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SUBRINGS OF $k[X, Y]$ GENERATED BY MONOMIALS

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1. Introduction. In this paper we study subrings A of $B = k[X, Y]$ generated by monomials over k. If A is normal and $A \subset B$ integral, we can completely characterize A. If $\dim A = 2$, we show that A is isomorphic to a subring A' of B generated by monomials with $A' \subset B$ integral. The author became interested in these rings while studying projective modules over subrings of $k[X, Y]$. For some applications, see [**1**].

In Section 4 we calculate $\mathrm{Cl}(A)$, the divisor class group of A. We also show that $\tilde{G}_0(A)$ is precisely $\mathrm{Cl}(A)$.

Rings generated by monomials may also be studied by considering the semigroup of the exponents of their monomials. For example, see [**4**].

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2. Subrings of $k[X, Y]$ generated by monomials. k will always denote a field, but in most cases it is clear that k could be any normal domain.

We first note that if A is a subring of $k[X, Y]$ generated by monomials, then A is a homogeneous or graded ring with the natural grading it inherits from $k[X, Y]$. In fact, A is also bihomogeneous, that is, $\sum a_{ij}X^iY^j$ is in A if and only if each $a_{ij}X^iY^j$ is in A. We also note that if A is any bihomogeneous subring of $k[X, Y]$ containing k, then necessarily A is generated by monomials over k.

First we study the case when $A \subset k[X, Y]$ is integral. Later (Proposition 2.8) we will see that we can always reduce to this case when (Krull) $\dim A = 2$. The following lemma is obvious.

LEMMA 2.1. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials, then $A \subset B$ is integral if and only if X^m and Y^n are in A for some positive m and n.*

If some power of X and Y is in A, clearly A is affine and $A \subset k[X, Y]$ is integral. Of course not all subrings of $k[X, Y]$ generated by monomials are affine; for example, consider $A = k[\{X^nY\}_{n=0}^{\infty}]$.

LEMMA 2.2. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Let X^m and Y^n be the smallest positive powers of X and Y in A. Then*

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(1) *if X^a and Y^b are in A, then $m|a$ and $n|b$.*

Also assume that $k[X^m, Y^n] \subsetneq A$, and let X^iY^j be in A with $ij \neq 0$ and i as small as possible. Then

(2) $0 < i < m$ *and* $i|m$;

(3) *if X^cY^d is in A, then $i|c$.*

Proof. We prove (1); the proofs of (2) and (3) are similar. Let X^a be in A, then $a = mq + r$ with $0 \leqq r < m$. Clearly $X^r = (X^a)(X^m)^{-q}$ is in the quotient field of A. But X^r is integral over A, and A is normal, so X^r is in A. m was chosen to be minimal, so $r = 0$, and thus $m|a$.

Let A be as in Lemma 2.2 and X^iY^j in A with i as small as possible. As above, by dividing by powers of Y^n, we can also assume that $0 < j < n$. So there is a X^iY^j in A with

(1) $0 < i < m$,

(2) $0 < j < n$, and

(3) i as small as possible.

It is easy to see that j is uniquely determined. These three monomials X^m, Y^n, and X^iY^j completely determine A.

PROPOSITION 2.3. *Let A be as above. Then $A = k(X^m, X^iY^j, Y^n) \cap k[X, Y]$.*

Proof. Let K be the quotient field of A and $R = k(X^m, X^iY^j, Y^n) \cap k[X, Y]$. $k(X^m, X^iY^j, Y^n) \subset K$, so $R = k(X^m, X^iY^j, Y^n) \cap k[X, Y] \subset K \cap k[X, Y]$. But A is normal and $A \subset k[X, Y]$ integral, so $K \cap k[X, Y] = A$. Thus $R \subset A$. Conversely, suppose that X^aY^b is in A. By arguments similar to Lemma 2.2, clearly X^aY^b is in $k(X^m, X^iY^j, Y^n)$. But thus $A \subset R$ since A is generated by monomials, so $A = R$.

COROLLARY 2.4. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Let X^m and Y^n be the least positive powers of X and Y in A. Suppose that $k[X^m, Y^n] \subsetneq A$, then let X^iY^j be in A with $0 < i < m$, $0 < j < n$, and i as small as possible. Then*

$$A = k[X^m, X^iY^j, X^{2i}Y^{\overline{2j}}, \ldots, X^{(q-1)i}Y^{\overline{(q-1)j}}, Y^n]$$

where $m = qi$ and overscoring denotes mod n.

Remarks. (1). We could carry out the above arguments with X^iY^j in A with $0 < i < m$, $0 < j < n$, and j as small as possible. Clearly we would obtain similar results.

(2) If gcd $(m, n) = 1$, then $A = k[X^m, Y^n]$. For if not, then there exists an X^iY^j in A with $0 < i < m$, $0 < j < n$, and i as small as possible. $i|m$, say $m = qi$, so $(X^iY^j)^q = X^mY^{jq}$ is in A. Thus Y^{jq} is in A, so $n|jq$. But gcd $(m, n) = 1$, so $n|j$, which is a contradiction.

(3) Assume that $m = n$ and $i = 1$, so $A = k[X^n, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$. By the usual arguments, gcd $(j, n) = 1$. Clearly any such j will work. So for fixed $m = n$ and $i = 1$, j can take on precisely $\phi(n)$ values, where ϕ is the Euler phi function. Distinct values of j may define isomorphic

subrings; but for distinct values of n, none of these rings are isomorphic (Theorem 4.4].

Our next result shows that we may actually assume, up to isomorphism, that $m = n$ and $i = 1$.

THEOREM 2.5. *Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Then A is isomorphic to either $B = k[X, Y]$ or $A' = k[X^n, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and* gcd $(j, n) = 1$.

Proof. If A is not isomorphic to $k[X, Y]$, then $A = k[X^m, X^iY^j, X^{2i}Y^{\overline{2j}}, \ldots, X^{(q-1)i}Y^{\overline{(q-1)j}}, Y^n]$ by Corollary 2.4. By Lemma 2.2, i divides all powers of X in any monomial of A. By replacing X by $X^{1/i}$, we may assume $i = 1$; thus $A = k[X^m, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{m-1}Y^{\overline{(m-1)j}}, Y^n]$. Let b be the smallest power of Y which appears in any monomial of A. Then b divides all powers of Y in any monomial of A. By replacing Y by $Y^{1/b}$, we may thus assume that both XY^j and X^aY are in A. Thus $(XY^j)^n$ is in A, so $m|n$. Similarly $n|m$, so $m = n$.

Remark. Let A be as above. Assume that X^m and Y^l are the lowest powers of X and Y in A. Also let X^iY^j and X^aY^b be the monomials in A for which $0 < i, a < m, 0 < j, b < n$, and i and b are as small as possible. Then in Theorem 2.5, $n = m/i = l/b$.

So far we have characterized the affine normal subrings A of $B = k[X, Y]$ with $A \subset B$ integral. We next consider the case when A is not normal. Let $\bar{A}$ be the integral closure of A in its quotient field K. Since A is generated by monomials, $K = k(X^m, X^iY^j, Y^n)$ where m and n are the least positive powers of X and Y in K respectively, and $0 \leqq i < m$ and $0 \leqq j < n$ with i as small as possible. Clearly $\bar{A} = k(X^m, X^iY^j, Y^n) \cap k[X, Y]$, so $\bar{A}$ is generated by monomials.

Let A and $\bar{A}$ be as above, and let $I = \{x \in \bar{A} | x\bar{A} \subset A\}$ be the conductor of $\bar{A}/A$. A and $\bar{A}$ have the same quotient field, and $\bar{A}$ is a finitely generated A-module; so it is well-known that $I \neq 0$. In fact, using a similar proof, it is easy to see that I actually contains a nonzero monomial X^aY^b. However, this also follows from the next lemma which shows that I is actually a bihomogeneous ideal.

LEMMA 2.6. *Let $A = \oplus A_{(m,n)}$ and $B = \oplus B_{(m,n)}$ be bihomogeneous commutative rings with $A \subset B$ and each $A_{(i,j)} \subset B_{(i,j)}$. Then $I = \{x \in B | xB \subset A\}$ is a bihomogeneous ideal.*

Proof. Let $\sum a_{ij}$ be in I, we must show that each a_{ij} is also in I. If x_{mn} is in B, then $(\sum a_{ij})x_{mn}$ is in A, and hence each $a_{ij}x_{mn}$ is in $A \cap B_{(i+m,j+n)} = A_{(i+m,j+n)}$. Thus a_{ij} is in I, so I is bihomogeneous.

Combining the previous remarks and the above lemma, we have proved the following proposition.

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PROPOSITION 2.7. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. Then $\bar{A}$, the integral closure of A, is also an affine subring of B generated by monomials. The conductor ideal is bihomogeneous and thus contains a nonzero monomial.*

Clearly not all affine subrings of $B = k[X, Y]$ generated by monomials have $A \subset B$ integral. By Lemma 2.1, this happens if and only if some power of X and Y is in A. If A is affine and contained in B, then $\dim A = \text{tr deg}_k A \leqq 2$. If $\dim A = 0$, then of course $A = k$. If $\dim A = 2$, we show that we can change variables so that A is isomorphic to an affine subring A' of B generated by monomials with $A' \subset B$ integral. Clearly this can happen only when $\dim A = 2$ because $A \subset B$ integral implies $\dim A = \dim B = 2$. If $\dim A = 1$, then we can change variables so that A is isomorphic to an affine subring A'' of $k[T]$ generated by monomials.

Let A be an affine subring of $B = k[X, Y]$ generated by monomials. Say $A = k[X^{a_1}Y^{b_1}, \ldots, X^{a_n}Y^{b_n}]$, and let $S = \{X^{a_1}Y^{b_1}, \ldots, X^{a_n}Y^{b_n}\}$. Pick X^aY^b in S with a/b maximum and X^cY^d in S with d/c maximum. (We define $a/0 > i/j$ if $j \neq 0$ or $j = 0$ and $a > i$.) We note that for X^iY^j in A, $aj \geqq bi$ and $di \geqq cj$.

Define a k-homomorphism $\phi : k[X, Y] \rightarrow k(X, Y)$ by

$$\phi(X) = X^d/Y^b \quad \text{and} \quad \phi(Y) = Y^a/X^c.$$

Then $\phi(X^aY^b) = X^{ad-bc}$, $\phi(X^cY^d) = Y^{ad-bc}$, and $\phi(X^iY^j) = X^{di-cj}Y^{aj-bi}$. Thus $\phi(A) \subset k[X, Y]$ and $\phi(A)$ is generated by monomials.

If $ad - bc = n > 0$, then X^n and Y^n are in $\phi(A)$, so $\phi(A) \subset k[X, Y]$ is integral. Thus $\dim \phi(A) = 2$, so $\phi|A$ is injective.

Next we show that $n = 0$ if and only if $\dim A \leqq 1$. If $\dim A \leqq 1$, then $\dim \phi(A) \leqq 1$, so $\phi(A) \subset B$ is not integral, and thus $n = 0$. Conversely suppose that $n = 0$. For any X^iY^j in A, $a/b \geqq i/j$ and $d/c \geqq j/i$, so $bi = aj$. Thus $(X^iY^j)^b = (X^aY^b)^j$. We define a k-homomorphism $\psi : A \rightarrow k[T]$ by $\psi(X^iY^j) = T^i$. By the above remarks ψ is injective, so $\dim A \leqq 1$.

PROPOSITION 2.8. *Let A be an affine subring of $B = k[X, Y]$ generated by monomials. If* $\dim A = 1$, *then A is isomorphic to an affine subring of $k[T]$ generated by monomials. If* $\dim A = 2$, *then A is isomorphic to an affine subring A' of B generated by monomials with $A' \subset B$ integral.*

Examples. (1) Let $A = k[X^2Y^2, X^3Y^3]$, $\dim A = 1$, so define $\psi : A \rightarrow k[T]$ by $\psi(X^2Y^2) = T^2$ and $\psi(X^3Y^3) = T^3$. Thus A is isomorphic to $k[T^2, T^3] \subset k[T]$.

(2) Let $A = k[XY, XY^{n-1}, X^{n-1}Y]$, $\dim A = 2$, so define $\phi : A \rightarrow k[X, Y]$ by $\phi(X) = X^{n-1}/Y$ and $\phi(Y) = Y^{n-1}/X$. Then $\phi(XY) = X^{n-2}Y^{n-2}$, $\phi(XY^{n-1}) = Y^{n(n-2)}$, and $\phi(X^{n-1}Y) = X^{n(n-2)}$. So A is isomorphic to $k[X^{n(n-2)}, X^{n-2}Y^{n-2}, Y^{n(n-2)}]$ which is isomorphic to $k[X^n, XY, Y^n] \subset k[X, Y]$.

3. Rings of invariants. Affine normal subrings A of $B = k[X, Y]$ generated by monomials arise naturally in two ways. First, let G be a finite subgroup of automorphisms of the form

$$\theta : \begin{cases} X \to aX \\ Y \to bY, \quad a, b \in k. \end{cases}$$

Then $A = B^G = \{f \in B | \theta(f) = f, \text{ for all } \theta \text{ in } G\}$ is an affine normal subring of B generated by monomials with $A \subset B$ integral. We give three examples.

Examples. (1) Let $\omega \in k$ be a primitive nth root of unity with gcd $(n, \text{char } k) = 1$. Define θ in $\text{Aut}_k(B)$ by $\theta(X) = \omega X$ and $\theta(Y) = \omega Y$; then $A = B^{\langle\theta\rangle} = k[X^n, XY^{n-1}, \ldots, X^{n-1}Y, Y^n]$.

(2) Let ω be as in (1), and define θ by $\theta(X) = \omega X$ and $\theta(Y) = \omega^{-1}Y$; then $A = B^{\langle\theta\rangle} = k[X^n, XY, Y^n]$.

(3) Let p be a prime not equal to char k and $\omega \in k$ a primitive pth root of unity. For fixed $1 \leqq i \leqq p - 1$, define θ by $\theta(X) = \omega X$ and $\theta(Y) = \omega^i Y$. Let j be the least positive integer such that $p | 1 + ij$. Such a j exists, and $0 < j < p$ because gcd $(i, p) = 1$. Then $A = B^{\langle\theta\rangle} = k[X^p, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{p-1}Y^{\overline{(p-1)j}}, Y^p]$.

Next, let k be a field with char $k = p \neq 0$ and $D : B \to B$ a k-derivation of the form $D(X) = aX$ and $D(Y) = bY$ with $a, b \in k$. Then $A = \ker D \subset B = k[X, Y]$ is an affine normal subring of B generated by monomials with $A \subset B$ integral. We note that $D(X^iY^j) = (ia + jb)X^iY^j$ and $k[X, Y]^p \subset A$. We give three examples.

Examples. (1) Let k be as above and D defined by $D(X) = X$ and $D(Y) = Y$, then $A = \ker D = k[X^p, XY^{p-1}, \ldots, X^{p-1}Y, Y^p]$.

(2) Let D be defined by $D(X) = X$ and $D(Y) = -Y = (p - 1)Y$, then $A = \ker D = k[X^p, XY, Y^p]$.

(3) Let i be a fixed integer with $1 \leqq i \leqq p - 1$. Define D by $D(X) = X$ and $D(Y) = iY$, then $A = \ker D = k[X^p, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{p-1}Y^{\overline{(p-1)j}}, Y^p]$ where j is the least positive integer such that $p | 1 + ij$.

These examples lead one to ask when an affine normal subring A of $B = k[X, Y]$ generated by monomials is either the ring of invariants of a k-automorphism of finite order or the kernel of a k-derivation of B. Of course, it is necessary to have $A \subset B$ integral. But there are still many subrings A which are not of these two types. We give three examples.

Examples. (1) k may not contain the necessary roots of unity. For example, $A = \mathbf{R}[X^4, XY, Y^4]$.

(2) If char $k = p \neq 0$, then for any k-derivation D of B, $k[X, Y]^p \subset A = \ker D$. For example $A = \mathbf{Z}/2\mathbf{Z}[X^4, XY, Y^4]$ cannot be obtained in this manner.

(3) Let $A = \mathbf{C}[X^4, X^2Y^2, Y^4]$, then A is not $B^{\langle\theta\rangle}$ for any θ in $\text{Aut}_{\mathbf{C}}(B)$. If such a θ exists, then necessarily $\theta_n(X) = \omega X$, $\theta_n(Y) = \omega^n Y$ where $\omega \in \mathbf{C}$ is a

primitive 4th root of unity and $n = 1$ or $n = 3$. If $n = 1$, then $B^{\langle\theta_1\rangle} = \mathbf{C}[X^4, XY^3, X^2Y^2, X^3Y, Y^4]$; while for $n = 3$, $B^{\langle\theta_3\rangle} = \mathbf{C}[X^4, XY, Y^4]$. Note that $A = B^G$ where G is generated by θ_1 and θ_3.

However, these are essentially the only types of exceptions. In many cases, even though A is not a ring of invariants or the kernel of a derivation, we can show that A is isomorphic to a ring of the desired type.

PROPOSITION 3.1. *Let k be a field with* char $k = p \neq 0$ *and A an affine normal subring of $B = k[X, Y]$ generated by monomials. If $k[X^p, Y^p] \subset A$ and A is not isomorphic to B, then there exists a k-derivation D of B with $A =$* ker D.

Proof. If X^n is in A, then $p|n$ since p is prime. Similarly for powers of Y, so by Corollary 2.4, $A = k[X^p, X^iY^j, X^{2i}Y^{\overline{2j}}, \ldots, X^{p-1}Y^{\overline{(p-1)j}}, Y^p]$ where $0 < i$, $j < p$ and $i|p$. p is prime, so $i = 1$. Also gcd $(j, p) = 1$, so there are integers c and d with $dp = 1 + cj$. Define a k-derivation D of B by $D(X) = X$ and $D(Y) = cY$. Then $D(XY^j) = (1 + cj)XY^j = 0$ since $p|1 + cj$ and char $k = p$. So $XY^j \in$ ker D, and thus $A =$ ker D.

The next proposition may be proved in a similar manner.

PROPOSITION 3.2. *Let k be a field with* char $k = p$ *and q a prime distinct from p. Assume that k contains a primitive* qth *root of unity. Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $k[X^q, Y^q] \subset A$ and A not isomorphic to B. Then there exists a k-automorphism θ of B of finite order with $A = B^{\langle\theta\rangle}$.*

PROPOSITION 3.3. *Let k be a field with* char $k = p$ *and $A = k[X^m, X^iY^j, X^{2i}Y^{\overline{2j}}, \ldots, Y^n]$. Let $l = m/i$ and assume that* gcd $(l, p) = 1$ *and that k contains a primitive* lth *root of unity. Then there exists a k-automorphism θ of $B = k[X, Y]$ of finite order with A isomorphic to $B^{\langle\theta\rangle}$.*

Proof. By Theorem 2.5 A is isomorphic to $A' = k[X^l, XY^q, X^2Y^{\overline{2q}}, \ldots, X^{l-1}Y^{\overline{(l-1)q}}, Y^l]$ where gcd $(l, q) = 1$. There exist integers c and d so that $cl = 1 + dq$. Let $\omega \in k$ be a primitive lth root of unity. Define θ by $\theta(X) = \omega X$ and $\theta(Y) = \omega^dY$. X^l and Y^l are in $B^{\langle\theta\rangle}$, and these are the smallest such positive powers because gcd $(l, d) = 1$. $\theta(XY^q) = \omega^{1+dq}XY^q = XY^q$ because $l|1 + dq$, so $XY^q \in B^{\langle\theta\rangle}$. Thus X^l, Y^l, and XY^q are in $B^{\langle\theta\rangle}$, so $B^{\langle\theta\rangle} = A'$.

4. The divisor class group of A. Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials. Then A is a Krull domain. In this section we calculate Cl (A), the divisor class group of A.

Let A be a Krull domain with quotient field K. Div (A) is the free abelian group on the height one prime ideals of A. Prin (A) is the subgroup of Div (A) generated by $\sum_P V_P(x)(P)$ for $0 \neq x \in K$. Cl (A), the divisor class group of A, is defined to be Div (A)/Prin (A).

There is another description of Cl (A) which we will use. A fractional ideal I is *divisorial* if it is an intersection of principal fractional ideals. Any prime ideal

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of height one is divisorial. A fractional ideal I is contained in a minimal divisorial ideal $\bar{I} = A : (A : I)$ where $A : I = \{x \in K | xI \subset A\}$. This defines an equivalence relation $\backsim$ on the set of fractional ideals of A with $I \backsim J$ if and only if $\bar{I} = \bar{J}$. Div (A) is just the abelian group of equivalence classes of fractional ideals with the usual multiplication. Prin (A) is then the subgroup of Div (A) generated by principal ideals.

If A is isomorphic to $k[X, Y]$, then A is factorial, so Cl $(A) = 0$. Otherwise $A = k[X^m, X^iY^j, X^{2i}Y^{\overline{2j}}, \ldots, X^{(q-1)i}Y^{\overline{(q-1)j}}, Y^n]$ where $m = qi$. We show that Cl (A) is isomorphic to $\mathbf{Z}/q\mathbf{Z}$.

Special cases of A have been calculated in other ways, and they depend on the field k. Suppose first that $A = B^G$ where G is a finite subgroup of $\mathrm{Aut}_k(B)$. If no height one prime ideal of B is ramified over A, then A is isomorphic to $H^1(G, B^*)$ [**3**, p. 82]. In our case, $B^* = k^*$, so if G is a finite cyclic group of order n, then Cl (A) is isomorphic to $\mathbf{Z}/n\mathbf{Z}$. For example, if k contains a primitive nth root of unity, then Cl $(k[X^n, XY, Y^n])$ is isomorphic to $\mathbf{Z}/n\mathbf{Z}$.

If char $k = p \neq 0$ and $A = \ker D$, where D is a k-derivation of B, then Cl (A) is isomorphic to L/L'. Here L and L' are the logarithmic derivatives, $L' = \{D(t)/t | t \in B^*\}$ and $L = \{D(t)/t \in B | t \in K^*\}$. In our special case, $B^* = k^*$, so $L' = 0$. Thus Cl (A) is isomorphic to L, which is isomorphic to $\mathbf{Z}/p\mathbf{Z}$ [**5**, p. 61].

Waterhouse [**7**] has combined these two theories using the cohomology theory of Hopf algebras. He has shown, for example, that if char $k = p \neq 0$, then $\mathrm{Cl}(k[X^{p^n}, XY, Y^{p^n}])$ is isomorphic to $\mathbf{Z}/p^n\mathbf{Z}$. Note that none of the above methods is applicable for calculating Cl $(\mathbf{Z}/2\mathbf{Z}[X^6, XY, Y^6])$, for example.

If $A = A_0 \oplus A_1 \oplus \ldots$ is a homogeneous Krull domain, it is well-known [**3**, p. 42] that Cl (A) is isomorphic to HDiv (A)/HPrin (A) where HDiv (A) is generated by the homogeneous prime divisorial ideals of A and HPrin (A) = HDiv $(A) \cap$ Prin (A). If $A = \oplus A_{(i,j)}$ is a bihomogeneous Krull domain, we show that Cl (A) is isomorphic to BDiv (A)/BPrin (A) where BDiv (A) is the subgroup generated by bihomogeneous prime divisorial ideals and BPrin (A) = BDiv $(A) \cap$ Prin (A).

LEMMA 4.1. *Let $A = \oplus A_{(i,j)}$ be a bihomogeneous ring and P a prime ideal of A. Let I be the ideal generated by the bihomogeneous elements of P. Then I is prime.*

Proof. Suppose that xy is in I, but neither x nor y is in I. Write $x = \sum a_{ij}$ and $y = \sum b_{ij}$, and assume that a_{ij} and b_{lk} are the first terms not in I. But then $a_{ij}b_{lk}$ is in $I \subset P$, so say a_{ij} is in P. Thus a_{ij} is also in I, a contradiction; so I must be prime.

LEMMA 4.2. *Let A be a bihomogeneous Krull domain and S the multiplicatively closed set of bihomogeneous elements of A. Then $S^{-1}A$ is factorial.*

Proof. Let $A = \oplus A_{(i,j)}$, then $S^{-1}A = \oplus_{i,j \in \mathbf{Z}} (S^{-1}A)_{(i,j)}$ where $(S^{-1}A)_{(i,j)} = \{x_{(m,n)}/y_{(l,k)} | m - l = i, n - k = j\}$. Clearly $F = (S^{-1}A)_{(0,0)}$ is a field. We may assume that the bigrading of A is not trivial, so there exists a bihomo-

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geneous U in $S^{-1}A$ of degree $(0, \alpha)$ with $\alpha > 0$ as small as possible. Let V in $S^{-1}A$ be a bihomogeneous element of degree (β, γ) with $\beta, \gamma > 0$ and (β, γ) as small as possible with respect to the lexicographic order on $\mathbf{N} \times \mathbf{N}$. Clearly U and V are transcendental over F, so $F[U, U^{-1}, V, V^{-1}] \subset S^{-1}A$. Let t in $S^{-1}A$ be bihomogeneous of degree (i, j); we may assume $i > 0$. Write $i = q\beta + r$ with $0 \leqq r < \beta$, so $V^{-q}t$ has degree $(r, j - q\gamma)$. Thus $r = 0$, so the degree of $V^{-q}t$ is $(0, j - q\gamma)$. But then for a suitable multiple of U, namely $p = (j - q\gamma)/\alpha$, $U^{-p}V^{-q}t \in F$. Hence $t \in F[U, V, U^{-1}, V^{-1}]$, so $S^{-1}A = F[U, V, U^{-1}, V^{-1}]$, which is factorial.

THEOREM 4.3. *Let A be a bihomogeneous Krull domain. Then* Cl (A) *is isomorphic to* BDiv $(A)/$BPrin (A).

Proof. Let $A = \oplus A_{(i,j)}$ and S be the multiplicatively closed set generated by the bihomogeneous elements of A. By Nagata's theorem [**3**, p. 36], there is a short exact sequence

$$0 \to \ker f \to \mathrm{Cl}(A) \xrightarrow{f} \mathrm{Cl}(S^{-1}A) \to 0$$

where $\ker f$ is generated by the prime divisorial ideals of A that meet S. By Lemma 4.1 these are precisely the height one prime ideals which are bihomogeneous. By Lemma 4.2, $S^{-1}A$ is factorial, so Cl $(S^{-1}A) = 0$. Thus Cl $(A) = \ker f$ is isomorphic to BDiv $(A)/$BPrin (A).

Remark. Theorem 4.3 clearly holds for more general gradings of A.

Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials with $A \subset B$ integral. We may assume that $A = k[X^n, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and gcd $(j, n) = 1$. Let P be a prime bihomogeneous ideal of height one. Then some X^aY^b is in P. $A \subset B$ is integral, so P can be lifted to a prime ideal $\bar{P}$ of B of height one. But $X^aY^b \in \bar{P}$, so $\bar{P}$ is either XB or YB. Thus $P = \bar{P} \cap A$ is either

$$P_1 = (X^n, XY^j, \ldots, X^{n-1}Y^{\overline{(n-1)j}}) \quad \text{or} \quad P_2 = (XY^j, \ldots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n).$$

So BDiv (A) is the free abelian group on (P_1) and (P_2). Let [ ] denote the image of an element in BDiv $(A)/$BPrin (A).

THEOREM 4.4. *Let k be a field and $A = k[X^n, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ where $0 < j < n$ and* gcd $(j, n) = 1$. *Then* Cl (A) *is isomorphic to* $\mathbf{Z}/n\mathbf{Z}$.

Proof. It is sufficient to show that
(1) $[P_1]^n = [P_2]^n = 0$,
(2) $[P_1][P_2]^j = 0$, and
(3) for $0 < m < n$, neither $[P_1]^m$ nor $[P_2]^m$ is 0.

Proof of (1). We show that $[P_1]^n = 0$. Let $I = P_1^n$, we show that $A : (A : I) = X^nA$. It is sufficient to show that any principal fractional ideal which contains I also contains X^nA. For $I \subset X^nA$ and $A : (A : I)$ is the intersection of all principal fractional ideals which contain I. Let $(f/g)A$ be a fractional ideal containing I. Since I is bihomogeneous we may assume $f = X^aY^a$ is in A. X^{n^2} is in I, so $X^{n^2} = (X^aY^b)(h/g)$ for some h in A. Thus $g = x^{a-n^2}Y^bh$, so $f/g = X^{n^2}/h$, and hence $I \subset (X^{n^2}/h)A$. Also X^nY^{jn} is in A, so $X^nY^{jn} = X^{n^2}(h'/h)$ for some h' in A. Thus $Y^{jn}|h'$, so $X^n = (X^{n^2}/h)(h'/Y^{jn})$ is in $(X^{n^2}/h)A = (f/g)A$. So $X^nA \subset (f/g)A$ and the proof is complete.

Proof of (2). Let $I = P_1P_2{}^j$, we show that $A : (A : I) = XY^jA$. It is sufficient to show that $I \subset (X^aY^b/h)A$ with X^aY^b and h in A implies that XY^j is also in $(X^aY^b/h)A$. XY^jY^{jn} is in I, so $hXY^{j+jn} = X^aY^bh'$ for some h' in A. Thus $X^aY^b/h = XY^{j+jn}/h'$. For some c, $X^cY \in P_2$ (see the remark after Theorem 2.5), so $X^{n+jc}Y^j$ is in I. Hence $h'X^{n+jc}Y^j = XY^{j+jn}g$ for some g in A. But thus $h'X^{n+jc} = XY^{jn}g$, so $Y^{jn}|h'$. Hence $(X^aY^b/h)A = (XY^j/h'')A$ for some h'' in A, so XY^j is in $(X^aY^b/h)A$.

Proof of (3). Let $I = P_1{}^m$ with $0 < m < n$, we show that $A : (A : I)$ is not principal. Clearly $A : (A : I) \subsetneq A$ because $I \subset P_1 \subset (X^n/X^{n-1}Y^{(n-1)j})A$ and $A \not\subset (X^n/X^{n-1}Y^{(n-1)j})A$. So it is sufficient to show that A is the only principal ideal of A containing I. But if $I \subset fA$ with $f = X^aY^b$ homogeneous, clearly $f = 1$, so (3) is proved.

COROLLARY 4.5. *Let R be factorial and $A = R[X^n, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ with $0 < j < n$ and* gcd $(j, n) = 1$. *Then* Cl (A) *is isomorphic to* $\mathbf{Z}/n\mathbf{Z}$.

Proof. If R is any Krull domain with quotient field K and $S = R\backslash 0$, then again by Nagata's theorem there is a short exact sequence

$$0 \to \ker f \to \mathrm{Cl}(A) \xrightarrow{f} \mathrm{Cl}(S^{-1}A) \to 0.$$

$\ker f$ is generated by the height one prime ideals of A which meet S. But these correspond to the height one primes of R, so $\ker f$ is isomorphic to Cl (R). Clearly $S^{-1}A = K[X^n, XY^j, \ldots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$, so Cl $(S^{-1}A)$ is isomorphic to $\mathbf{Z}/n\mathbf{Z}$. If R is factorial, then Cl $(R) = 0$, so Cl (A) is isomorphic to $\mathbf{Z}/n\mathbf{Z}$.

Theorem 4.4 may be used to calculate $G_0(A)$. Recall that $G_0(A)$ is the Grothendieck group with generators $[M]$ for isomorphism classes of finitely generated A-modules and relations $[M] = [M'] + [M'']$ for each short exact sequence $0 \to M' \to M \to M'' \to 0$. $\tilde{G}_0(A)$ is $G_0(A)$ modulo the subgroup generated by $[A]$. If A is a domain, then $G_0(A)$ is naturally isomorphic to $\mathbf{Z} \oplus \tilde{G}_0(A)$. When A is a Krull domain, there is a natural epimorphism $\tilde{G}_0(A) \to$ Cl (A) [**2**, p. 500]. In general this map is not an isomorphism.

Let A be an affine normal subring of $B = k[X, Y]$ generated by monomials. In [**1**] it is shown that all finitely generated projective A-modules are free, so $K_0(A)$ is just $\mathbf{Z}$. Here we show that $\tilde{G}_0(A)$ is isomorphic to Cl (A), so $G_0(A)$ is $\mathbf{Z} \oplus$ Cl (A).

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THEOREM 4.6. *Let* $A = k[X^n, XY^j, X^2Y^{\overline{2j}}, \ldots, X^{n-1}Y^{\overline{(n-1)j}}, Y^n]$ *where* $0 < j < n$ *and* gcd $(j, n) = 1$. *Then* $\tilde{G}_0(A)$ *is isomorphic to* Cl (A) (*isomorphic to* $\mathbf{Z}/n\mathbf{Z}$.)

Proof. For $s \in A$ and $S = \{1, s, s^2, \ldots\}$, by [**6**, p. 122] the following localization sequence

$$\tilde{G}_0(A/sA) \to \tilde{G}_0(A) \to \tilde{G}_0(A_S) \to 0$$

is exact. Let $s = X^n$, then $A_S = A[1/X^n] = k[X^n, X^cY][1/X^n]$, so $\tilde{G}_0(A_S) = 0$. Let $R = A/X^nA$ and $B = R/\text{nil }(R)$ where nil (R) is the nilradical of R. The natural map $G_0(B) \to G_0(R)$ is an isomorphism [**2**, p. 454]. Clearly $B = k[\bar{Y}^n]$, so $G_0(B) = \mathbf{Z}$ on $[B]$. Thus $G_0(R) = \mathbf{Z}$ on $[B]$ also. As a B-module,

$$R = B \oplus \bar{X}\bar{Y}^jB \oplus \ldots \oplus \bar{X}^{n-1}\bar{Y}^{\overline{(n-1)j}}B,$$

so $[R] = n[B]$ in $G_0(R)$. Hence $n[B] = 0$ in $\tilde{G}_0(R)$, so $|\tilde{G}_0(R)|\,|n$. Thus $|\tilde{G}_0(A)|\,|n$, but $\tilde{G}_0(A) \to \text{Cl }(A) \approx \mathbf{Z}/n\mathbf{Z}$ is surjective; so it must be an isomorphism.

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