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A characterization of the Hall planes of odd order

P.B. Kirkpatrick

The Hall projective planes of odd order are characterized in terms of their translations, collineations which fix all the points of a Baer subplane, and involutory homologies.

1. Introduction

Let Π be a projective plane, l_∞ a line of Π and Π_0 a Baer subplane of Π such that l_∞ is a line of Π_0. We call Π a *generalized Hall plane* with respect to l_∞, Π_0 if

(1) Π is a translation plane with respect to l_∞, and

(2) Π has a group of collineations which is transitive on the points of l_∞ not in Π_0, and fixes every point in Π_0.

The object of this paper is to prove:

THEOREM. *A projective plane* Π *is a Hall plane of odd order if and only if*

(a) Π *is a finite generalized Hall plane with respect to some line* l_∞ *and Baer subplane* Π_0 *containing* l_∞, *and*

(b) each point of $\underline{\underline{M}} = \{M \mid M \in l_\infty$ *and* $M \in \Pi_0\}$ *is the centre of an involutory homology with axis in* Π_0.

The necessity of conditions *(a)* and *(b)* was proved by Hughes [6]

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in 1959.

All finite planes with involutory homologies have odd order.

We use the terminology of Dembowski [2], except that we denote points by capital letters and the coordinate quadrangle $(0, 0)$, $(1, 1)$, (0), (∞) by O, I, X, Y instead of o, e, u, v.

2. Preliminary results and lemmas

If O, I, X, Y is any coordinate quadrangle of a generalized Hall plane Π, with $O, I, X, Y \in \Pi_0$ and $X, Y \in l_\infty$, then the corresponding ternary ring determines a quasifield F which has a sub-quasifield F_0 such that whenever $z \in F \backslash F_0$ and $w \in F$ then $w = z\alpha + \beta$ for exactly one pair $(\alpha, \beta) \in F_0 \times F_0$. F has a group of automorphisms which is transitive on $F \backslash F_0$ and fixes every element of F_0. Consequently, there exist four maps $f, g, h, k : F_0 \to F_0$ with

$$(z\alpha)z = zf(\alpha) + g(\alpha) \quad \text{and} \quad \beta z = zh(\beta) + k(\beta)$$

for all $z \in F \backslash F_0$, $\alpha \in F_0$, $\beta \in F_0$.

In addition to these facts, we shall need the classification of the subgroups of $PSL(2, q)$ by Dickson ([3], Second Part, Chapter XII), and the following two results:

RESULT 1 (Kirkpatrick [7]). If Π is a generalized Hall plane of odd order then F_0 is a field and F is a right vector space (of dimension two) over F_0 with respect to the operations induced by the quasifield structure of F.

RESULT 2 (André [1]). If $\underline{H}_1$ and $\underline{H}_2$ are non-trivial homologies in a finite projective plane, and if $\underline{H}_1$ and $\underline{H}_2$ have the same axis but distinct centres, then the group $\langle \underline{H}_1, \underline{H}_2 \rangle$ contains a non-trivial elation.

Let us assume throughout the remainder of the paper that Π is a generalized Hall plane of odd order, with special line l_∞ and special subplane Π_0; and that all coordinate systems mentioned shall have

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$O, I \in \Pi_0$ and $X, Y \in \underline{\underline{M}} = \{M \mid M \in l_\infty$ and $M \in \Pi_0\}$. Then $\underline{\underline{M}} = \{(\alpha) \mid \alpha \in F_0\} \cup \{(\infty)\}$ and, by Result 1,

$$(z\alpha+\beta)z = z[f(\alpha)+h(\beta)] + g(\alpha) + k(\beta),\ \forall z \in F\backslash F_0,\ \alpha \in F_0,\ \beta \in F_0\ ,$$

and the four maps are additive homomorphisms.

LEMMA 1. *Every elation of* Π *with centre in* $\underline{\underline{M}}$ *is a translation.*

Proof. Suppose there is an elation with centre $M \in \underline{\underline{M}}$ which is not a translation. There is an allowable coordinate system with $Y = M$. The non-trivial (Y, OY)-elations are in one-to-one correspondence with the non-zero $d \in F$ such that

$$x(d+y) = xd + xy,\ \forall x, y \in F\ .$$

If there exists such an element d, then, for some $\beta \in F_0$, $\beta \neq 0$, and $z \in F\backslash F_0$, $z(\beta+z) = z\beta + zz$ and $\beta(\beta+z) = \beta\beta + \beta z$. Now

$$\begin{aligned}\beta(\beta+z) = \beta\beta + \beta z &\Rightarrow (\beta+z)h(\beta) + k(\beta) = \beta\beta + zh(\beta) + k(\beta)\\ &\Rightarrow h(\beta) = \beta\ ,\end{aligned}$$

and

$$\begin{aligned}z(\beta+z) = z\beta + zz &\Rightarrow (\beta+z-\beta)(\beta+z) = z\beta + zf(1) + g(1)\\ &\Rightarrow (\beta+z)f(1) + g(1) + (\beta+z)h(-\beta) + k(-\beta) = z[\beta+f(1)] + g(1)\\ &\Rightarrow f(1) + h(-\beta) = \beta + f(1)\\ &\Rightarrow h(\beta) = -\beta\ .\end{aligned}$$

This contradiction establishes the lemma.

LEMMA 2. Π *has an involutory* (X, OY)*-homology if and only if* $f = k = 0$ *(the zero map).*

Proof. The (X, OY)-homologies are given by $(x, y) \mapsto (xb, y)$ where $b \in N_m$,

$$N_m = \{b \mid a(bc) = (ab)c;\ \forall a, c \in F\}\ .$$

Suppose $(x, y) \mapsto (xz, y)$, where $z \in F\backslash F_0$, is an involutory homology. Then $(xz)z = x$ if $x \in F$, so that $zz = 1$; and $z(z\alpha) = (zz)\alpha = \alpha$ if $\alpha \in F_0$, so that $f = 0$ and $g(\alpha) = \alpha^{-1}$ if $\alpha \neq 0$. But g is an

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additive homomorphism and F_0 is a field of odd order, so we have a contradiction.

Now the multiplicative group of F_0 is cyclic, with unique involution -1 , and it is easily verified that if $f = k = 0$ then $-1 \in N_m$. Suppose, on the other hand, that $-1 \in N_m$, $\alpha \in F_0$ and $z \in F \backslash F_0$. Then

$$\alpha(-z) = (-\alpha)z = -(\alpha z) ,$$

and so $(-z)h(\alpha) + k(\alpha) = -[zh(\alpha)+k(\alpha)]$. Thus $k = 0$, and, if $\beta \in F_0$,

$$\begin{aligned}[(z\alpha+\beta)(-1)]z &= -[(z\alpha+\beta)z] \\ &= -z[f(\alpha)+h(\beta)] - g(\alpha) ,\end{aligned}$$

whereas

$$\begin{aligned}(z\alpha+\beta)(-z) &= [(-z)(-\alpha)+\beta](-z) \\ &= (-z)f(-\alpha) + g(-\alpha) + (-z)h(\beta) ,\end{aligned}$$

that is, $f = 0$ also.

LEMMA 3. *Suppose* Π *has an involutory* (X, OY)*-homology* $\underline{\underline{H}}_{0,\infty}$ *and an involutory* $\big((\alpha), O(\beta)\big)$*-homology* $\underline{\underline{H}}_{\alpha,\beta}$ *, where* α *and* β *are (distinct) non-zero elements of* F_0 *. Then* $\underline{\underline{H}}_{\alpha,\beta}$ *maps* (z) *to* $(-z)$ *, for all* z *in* $F \backslash F_0$ *, and*

$$g(\tau) = \alpha\beta\gamma h^{-1}\left(\gamma^{-1}\tau\right) \quad \textit{for all} \quad \tau \in F_0 ,$$

where $\gamma = 2(\alpha-\beta)^{-1}$.

Proof. $\underline{\underline{H}}_{\alpha,\beta}$ maps Y to (σ) , where $\sigma = \frac{1}{2}(\alpha+\beta)$, and therefore $\underline{\underline{H}}_{\alpha,\beta}$ maps any affine point (x, y) of Π to $\left(x_1, y_1\right)$, where

$$y_1 - x\beta = \left(x_1-x\right)\sigma \quad \text{and} \quad y_1 - y = \left(x_1-x\right)\alpha .$$

The line $y = xz$ is mapped to a line $y = xw$. Substituting $y = xz$ and $y_1 = x_1w$ in the above equations, we derive:

$$x_1w - x_1\sigma = x\beta - x\sigma \quad \text{and} \quad x_1w - x_1\alpha = xz - x\alpha .$$

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These imply:

$$x_1 = x + (x\beta - xz)(\alpha-\sigma)^{-1} \quad \text{and} \quad x_1 w = x_1\sigma + x(\beta-\sigma) .$$

Thus

$$(1) \qquad [x+(x\beta-xz)\gamma]w = (x\beta-xz)\gamma\sigma + x\beta \quad \text{for all} \quad x \in F ,$$

where $\gamma = (\alpha-\sigma)^{-1} = 2(\alpha-\beta)^{-1}$.

There exists an automorphism of F which maps z to w , while fixing each element of F_0 . Suppose this maps w to v . Then we deduce from (1):

$$[x+(x\beta-xw)\gamma]v = (x\beta-xw)\gamma\sigma + x\beta \quad \text{for all} \quad x \in F .$$

But $\underline{\underline{H}}_{\alpha,\beta}$ maps (w) to (z) , so (1) also yields:

$$[x+(x\beta-xw)\gamma]z = (x\beta-xw)\gamma\sigma + x\beta \quad \text{for all} \quad x \in F .$$

It follows that $v = z$, whence $w = -z + \lambda$ for some λ in F_0 .

Now let G be the cyclic group $\langle \underline{\underline{H}}_{0,\infty}\underline{\underline{H}}_{\alpha,\beta} \rangle$. There is a homomorphism $\varphi : G \to (F_0, +)$ which maps any $\underline{\underline{K}}$ in G to the μ in F_0 such that $(z+\mu) = (z)^{\underline{\underline{K}}}$. We shall prove that φ is trivial. Write p equals the characteristic of F_0 , $q = |F_0|$, and suppose that G contains an element $\underline{\underline{K}}$ of order p . Since $\underline{\underline{K}}$ fixes O , and $(q^2-1, p) = 1$, $\underline{\underline{K}}$ fixes another affine point A of Π_0 . It follows that $\underline{\underline{K}}$ is a central collineation with axis OA , when restricted to the subplane Π_0 . We readily conclude from this that either the two homologies have the same centre or they have the same axis. This contradiction shows that $(|G|, |F_0|) = 1$.

Since φ is trivial and $\underline{\underline{H}}_{0,\infty}$ maps (z) to $(-z)$, $\underline{\underline{H}}_{\alpha,\beta}$ also maps (z) to $(-z)$.

Substituting $w = -z$, and restricting x to F_0 , we may simplify equation (1) to:

$$-zh(x+x\beta\gamma) + g(h(x)\gamma) = x\beta(\gamma\sigma+1) - zh(x)\gamma\sigma .$$

It follows that $g\big(h(x)\gamma\big) = x\beta(\gamma\sigma+1)$ for all x in F_0 . Since $\gamma\sigma + 1 = \alpha\gamma$, the lemma is proved.

3. Proof of the theorem

We now assume that each point of $\underline{\underline{M}}$ is the centre of an involutory homology with axis in Π_0 . The theorem is to be proved by considering separately the cases $q = |F_0| \equiv 3 \pmod 4$ and $q \equiv 1 \pmod 4$. We shall refer to the unordered pairs $\{(\alpha), (\beta)\}$ and $\{X, Y\}$ corresponding to the involutory $\big((\alpha), O(\beta)\big)$-, (X, OY)- and (Y, OX)-homologies as "special pairs". These special pairs partition $\underline{\underline{M}}$ and are permuted by each of the homologies (by Result 2 and Lemma 1).

Case 1: $q \equiv 3 \pmod 4$. The permutation induced by $\underline{\underline{H}}_{0,\infty}$ on the set of $\frac{1}{2}(q+1)$ special pairs fixes $\{X, Y\}$ and therefore fixes at least one other special pair $\{(\alpha), (\beta)\}$. The corresponding homologies $\underline{\underline{H}}_{\alpha,\beta}$ and $\underline{\underline{H}}_{\beta,\alpha}$ both interchange X and Y . We can change coordinates so that $\underline{\underline{H}}_{\alpha,\beta}$ becomes $\underline{\underline{H}}_{1,-1}$. By Lemma 3,

$$g(\tau) = -h^{-1}(\tau) \quad \text{for all} \quad \tau \in F_0 .$$

Now consider any $\underline{\underline{H}}_{\alpha,\beta}$ with $\alpha \neq \pm 1$. Since $\underline{\underline{H}}_{1,-1}^{-1}\underline{\underline{H}}_{\alpha,\beta}\underline{\underline{H}}_{1,-1} = \underline{\underline{H}}_{\alpha^{-1},\beta^{-1}}$, we have, from Lemma 3 again

$$g(\tau) = \alpha\beta\gamma h^{-1}\big(\gamma^{-1}\tau\big) = \alpha^{-1}\beta^{-1}(-\alpha\beta\gamma)h^{-1}\big(-\alpha^{-1}\beta^{-1}\gamma^{-1}\tau\big) ,$$

and so $\alpha\beta h^{-1}\big(\gamma^{-1}\tau\big) = h^{-1}\big(\alpha^{-1}\beta^{-1}\gamma^{-1}\tau\big)$. Putting $\tau = \gamma\rho$, we deduce that

$$h^{-1}\big(\alpha^{-1}\beta^{-1}\rho\big) = \alpha\beta h^{-1}(\rho) \quad \text{for all} \quad \rho \in F_0 .$$

It follows that either $\alpha\beta$ is in the prime subfield $GF(p)$ of F_0 , or h^{-1} induces an involutory automorphism of the extension of $GF(p)$ by $\alpha\beta$. Since $q \equiv 3 \pmod 4$, the first alternative is the only possibility. Thus $\alpha\beta = \pm 1$, and $g(\gamma) = \pm\gamma h^{-1}(1) = \pm\gamma$ (since $1.z = z.1$), whence $h^{-1}(\gamma) = \mp\gamma$. But γ ranges over exactly half of the non-zero elements of

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F_0 , since $\gamma = 2(\alpha-\beta)^{-1}$ and if $\gamma' = \gamma$ then $\alpha'\beta' = \alpha\beta$ (Lemma 3). So we may choose a basis for F_0 (as a vector space over $GF(p)$) from among the values taken by γ , and relative to this basis h has a diagonal matrix whose diagonal entries are ± 1 .

If one or more of these entries is -1 , then a contradiction results. For $h(1) = 1$ implies that at least one entry is $+1$, and so h has only $p^i + p^j - 1$ eigenvectors, for some i, j with $p^{i+j} = p^n = q$, $i > 0$, $j > 0$. But h has at least $\frac{1}{2}(p^n-1)$ eigenvectors, so $p^i + p^j - 1 \geq \frac{1}{2}(p^n-1)$. Simple calculations now show that (since p is an odd prime) $q = 9$, contradicting $q \equiv 3 \pmod 4$.

Thus $h(\tau) = \tau$ and $g(\tau) = -\tau$ for all τ in F_0 , that is F is a Hall system.

Case 2: $q \equiv 1 \pmod 4$. We show first that $\underline{H}_{0,\infty}$ does not interchange the two points of any special pair. Suppose the contrary. Then choose coordinates so that $\{(1), (-1)\}$ is a special pair. Let ε be a square root of -1 in F_0 , and $\{(\varepsilon), (\beta)\}$ the corresponding special pair. Each of $\underline{H}_{0,\infty}$ and $\underline{H}_{1,-1}$ maps this special pair to a special pair. So $\{(-\varepsilon), (-\beta)\}$ and $\left\{(-\varepsilon), (\beta^{-1})\right\}$ are special pairs, that is $\beta^2 = -1$, and $\{(\varepsilon), (-\varepsilon)\}$ is a special pair. By Lemma 3,

$$-h^{-1}(\tau) = g(\tau) = -\varepsilon h^{-1}(\varepsilon\tau) \quad \text{for all} \quad \tau \in F_0 .$$

Thus $h^{-1}(\varepsilon) = -\varepsilon$, and so ε does not lie in $GF(p)$, and h^{-1} induces an involutory automorphism of the extension of $GF(p)$ by ε . This field contains an element ρ such that $h(\rho)\rho = -1$. But

$$(z+\rho)z = g(1) + zh(\rho) = -1 - z\rho^{-1} = (z+\rho)\left(-\rho^{-1}\right)$$

if $z \in F\backslash F_0$. This contradiction proves our original assertion.

It follows that the collineation group K generated by our involutory homologies is transitive on the set of $\frac{1}{2}(q+1)$ special pairs.

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Let K^* be the group induced on the points of $\underline{\underline{M}}$ by K, and let H^* be the subgroup of K^* generated by all the products of two non-trivial elements of K^*. Then K^* is a subgroup of $\mathrm{PSL}(2, q)$, since $q \equiv 1 \pmod 4$. Also $[K^* : H^*] = 2$ and $\frac{1}{2}(q+1) \,\Big|\, |K^*|$. But $\frac{1}{2}(q+1)$ is odd, so $q + 1 \,\Big|\, |K^*|$ and H^* is transitive on the set of special pairs.

Now $(|K^*|, q) = 1$, by an argument used in the proof of Lemma 3; also $K^* \le \mathrm{PSL}(2, q)$, and $q + 1 \,\Big|\, |K^*|$. So either K^* is dihedral of order $q + 1$ or K^* is isomorphic to A_4, S_4 or A_5 (Dickson [3]). Since $q \equiv 1 \pmod 4$, K^* is dihedral of order $q + 1$, and H^* is cyclic.

The situation, then, is that our involutory homologies, restricted to the affine portion of Π_0, are the reflections in an orthogonal group $O(2, q)$, and therefore $\alpha\beta$ has the same value for all $\underline{\underline{H}}_{\alpha,\beta}$. Since $g(\gamma) = \alpha\beta\gamma$ for all γ corresponding to homologies $\underline{\underline{H}}_{\alpha,\beta}$, an argument used in Case 1 shows that $g(\tau) = \alpha\beta\tau$ for all τ in F_0. But $g(\tau) = \alpha\beta\gamma h^{-1}\left(\gamma^{-1}\tau\right)$, and so $h(\tau) = \tau$ for all τ in F_0. Thus F is once again a Hall system, and the theorem is proved.

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Department of Pure Mathematics,
University of Sydney,
Sydney,
New South Wales.