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A NOTE ON PERMANENTS

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Let $A=(a_{ij})$ be an $m\times n$ matrix and let $K=\{s_1,\ldots,s_k\}$ be a k-subset from $\{1,2,\ldots,n\}$. For $0\le t\le k\le n$ define the (t,K)-permanent of A to be

(1) $$\mathrm{per}_{(t,K)}(A) = \sum a_{1i_1}a_{2i_2}\ldots a_{mi_m}$$

the summation taken over all m-tuples $(i_1, i_2,\ldots, i_m)$ (repetitions allowed) of $1, 2,\ldots, n$ each containing exactly t distinct entries from K and any number of distinct entries from the remaining $n-k$ integers. For example, (4, 4, 7, 1, 1, 2), (4, 4, 6, 6, 6, 5) are 6-tuples, each containing exactly two distinct entries from $K=\{2, 4, 5\}$ for $n\ge 7$. We define the t-permanent of A to be the case $K=\{1,2,\ldots,n\}$ and write

(2) $$\mathrm{per}_t(A) = \mathrm{per}_{(t,K)}(A),\quad K=\{1,2,\ldots,n\},$$

each m-tuple $(i_1,\ldots,i_m)$ in the summation in (1) containing exactly t distinct entries of $\{1,2,\ldots,n\}$. When $t=m=n$, (2) is the "ordinary" permanent of a square matrix (see [1] for a survey article), while $t=m\le n$ is the generalization to rectangular matrices described by Ryser [3].

Let A_r denote a matrix obtained from A by replacing r of the k columns $s_1, s_2,\ldots, s_k$ of A by zeros, $S(A_r)$ the product of the row sums of A_r and $\sum S(A_r)$ the sum of all the $\binom{k}{r}$ numbers $S(A_r)$. Then $\mathrm{per}_{(t,K)}(A)$ can be evaluated by

(3) $$\mathrm{per}_{(t,k)}(A) = \sum_{i=0}^{t}(-1)^k\binom{k-t+i}{i}\sum S(A_{k-t+i})$$

and hence

(4) $$\mathrm{per}_t(A) = \sum_{i=0}^{t}(-1)^i\binom{n-t+i}{i}\sum S(A_{n-t+i}).$$

Formula (4) in the case $m=t\le n$ was first observed by Ryser [3, p. 26] and his elegant proof (based on the Principle of Inclusion and Exclusion) suffices, with minor modification, to establish (3), so we omit the proof.

When $t=k$, (3) becomes

(5) $$\mathrm{per}_{(k,K)}(A) = \sum_{i=0}^{k}(-1)^i\sum S(A_i).$$

Let ${}_tA$ be a submatrix of A obtained by deleting $n-t$ columns of A. It follows that

(6) $$\mathrm{per}_t(A) = \sum \mathrm{per}_t({}_tA),$$

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the summation taken over all the $\binom{n}{t}$ choices of ${}_tA$. In the case $t=m\le n$, $\mathrm{per}_m(A)$ is therefore equal to the sum of permanents of square submatrices of order m. It is clear that

(7) $$\sum_{t=1}^{n} \mathrm{per}_t(A) = A_0,$$

A_0 being the product of the row sums of A. In the case $m<t\le n$, $\mathrm{per}_t(A)=0$.

We describe several of many possible applications.

Suppose that m distinct objects $d_1, \ldots, d_m$ are to be distributed into n distinct cells $c_1, \ldots, c_n$. An $m\times n$ (0, 1) matrix $A=(a_{ij})$ can be interpreted as describing restrictions on the distribution, namely object d_i can be placed in cell c_j if and only if $a_{ij}=1$. Suppose furthermore that t is given, and we insist that exactly t cells are nonempty. Then the number of such distributions (each satisfying conditions described by A and exactly t cells nonempty) is $\mathrm{per}_t(A)$. For a fixed subset $K=\{s_1, \ldots, s_k\}$ of $\{1, 2, \ldots, n\}$, $\mathrm{per}_{(t,k)}(A)$ is the number of distributions each satisfying conditions described by A and exactly t of the cells $c_{s_1}, c_{s_2}, \ldots, c_{s_k}$ nonempty.

It is easy to see that $\mathrm{per}_n(A)>0$ if and only if n is equal to the term rank of A, i.e., there is a way of distributing the objects with no cell empty if and only if the maximal number of 1's in A, no two in a row or column, is n. Equivalently, $\mathrm{per}_n(A)>0$ if and only if the n subsets of $\{1, 2, \ldots, n\}$ whose incidence matrix is the transpose of A have a system of distinct representatives.

If any object is permitted into any cell, the matrix A becomes $J_{m,n}$ all of whose entries are 1. Then the number of distributions each with exactly t of the first k cells nonempty (the remaining $n-k$ cells may or may not be empty) is, by (3) with $K=\{1, 2, \ldots, k\}$, $k\le n$,

(8) $$\mathrm{per}_{(t,K)}(J_{m,n}) = \sum_{i=0}^{t} (-1)^i \binom{k-t+i}{i}\binom{k}{k-t+i}(t+n-i-k)^m$$
$$= \binom{k}{t}\sum_{i=0}^{t} (-1)^i \binom{t}{i}(t+n-i-k)^m.$$

In the case $t=k$, (8) becomes,

(9) $$\mathrm{per}_{(k,K)}(J_{m,n}) = \sum_{i=0}^{t} (-1)^i \binom{k}{i}(n-i)^m = \begin{cases} 0 & \text{if } 1\le m<k \\ k! & \text{if } m=k. \end{cases}$$

Formula (9) may also be obtained directly using the principle of inclusion and exclusion or by noting that $\mathrm{per}_{(k,K)}(J_{m,n})$ is the coefficient of $x^m/m!$ in the expression $(e^x-1)^k(e^x)^{n-k}$, $k\le n$. The number of distributions with none of the n cells empty is formula (9) with $k=n$, namely the well-known formula

(10) $$\mathrm{per}_n(J_{m,n}) = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i}(n-i)^m = \begin{cases} 0 & \text{if } 1\le m<\text{n}, \\ n! & \text{if } m=n. \end{cases}$$

This is also the number of ordered partitions of a finite set of m objects into n disjoint nonempty subsets, while the number of unordered partitions is

(11) $$S(m, n) = (1/n!)\,\mathrm{per}_n(J_{m,n})$$

the numbers $S(m, n)$ usually being called Stirling numbers of the second kind [2]. By (6) and (7) we have

$$(12)\qquad \sum_{t=1}^{n} \operatorname{per}_t(J_{m,n}) = \sum_{t=1}^{n} \binom{n}{t} \operatorname{per}_t(J_{m,t}) = n^m$$

and therefore using (11)

$$(13)\qquad \sum_{t=1}^{n} S(m, t)(n)_t = n^m, \quad (n)_t = n(n-1)\ldots(n-t+1), \quad \text{(see [2])}.$$

If object d_i is placed into cell c_j with probability p_{ij}, then, letting $P=(p_{ij})$, the probability that exactly t of the cells are nonempty is $\operatorname{per}_t(P)$.

A generalization of Montmort's "problème des rencontres" is obtained by taking $A=(a_{ij})$ with $a_{ii}=0$, $i=1, 2, \ldots, k$, $k \le \min(m, n)$ and $a_{ij}=1$ otherwise. Then the number of distributions of m objects into the n cells such that for $i=1, 2, \ldots, k$ cell c_i is nonempty and object d_i is not in cell c_i, is by (5) with $K=\{1, 2, \ldots, k\}$,

$$(14)\qquad \operatorname{per}_{(k,K)}(A) = D_k(m, n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n-i)^{m-k+i}(n-i-1)^{k-i},$$

with $D_0(m, n)=n^m$ while,

$$(15)\qquad D(m, n) = \operatorname{per}_n(A) = D_n(m, n)$$
$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^{m-n+i}(n-i-1)^{n-i}, \quad m \ge n,$$

is the number of distributions with none of the cells empty and object d_i not in cell c_i, $i=1, 2, \ldots, n$. Thus, the ordinary rencontres numbers are given by

$$(16)\qquad D(n, n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^i (n-i-1)^{n-i},$$

an expression noted by Ryser [3, p. 28]. It is easily seen that

$$(17)\qquad \sum_{t=1}^{n} \binom{n}{t} D(m, t) = (n-1)^n n^{m-n}.$$

The expression for $D(m, n)$ given by (15) may be easily obtained directly using the principle of inclusion and exclusion. Denote by $g(m, n)$ the number of distributions of m objects into n cells with object d_i not in cell c_i, $i=1, \ldots, n$. Clearly

$$g(m, n) = (n-1)^n n^{m-n} \quad \text{and} \quad D(m, n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} g(m, n-i), \quad m \ge n,$$

giving (15). Similarly (14) may be obtained directly. A second expression may be obtained for $D(m, n)$ (also by the use of inclusion and exclusion), namely

$$(18)\qquad D(m, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \operatorname{per}_{(n-k,K)}(J_{m-k,n}), \quad K = \{1, 2, \ldots, n-k\},$$
$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)^{m-k},$$

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so

$$(19)\qquad D(n, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} (n-j)^{n-k}$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} (-1)^k/k!.$$

The last expression of (19) is the most common one given for the rencontres numbers [3, p. 23]. Denote by $D(m, n, r)$, $m \geq n$, the number of m-permutations of $1, 2, \ldots, n$, repetitions allowed, such that each of the n integers appears at least once in each permutation with exactly r of them being in natural position. Then $D(m, n, 0) = D(m, n)$ and

$$(20)\qquad D(m, n, r) = \binom{n}{r} D_{n-r}(m-r, n), \quad r = 0, 1, \ldots, n,$$

while

$$(21)\qquad \sum_{r=0}^{n} D(m, n, r) = n!\, S(m, n)$$

the numbers $D_k(m, n)$ and $S(m, n)$ given by (14) and (11) respectively.

In the case A is a (0, 1) matrix with $m < n$ and all entries on the main diagonal zero and one elsewhere,

$$(22)\qquad \mathrm{per}_n(A) = \sum_{k=0}^{n-1} (-1)^k \sum_{u=0}^{k} \binom{m}{u} \binom{n-m}{k-u} (n-k)^u (n-k-1)^{m-u} = 0.$$

With regard to matching problems we have the following. Let $A_1, \ldots, A_m$ be subsets of an n-set S and define a system of t-representatives of $(A_1, \ldots, A_m)$ to be an m-tuple $(a_1, \ldots, a_m)$ containing exactly t distinct elements of S with $a_i \in A_i$, $i = 1, \ldots, m$. Then the number of systems of t-representatives of $(A_1, \ldots, A_m)$ is equal to the t-permanent of the corresponding (0, 1) incidence matrix of size m by n. Putting $t = \min(m, n)$, we have the number of systems of maximum distinct representatives and when $t = m \leq n$, the number of systems of distinct representatives [3, p. 54, Theorem 4.1].

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