see that Lemma 1 implies

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THEOREM 2. *For $r = 2, 4, 6, 8, 10, 14, 26$ $p_r(n)$ vanishes for arbitrarily long strings of consecutive values of n, arbitrarily many in number.*

We remark that the same is true for $p_1(n)$, $p_3(n)$, because of the classical identities

$$\prod (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}(3n^2+n)}$$
$$\prod (1 - x^n)^3 = \sum (-1)^n (2n + 1) x^{\frac{1}{2}(n^2+n)},$$

due respectively to Euler and Jacobi.

Finally, we state without proof some additional identities derivable in the same way that (1) was derived in **(1)**; p is a prime in what follows.

(4) $p_2\left(np + \frac{1}{12}(p^2 - 1)\right) = (-1)^{\frac{1}{2}(p+1)} p_2\left(\frac{n}{p}\right), \quad p \not\equiv 1 \pmod{12}, \qquad p > 3.$

(5) $p_6(3n + 2) = 9p_6(\frac{1}{3}n).$

(6) $p_8(2n + 1) = -8p_8(\frac{1}{2}n)$ (due to van der Pol **(2)**)

(7) $p_{10}\left(np + \frac{5}{12}(p^2 - 1)\right) = p^4 p_{10}\left(\frac{n}{p}\right), \qquad p \equiv 7 \pmod{12}.$

(8) $p_{14}\left(np + \frac{7}{12}(p^2 - 1)\right) = -p^6 p_{14}\left(\frac{n}{p}\right), \qquad p \equiv 5 \pmod{12}.$

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