

HILBERT SPACES HAVE THE BANACH-STONE PROPERTY FOR BOCHNER SPACES

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Let $(\Omega_i, \Sigma_i, \mu_i)$ be two positive finite measure spaces, V a non-zero Hilbert space, and $1 \leq p < \infty$, $p \neq 2$. In this article it is shown that each surjective linear isometry between the Bochner spaces $L^p(\mu_i, V)$ induces a Boolean isomorphism between the measure algebras Σ_i/μ_i , thus generalizing a result of Cambern's for separable Hilbert spaces.

This Banach-Stone type theorem is achieved via a description of the L^p -structure of $L^p(\mu_i, V)$.

1. Introduction

In analogy to the Banach-Stone property [1, p. 142] a Banach space V is said to have the *p-Banach-Stone property*, if for any pair of positive finite measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ the existence of a surjective linear isometry between the Bochner spaces $L^p(\mu_i, V)$ implies that the measure algebras Σ_i/μ_i are isomorphic. (See [5] for the definition and properties of $L^p(\mu_i, V)$.) By Lamperti's extension of Banach's classical result [8] the scalar field \mathbb{K} has the *p-Banach-Stone property* for

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$p \neq 2$. The first vector-valued generalization is due to Cambern who showed that separable Hilbert spaces have this property for $1 \leq p < \infty$, $p \neq 2$ [3] and for $p = \infty$ [4]. Cambern's theorem has been extended to larger classes of separable spaces [9], [6], and [7]; however, for all these Banach-Stone type theorems the separability of V was essential. In this paper we give a generalization of Cambern's theorem for non-separable Hilbert spaces.

THEOREM 1. *All non-zero Hilbert spaces have the p -Banach-Stone property ($1 \leq p < \infty$, $p \neq 2$).*

We shall achieve this result via a description of the L^p -structure of $L^p(\mu, V)$ (see Theorem 2).

As to our method, the proofs in [6], [7], and [9] use some kind of lifting of $L^p(\mu, V)$ in order to deal with functions instead of equivalence classes. In this paper we employ a representation of $L^p(\mu, V)$ as a space of L^p -like functions with values in Banach spaces V_k that may vary from point to point, a so-called "integral module", introduced by R. Evans. The virtue of this representation is the fact that all norm-functions $k \mapsto \|x(k)\|$ are continuous, which might turn out to be useful also in the investigations of other geometric relations between V and $L^p(\mu, V)$.

2. An integral module representation

We need some definitions and notations. From now on let $1 \leq p < \infty$, $p \neq 2$. By χ_M we denote the characteristic function of a set as well as the corresponding multiplication operator; the constant function with value v is denoted by v . In both cases the domains of the functions will be clear from the context. A projection P in a Banach space X is called an L^p -projection, if

$$(1) \quad \|x\|^p = \|Px\|^p + \|x-Px\|^p$$

for all $x \in X$. The set of all L^p -projections of X is a complete Boolean algebra [2, p. 11] denoted by $\mathbf{P}_p(X)$.

REPRESENTATION THEOREM. Let (Ω, Σ, μ) be a finite measure space and V a real Banach space, K the Stonean space of the measure algebra Σ/μ , and Φ the Boolean isomorphism of Σ/μ onto the clopen subsets of K . Then there is a regular Borel measure m on K satisfying

$$(2) \quad m(\Phi M) = \mu(M)$$

for all $M \in \Sigma$ and

$$(3) \quad m(D) = 0$$

for all measurable nowhere dense $D \subset K$. Furthermore, there is a system $(V_k)_{k \in K}$ of Banach spaces containing V and an embedding

$$x \mapsto \langle x \rangle$$

of $L^p(\mu, V)$ into the cartesian product $\prod_{k \in K} (V_k \dot{\cup} \{\infty\})$ such that

(i) for each $x \in L^p(\mu, V)$ the norm function

$$k \mapsto |x|(k) := \|\langle x \rangle(k)\|$$

$(\|\infty\| := \infty)$ is a continuous numerical $L^p(m)$ -function with L^p -norm equal to $\|x\|$,

(ii) addition and scalar multiplication on $L^p(\mu, V)$, when embedded into the cartesian product, coincide with the m -almost everywhere pointwise operations, precisely: for each $k \in K$ the evaluation mapping

$$x \mapsto \langle x \rangle(k)$$

is linear on the preimage of V_k (observe that

$$\langle x \rangle(k) \in V_k \text{ } m\text{-almost everywhere since } |x| \in L^p(m) \text{),}$$

(iii) for each $v \in V$ the constant function v on Ω is mapped onto the constant function v on K ,

(iv) for each $x \in L^p(m, V)$ and $M \in \Sigma$,

$$\langle \chi_M x \rangle = \chi_{\Phi M} \langle x \rangle \quad (0 \cdot \infty := 0) .$$

The proof is a simple combination of Theorems 3.7 and 7.3 in [2]. We

only have to verify the assumption that the set A of characteristic projections χ_M ($M \in \Sigma$) in $L^p(\mu, V)$ is a *complete* Boolean algebra of projections in the sense that each increasing net converges strongly to its supremum. This, however, is an immediate consequence of the fact that the set of characteristic projections χ_M ($M \in \Sigma$) in the scalar-valued $L^p(\mu)$ has this property as it is the whole of $\mathbb{P}_p(L^p(\mu))$ [2, 4.9 and 1.6]. \square

The image of $L^p(\mu, V)$ under $\langle \cdot \rangle$ is called an *integral module*; clearly it is a Banach space isometrically isomorphic to $L^p(\mu, V)$.

PROPOSITION. *For each $x \in L^p(\mu, V)$ there is an open dense subset U of K such that*

(v) $\langle x \rangle|_U$ takes its values in V ,

(vi) $\langle x \rangle|_U$ is continuous.

Proof. By (iii) and (iv) the simple functions are represented as continuous V -valued simple functions, and the latter ones are dense in the integral module. An application of Egorov's theorem shows that $\langle x \rangle$ is an m -almost uniform limit of continuous functions. Thus for each $\varepsilon > 0$ there is a set U_ε such that $\langle x \rangle$ restricted to U_ε is continuous and $m(K \setminus U_\varepsilon) < \varepsilon$. By the regularity of m we may choose U_ε clopen.

(Observe that an open set and its closure have the same measure.) The union U of all U_ε 's has the desired properties: its complement has measure zero, so that it cannot contain a non-void clopen subset. As K is 0-dimensional, this means that $K \setminus U$ has void interior. \square

3. The L^p -structure of $L^p(\mu, V)$

We prove the following theorem.

THEOREM 2. *Let V be a non-zero Hilbert space, (Ω, Σ, μ) a finite positive measure space, $1 \leq p < \infty$, $p \neq 2$. Then*

$$\mathbb{P}_p(L^p(\mu, V)) \cong \Sigma/\mu.$$

Since each surjective isometry T between two Banach spaces X_1 and X_2 induces a Boolean isomorphism Ψ of $\mathbf{P}_p(X_1)$ onto $\mathbf{P}_p(X_2)$ via

$$\Psi P := T \circ P \circ T^{-1} ,$$

we have an improvement of Theorem 1 as an immediate consequence of Theorem 2.

COROLLARY. *For each isometry T of $L^p(\mu_1, V)$ onto $L^p(\mu_2, V)$ there is a Boolean isomorphism Ψ of Σ_1/μ_1 onto Σ_2/μ_2 such that*

$$\chi_{\Psi M} \circ T = T \circ \chi_M$$

for all $M \in \Sigma_1$. That is, the restriction of T to $\chi_M \cdot L^p(\mu_1, V)$ is an isometry onto $\chi_{\Psi M} L^p(\mu_2, V)$.

Proof of Theorem 2. Since the characteristic projections χ_M are trivially L^p -projections, it remains to show that for each $P \in \mathbf{P}_p(L^p(\mu, V))$ there is an $M \in \Sigma$ such that $Px = \chi_M x$ for all $x \in L^p(\mu, V)$. Of course we may assume without loss of generality that the scalars are real. We represent $L^p(\mu, V)$ as an integral module as in Section 2 and we show first that P splits into L^p -projections P_k of V_k , that is,

$$(4) \quad \langle Px \rangle(k) = P_k(\langle x \rangle(k)) \text{ almost everywhere}$$

for all $x \in L^p(\mu, V)$. Since the L^p -projections P and χ_M commute ($M \in \Sigma$), we have

$$\int_{\Phi M} |x|^p dm = \|\chi_M x\|^p = \|P\chi_M x\|^p + \|\chi_M x - \chi_M Px\|^p = \int_{\Phi M} (|Px|^p + |x - Px|^p) dm .$$

Observe that ΦM runs through all clopen subsets of K . Consequently the continuous (numerical) integrands coincide,

$$(5) \quad |x|^p = |Px|^p + |x - Px|^p .$$

In particular $\langle Px \rangle(k) \neq \infty$ for $\langle x \rangle(k) \neq \infty$ and

$$\langle Px \rangle(k) - \langle Py \rangle(k) = \langle P(x-y) \rangle(k) = 0$$

for $\langle x \rangle(k) = \langle y \rangle(k) \neq \infty$. Thus

$$P_k \langle x \rangle(k) := \langle Px \rangle(k)$$

is well-defined on $V_k = \{\langle x \rangle(k) \mid x \in L^p(\mu, V) \setminus \{\infty\}\}$ [2, 3.10], and (5)

says that the projection P_k is an L^p -projection on V_k . Since $\langle x \rangle(k) = \infty$ only on a set of measure zero, we have (4).

Now we claim $P_k \in \{0, \text{Id}\}$ for all $k \in K$ (Id the identity operator). Then $B := \{k \mid P_k = \text{Id}\}$ is clopen, for if we choose $v \in V$ with $\|v\| = 1$, then

$$B = \{k \mid P_k v = v\} = \{k \mid |Pv|(k) = 1\},$$

and $|Pv|$ is a continuous function into the discrete set $\{0, 1\}$. Now take $M \in \Sigma$ with $\Phi M = B$. From (ii) and the definition of B we have m -almost everywhere, namely for all k with $\langle x \rangle(k) \neq \infty$,

$$|Px - \chi_M x|(k) = \|\langle Px \rangle(k) - \chi_B \langle x \rangle(k)\| = \|P_k(\langle x \rangle(k)) - \chi_B(k) \cdot \langle x \rangle(k)\| = 0;$$

that is, $\|Px - \chi_M x\| = 0$.

Now let us prove our claim. Hilbert spaces have only the trivial L^p -projections 0 and Id [2, 1.3]. So it suffices to verify the parallelogram equality for all components V_k . Let $\langle x \rangle(k), \langle y \rangle(k)$ be two arbitrary elements of V_k . By (v), for all l in a dense subset U of K , $\langle x \rangle(l)$ and $\langle y \rangle(l)$ are elements of the Hilbert space V . Thus

$$|x+y|(l)^2 + |x-y|(l)^2 = 2(|x|(l)^2 + |y|(l)^2)$$

for all $l \in U$, hence, by continuity, everywhere. This shows that $\langle x \rangle(k)$ and $\langle y \rangle(k)$ satisfy the parallelogram equality, thus completing the proof of Theorem 2. \square

REMARK 1. The method of this proof applies to all Banach spaces V where a norm (in-)equality involving only finitely many vectors can be

shown to imply trivial L^p -structure, for example all L^q -spaces ($q \neq p$).

REMARK 2. The finiteness of the measures in Theorems 1 and 2 is not essential. They hold for σ -finite measures as well and, if we replace Σ/μ by $\mathbf{P}_p(L^p(\mu))$, even for arbitrary positive measures. To see this,

observe that each $L^p(\mu)$ is an L^p -direct sum $\sum_{i \in I}^p L^p(\mu_i)$ with finite measures μ_i . Evidently we have also $L^p(\mu, \nu) = \sum_{i \in I}^p L^p(\mu_i, \nu)$. The

fact that a projection in an L^p -direct sum is an L^p -projection if and only if its restrictions to the summands are L^p -projections of the summands [2, 2.6] and an application of Theorem 2 show that

$\mathbf{P}_p(L^p(\mu, \nu)) \cong \mathbf{P}_p(L^p(\mu))$. For σ -finite μ the latter is Σ/μ [2, 4.9].

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