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STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION WITH ALMOST SPACE–TIME WHITE NOISE

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Abstract

We study the stochastic cubic nonlinear Schrödinger equation (SNLS) with an additive noise on the one-dimensional torus. In particular, we prove local well-posedness of the (renormalized) SNLS when the noise is almost space–time white noise. We also discuss a notion of criticality in this stochastic context, comparing the situation with the stochastic cubic heat equation (also known as the stochastic quantization equation).

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1. Introduction

1.1. Stochastic nonlinear Schrödinger equation. We consider the Cauchy problem of the following stochastic cubic nonlinear Schrödinger equation (SNLS) with an additive noise on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$:

$$\begin{cases} i\partial_t u - \partial_x^2 u + |u|^2 u = \phi\xi, \\ u|_{t=0} = u_0, \end{cases} \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{T}, \tag{1.1}$$

where $\xi(t, x)$ denotes a (Gaussian) space–time white noise on $\mathbb{R}_+ \times \mathbb{T}$ and ϕ is a bounded linear operator on $L^2(\mathbb{T})$. In view of the time reversibility of the deterministic nonlinear Schrödinger equation, one can also consider (1.1) on $\mathbb{R} \times \mathbb{T}$ by extending the white noise ξ onto $\mathbb{R} \times \mathbb{T}$. For simplicity, however, we only consider positive times in the following.

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The SNLS (1.1) has a wide range of applications, ranging from nonlinear optics and plasma physics to solid state physics and quantum statistics [1, 21, 49]. In the context of nonlinear fiber optics [1, 33], the nonlinear Schrödinger equation (NLS), namely, $\phi = 0$ in (1.1) (see (1.5) below), when derived from the Maxwell equations, describes transmission of a signal along a fiber line, where the roles of the variables t and x are switched from the 'standard' interpretation, namely, in this particular application for nonlinear fiber optics, t denotes the (rescaled) propagation distance and x denotes the (rescaled and translated) time (at least locally). See [1, 33] for further details. In the following, however, we stick to the standard convention, namely, we always refer to x as the spatial variable in $\mathbb{T}$ and t as the temporal variable in $\mathbb{R}_+$ in the remaining part of this paper.

When ϕ is the identity operator, the stochastic forcing in (1.1) reduces to the space–time white noise ξ. The Cauchy problem (1.1) in this case is of particular interest in terms of applications [20, 21, 49] as well as its analytical difficulty since the problem is then *critical*. See Section 1.2 for a further discussion.

We say that u is a solution to (1.1) if it satisfies the following mild formulation (= Duhamel formulation):

$$u(t) = S(t)u_0 + i \int_0^t S(t-t')|u|^2 u(t')\,dt' - i \int_0^t S(t-t')\phi\xi(dt'), \tag{1.2}$$

where $S(t) = e^{-it\partial_x^2}$ denotes the linear Schrödinger propagator. The last term on the right-hand side of (1.2) is the so-called stochastic convolution, representing the effect of the random forcing. In the following, we set

$$\Psi(t) := \int_0^t S(t-t')\phi\xi(dt'). \tag{1.3}$$

If $\phi \in HS(L^2; H^s)$, namely, it is a Hilbert–Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$, then a standard argument [17] shows that $\Psi \in C(\mathbb{R}_+; H^s(\mathbb{T}))$ almost surely. When $\phi = \mathrm{Id}$, namely when the noise is given by the space–time white noise ξ, we have $\Psi \in C(\mathbb{R}_+; H^s(\mathbb{T}))$ almost surely if and only if $s < -\frac{1}{2}$. This roughness (in space) of the stochastic convolution is the source of difficulty in studying the SNLS (1.1) with the space–time white noise.

Given $\phi \in HS(L^2; H^s)$ for some $s > \frac{1}{2}$, local well-posedness of (1.1) in $H^s(\mathbb{T})$ easily follows from the algebra property of $H^s(\mathbb{T})$ and the unitarity of the linear Schrödinger propagator $S(t)$ on $H^s(\mathbb{T})$. For lower regularities, however, one needs to employ the Fourier restriction norm method due to Bourgain [6]. In particular, it was shown in [11] that (1.1) is locally well-posed in $L^2(\mathbb{T})$, provided that $\phi \in HS(L^2; L^2)$. The argument in [11] is based on (a slight modification of) the L^2-local theory by Bourgain [6] and controlling the stochastic convolution in the relevant $X^{s,b}$-norm (see Lemma 3.1 below). A standard application of Ito's lemma combined with the conservation of the L^2-norm for the deterministic NLS (1.5) yields an *a priori* bound on the L^2-norm of a solution and thus global well-posedness of (1.1) in $L^2(\mathbb{T})$. See [11] for details.

See also [19] for a related argument in the context of the stochastic Korteweg-de Vries (KdV) equation. We also mention the well-posedness results [18, 36] of the SNLS (1.1) on the Euclidean space $\mathbb{R}^d$, where the Strichartz estimates and the dispersive estimate play an important role.

Our main goal in this paper is to study (1.1) when ϕ is almost the identity operator. Given $\alpha \in \mathbb{R}$, let ϕ be the Bessel potential of order α given by

$$\phi = \langle \partial_x \rangle^{-\alpha} := (1 - \partial_x^2)^{-(\alpha/2)}. \tag{1.4}$$

Namely, the operator ϕ in (1.4) is the Fourier multiplier operator with the multiplier given by $\langle n \rangle^{-\alpha}$:

$$\widehat{\phi f}(n) = \langle n \rangle^{-\alpha} \widehat{f}(n)$$

for $n \in \mathbb{Z}$, where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. Then we prove that (a renormalized version of) the SNLS (1.1) is locally well-posed, provided that $\alpha > 0$. See Theorem 1.1 below for a precise statement. Note that our main result (Theorem 1.1) handles the case of *almost* space–time white noise forcing since the operator ϕ in (1.4) reduces to the identity operator when $\alpha = 0$.

Let ϕ be as in (1.4). Then it is easy to see that $\phi \in HS(L^2; H^s)$ if and only if

$$s < \alpha - \tfrac{1}{2}.$$

In particular, when $\alpha > \frac{1}{2}$, the L^2 well-posedness theory in [11] is readily applicable and we conclude that (1.1) is globally well-posed in $L^2(\mathbb{T})$ in this case. When $\alpha \le \frac{1}{2}$, however, the stochastic convolution lies almost surely outside $L^2(\mathbb{T})$ (for fixed $t \neq 0$), which causes a serious issue in studying (1.1) with rough noises.

Before proceeding further, let us first discuss the situation for the (deterministic) cubic nonlinear Schrödinger equation

$$i\partial_t u - \partial_x^2 u + |u|^2 u = 0. \tag{1.5}$$

By introducing the Fourier restriction norm method, Bourgain [6] proved that (1.5) is locally well-posed in $L^2(\mathbb{T})$, which was immediately extended to global well-posedness thanks to the conservation of the L^2-norm. On the other hand, it is known that (1.5) is ill-posed in negative Sobolev spaces [13, 25, 31]. In order to overcome this issue, the following renormalized NLS:

$$i\partial_t u - \partial_x^2 u + \Big(|u|^2 - 2 \int_{\mathbb{T}} |u|^2 \, dx \Big) u = 0 \tag{1.6}$$

has been proposed as an alternative model to (1.5) outside $L^2(\mathbb{T})$ [12, 14, 24, 25, 39]. We point out that (1.5) and (1.6) are equivalent in $L^2(\mathbb{T})$ in the sense that the following invertible gauge transformation:

$$u(t) \longmapsto \mathcal{G}(u)(t) := e^{-2it \int |u|^2 \, dx} u(t) \tag{1.7}$$

allows us to freely convert solutions to (1.5) to those to (1.6), provided that they belong to $C(\mathbb{R}; L^2(\mathbb{T}))$. The renormalized NLS (1.6) first appeared in the work of Bourgain [8]

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in studying the invariant Gibbs measure for the defocusing cubic NLS on $\mathbb{T}^2$. In [8], it was introduced as a model equivalent to the Hamiltonian dynamics corresponding to the Wick ordered Hamiltonian arising in Euclidean quantum field theory. See [40] for a further discussion on the Wick renormalization in the context of the NLS on $\mathbb{T}^2$. For this reason, the equation (1.6) is often referred to as the Wick ordered NLS. The gauge transformation (1.7) removes a certain singular component from the cubic nonlinearity in (1.5); see (4.1) and (4.2) below. As a result, the Wick ordered NLS (1.6) behaves better than the cubic NLS (1.5) outside $L^2(\mathbb{T})$, while they are equivalent in $L^2(\mathbb{T})$. In particular, we proposed in [39] that the Wick ordered NLS (1.6) is the right model to study outside $L^2(\mathbb{T})$.

In an analogous manner, we propose to study the following renormalized SNLS:

$$\begin{cases} i\partial_t u - \partial_x^2 u + \Big(|u|^2 - 2\displaystyle\int_{\mathbb{T}} |u|^2\,dx\Big)u = \phi\xi, \\ u|_{t=0} = u_0 \end{cases} \tag{1.8}$$

for rough noises: $\phi \notin HS(L^2; L^2)$. For simplicity, we refer to (1.8) as the Wick ordered SNLS in the following. Our main goal is to establish local well-posedness of (1.8) for ϕ given by (1.4) with $\alpha > 0$ arbitrarily close to 0. For this purpose, we now go over the known results on the Wick ordered NLS (1.6) outside $L^2(\mathbb{T})$. It was observed in [14] that the Wick ordered NLS (1.6) is mildly ill-posed in negative Sobolev spaces in the sense of the failure of local uniform continuity of the solution map. This in particular implies that we cannot apply a contraction argument to construct solutions to (1.6) in negative Sobolev spaces. In [25], the second author (with Guo) employed a more robust energy method (in the form of the short-time Fourier restriction norm method) and proved local existence of solutions (without uniqueness) to the Wick ordered NLS (1.6) in $H^s(\mathbb{T})$, $-\frac{1}{8} < s < 0$. This local existence result in [25] can be extended to global existence. See [29, 43]. The result in [25] leaves a substantial gap to the desired regularity $s \approx -\frac{1}{2}$, corresponding to $\alpha \approx 0$. More importantly, the question of uniqueness for the Wick ordered NLS (1.6) in negative Sobolev spaces still remains as a very challenging open question in the field of nonlinear dispersive partial differential equations. Hence, the approach in [25] does not seem to be suitable for studying the Wick ordered SNLS (1.8).

In [14], the second author (with Colliander) studied the Wick ordered NLS (1.6) with random initial data of the form:

$$u_0(x;\omega) = u_0^\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{2\pi i n x}, \tag{1.9}$$

where $\{g_n\}_{n\in\mathbb{Z}}$ is a sequence of independent standard complex-valued Gaussian random variables. (In the following, we may drop the harmless factor of 2π when it plays no important role.) Denoting by η the spatial white noise on $\mathbb{T}$, we have $u_0^\omega = \langle\partial_x\rangle^{-\alpha}\eta$, namely, for $\alpha > 0$, the random initial data u_0^ω corresponds to a smoothed spatial white noise. By exploiting a gain of space–time integrability of the random linear solution $S(t)u_0^\omega$, we proved that the Wick ordered NLS (1.6) is locally well-posed

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almost surely with respect to the random initial data (1.9), provided that $\alpha > \frac{1}{6}$. By slightly modifying the argument in [14], we can show that the Wick ordered SNLS (1.8) is locally well-posed with (deterministic) initial data in $L^2(\mathbb{T})$, provided that $\phi = \langle \partial_x \rangle^{-\alpha}$ for $\alpha > \frac{1}{6}$. The main idea would be to apply the so-called Da Prato–Debussche trick [15], namely, write $u = v + \Psi$ with Ψ as in (1.3) and study the fixed point problem for the residual term $v := u - \Psi$. In order to lower the regularity of the noise below $\alpha = \frac{1}{6}$, however, we would need to employ higher order expansions [4, 41]. In particular, since our main goal is to handle an almost white noise (that is, arbitrarily small $\alpha > 0$), we would need to (at least) consider higher order expansions of arbitrarily high orders (see [41]), which seems to be out of reach at this point.

In order to overcome this difficulty, we leave the realm of the L^2-based Sobolev spaces. More precisely, we study the Wick ordered SNLS (1.8) in the Fourier–Lebesgue spaces $\mathcal{F}L^{s,p}(\mathbb{T})$ in the following. Here, the Fourier–Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$ is defined by the norm

$$\|f\|_{\mathcal{F}L^{s,p}(\mathbb{T})} := \|\langle n \rangle^s \widehat{f}(n)\|_{\ell^p_n(\mathbb{Z})}. \tag{1.10}$$

When $s = 0$, we simply set $\mathcal{F}L^p(\mathbb{T}) = \mathcal{F}L^{0,p}(\mathbb{T})$. Recall the following embedding: $\mathcal{F}L^{p_1}(\mathbb{T}) \subset L^2(\mathbb{T}) \subset \mathcal{F}L^{p_2}(\mathbb{T})$ for $p_1 \le 2 \le p_2$. Let us first go over the known results for the Wick ordered NLS (1.6) in the Fourier–Lebesgue spaces. In [12], by a power series expansion, Christ constructed a solution to (1.6) (without uniqueness) with initial data in $\mathcal{F}L^p(\mathbb{T})$, $1 \le p < \infty$. In [24], Grünrock–Herr adapted the Fourier restriction norm method to the Fourier–Lebesgue space setting (see Section 2) and proved local well-posedness of (1.6) in $\mathcal{F}L^p(\mathbb{T})$ for $1 \le p < \infty$ by a standard contraction argument. We also mention the work [45] on the construction of solutions to (1.6) in $\mathcal{F}L^p(\mathbb{T})$, $1 \le p < \infty$, based on a normal form method. These results allow us to handle almost spatial white noise, that is, u_0^ω in (1.9) with arbitrarily small $\alpha > 0$, suggesting that the Fourier–Lebesgue setting is an appropriate framework for studying the Wick ordered SNLS (1.8).

In order to study the Wick ordered SNLS (1.8) in the Fourier–Lebesgue spaces, we first need to extend the notion of Hilbert–Schmidt operators to the Banach space setting. Given $s \in \mathbb{R}$ and $1 \le p < \infty$, we say that ϕ is a γ-radonifying operator from $L^2(\mathbb{T})$ to $\mathcal{F}L^{s,p}(\mathbb{T})$ if the $\gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$-norm defined by

$$\|\phi\|_{\gamma(L^2;\mathcal{F}L^{s,p})} := \bigg\| \bigg(\sum_{k\in\mathbb{Z}} |\langle n \rangle^s \widehat{\phi(e_k)}(n)|^2 \bigg)^{1/2} \bigg\|_{\ell^p_n(\mathbb{Z})} \tag{1.11}$$

is finite, where $e_k(x) = e^{2\pi i k x}$, $k \in \mathbb{Z}$. We denote the collection of γ-radonifying operators by $\gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$. Note that when $p = 2$, the norm in (1.11) reduces to the standard Hilbert–Schmidt $HS(L^2; H^s)$-norm. For readers' convenience, we present basic definitions and properties of γ-radonifying operators in Appendix A.

We now state our main result of this paper.

THEOREM 1.1. *Let $s > 0$ and $1 < p < \infty$. Then, given*

$$\phi \in \gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T})),$$

the Wick ordered SNLS (1.8) is pathwise locally well-posed in $\mathcal{F}L^{s,p}(\mathbb{T})$. More precisely, given $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$, there exist a stopping time $T = T(\|u_0\|_{\mathcal{F}L^{s,p}}, \Psi)$, which is positive almost surely, and a unique solution u to (1.8) in the class

$$C([0,T];\mathcal{F}L^{s,p}(\mathbb{T})) \cap X_p^{s,b}([0,T])$$

for some $b > 0$ such that $(b-1)p < -1$.

Here, $X_p^{s,b}([0,T])$ denotes the local-in-time version of the $X^{s,b}$-space adapted to $\mathcal{F}L^{s,p}(\mathbb{T})$. See Section 2 for the precise definition.

Let $\phi = \langle \partial_x \rangle^{-\alpha}$ be as in (1.4). Then a direct computation with (1.11) shows that $\phi \in \gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$ if and only if

$$s < \alpha - \frac{1}{p}. \tag{1.12}$$

Hence, given $\alpha > 0$, we can choose sufficiently small $s > 0$ and large $p \gg 1$ such that (1.12) holds and thus Theorem 1.1 is applicable. This establishes local well-posedness of the Wick ordered SNLS (1.8) with almost space–time white noise.

By writing (1.8) in the mild formulation,

$$u(t) = S(t)u_0 + i\int_0^t S(t-t')\mathcal{N}(u)(t')\,dt' - i\Psi,$$

where $\mathcal{N}(u) = (|u|^2 - 2\int_{\mathbb{T}} |u|^2\,dx)u$ denotes the Wick ordered nonlinearity in (1.8) and Ψ denotes the stochastic convolution in (1.3). The proof of Theorem 1.1 is based on a two-step argument. (i) We construct a solution u in $X_p^{s,b}([0,T])$ by a standard contraction argument. The local-in-time regularity of the stochastic convolution (= the local-in-time regularity of the Brownian motion) forces us to choose the temporal regularity b such that $(b-1)p < -1$. We point out that Grünrock–Herr [24] imposed a stronger regularity assumption: $(b-1)p > -1$ and therefore their trilinear estimate [24, Proposition 1.3] is not directly applicable to our problem. We make up this loss of temporal regularity (as compared to [24]) by taking the spatial regularity s to be slightly positive. (ii) For this particular choice of the temporal regularity, we have $X_p^{s,b}([0,T]) \not\subset C([0,T];\mathcal{F}L^{s,p}(\mathbb{T}))$. Hence, we need to show *a posteriori* the continuity in time of the solution u constructed in Step (i).

REMARK 1.2. (i) Our main goal is to handle the case of almost space–time white noise, namely, small $s > 0$ and large $p \gg 1$ such that (1.12) holds. As such, our proof of Theorem 1.1 is tailored for this purpose. For example, our argument does not treat the $p = 1$ case. Note that, when $p = 1$, local well-posedness of (1.8) follows from the algebra property of $\mathcal{F}L^1(\mathbb{T})$, the unitarity of $S(t)$ on $\mathcal{F}L^1(\mathbb{T})$ and Lemma 3.3 and there is no need to resort to the Fourier restriction norm method.

(ii) As we see in the next subsection, the SNLS with an additive space–time white noise is critical. While Theorem 1.1 establishes an almost critical local well-posedness result, the problem with $\phi = \mathrm{Id}$ seems to be out of reach at this point.

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(iii) In the case of a spatially homogeneous noise, namely, when ϕ is a convolution operator with a kernel K, then the $\gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$-norm in (1.11) reduces to the $\mathcal{F}L^{s,p}$-norm of the kernel function K. In [34], the second author used such a characterization of γ-radonifying operators and proved local well-posedness of the stochastic KdV equation with an additive space–time white noise.

(iv) Given $N \in \mathbb{N}$, consider the following SNLS with a truncated nonlinearity (but with a full space–time white noise):

$$i\partial_t u - \partial_x^2 u + \mathbf{P}_N(|\mathbf{P}_N u|^2 \mathbf{P}_N u) = \xi, \tag{1.13}$$

where $\mathbf{P}_N$ denotes the Dirichlet projection onto the frequencies $\{|n| \le N\}$. Let u be a solution to (1.13) with initial data given by the (spatial) white noise, that is, u_0^ω in (1.9) with $\alpha = 0$. Then, by exploiting invariance of the spatial white noise under the deterministic NLS (with a truncated nonlinearity; see [38]), we can show that the solution u at time $t > 0$ is given by the spatial white noise of variance $1 + t$. The same result also holds for the Wick ordered SNLS with a truncated nonlinearity. This would provide a basis for applying a modification of Bourgain's invariant measure argument [7, 8] in constructing global-in-time dynamics for the Wick ordered SNLS (1.8) with $\phi = \mathrm{Id}$. See [37] for details. Unfortunately, we do not know how to construct local-in-time dynamics for the Wick ordered SNLS (1.8) with $\phi = \mathrm{Id}$.

Remark 1.3. In a recent paper [44], the second and third authors exploited the completely integrable structure of the equation and proved global well-posedness of the renormalized (deterministic) NLS (1.6) in $\mathcal{F}L^{s,p}(\mathbb{T})$ for $s \ge 0$ and $1 \le p < \infty$. It would be of interest to investigate the global-in-time behavior of solutions to the renormalized SNLS (1.8) constructed in Theorem 1.1.

1.2. On the criticality of SNLS with space–time white noise. In this subsection, we discuss a notion of criticality for SNLS (1.2) with an additive space–time white noise forcing (that is $\phi = \mathrm{Id}$); see also a discussion in [5]. Before doing so, let us first go over Hairer's notion of local (sub)criticality [26] by considering the following stochastic cubic heat equation with an additive space–time white noise forcing on the d-dimensional spatial domain:

$$\partial_t u - \Delta u + u^3 = \xi. \tag{1.14}$$

Here, u is a real-valued function/distribution and ξ denotes a space–time white noise. We have $\mathbb{T}^d$ in mind but we intentionally remain vague about the underlying spatial domain for the purpose of scaling in this formal discussion. The equation (1.14) is also known as the stochastic quantization equation (SQE). The white noise scaling tells us that

$$\xi_\lambda(t, x) = \lambda^{(d+2)/2} \xi(\lambda^2 t, \lambda x) \tag{1.15}$$

is also a space–time white noise. (Recalling that $\mathbb{E}[\xi(t_1, x_1)\xi(t_2, x_2)] = \delta(t_1 - t_2)\delta(x_1 - x_2)$, we see that the white noise behaves like a square root of the Dirac delta function, which gives an intuition for the white noise scaling (1.15). Moreover, in view of the linear part of the equation, we count one temporal dimension as two spatial dimensions.) By applying the following scaling to the unknown:

$$u_\lambda(t,x) = \lambda^{(d/2)-1} u(\lambda^2 t, \lambda x), \tag{1.16}$$

we see that the scaled function u_λ satisfies the following equation:

$$\partial_t u_\lambda - \Delta u_\lambda + \lambda^{4-d} u_\lambda^3 = \xi_\lambda. \tag{1.17}$$

As $\lambda \to 0$, namely, studying the behavior of a solution at smaller and smaller scales, we see that, at least at a formal level, the nonlinearity vanishes and (1.17) reduces to the following stochastic heat equation:

$$\partial_t u - \Delta u = \xi,$$

provided that $4 - d > 0$. This formal discussion shows that SQE (1.14) in dimensions $d = 1, 2$ and 3 is locally subcritical, while it is locally critical when $d = 4$. See [26, Section 8] for a more rigorous definition of local subcriticality. Indeed, when $d \le 3$, SQE (1.14) is known to be well-posed (after an appropriate renormalization for $d = 2, 3$); see [10, 16, 26, 32]. As we see below, while this notion of local criticality is suitable for studying the heat equation, it is not a suitable concept for studying the Schrödinger equation.

By repeating a similar scaling argument for the SNLS with an additive space–time white noise:

$$i\partial_t u - \Delta u + |u|^2 u = \xi \tag{1.18}$$

with (1.15) and (1.16), we arrive at the scaled equation

$$i\partial_t u_\lambda - \Delta u_\lambda + \lambda^{4-d} |u_\lambda|^2 u_\lambda = \xi_\lambda. \tag{1.19}$$

It is tempting to conclude that, by taking $\lambda \to 0$, we may neglect the effect of the nonlinearity when $d \le 3$ as in the case of the SQE. This is, however, not quite correct since, in order to compare the sizes of the terms in (1.19), we need to measure them in a norm compatible with the Schrödinger equation. For example, in terms of the L^2-based homogeneous Sobolev spaces, the $\dot{H}^{-(d/2)-}$-norm captures the (spatial) regularity of the white noise. (We use $a-$ to denote $a - \varepsilon$ for arbitrarily small $\varepsilon \ll 1$, where a relevant norm diverges as $\varepsilon \to 0$.) On the other hand, the scaling (1.16) preserves the $\dot{H}^1$-norm, while the $\dot{H}^{-(d/2)-}$-norm scales by a factor of $\lambda^{-(d/2)-1}$. (Strictly speaking, we have a factor of $\lambda^{-(d/2)-1+}$ here. We, however, use $\lambda^{-(d/2)-1}$ for simplicity. Recall also the end point Besov regularity of the (spatial) white noise in $B^{-(d/2)}_{2,\infty}$; see [3, 46].) By combining the factor λ^{4-d} in (1.19) with $(\lambda^{-(d/2)-1})^2$, we essentially have λ^{2-2d} as the size of the nonlinearity (relative to the linear term u_λ). This shows that SNLS (1.18) with an additive space–time white noise is *critical* when $d = 1$.

Hairer's notion of local criticality is useful for studying the heat equation since it is adapted to the Hölder spaces $C^s = B^s_{\infty,\infty}$. Recall that the stochastic convolution

$$\Psi_{\text{heat}} = \int_0^t e^{(t-t')\Delta} \xi(dt')$$

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for the problem (1.14) has the spatial regularity $-(d/2)+1-$ almost surely. We expect a solution u to (1.14) to have the same regularity. Noting that the scaling (1.16) (essentially) preserves the relevant $\dot{B}^{-(d/2)+1-}_{\infty,\infty}$-norm, we conclude from (1.17) that SQE (1.14) is critical when $d = 4$. On the other hand, our discussion above on a notion of criticality for the SNLS was based on the L^2-Sobolev spaces. We may also carry out a similar analysis in terms of the homogeneous Fourier–Lebesgue spaces $\dot{\mathcal{F}L}^{s,p}$ defined by the norm

$$\|f\|_{\dot{\mathcal{F}L}^{s,p}(\mathbb{R}^d)} := \|\,|\zeta|^s \widehat{f}(\zeta)\|_{L^p(\mathbb{R}^d)}.$$

Note that the $\dot{\mathcal{F}L}^{s,p}$-norm with $s = -(d/p)-$ captures the (spatial) regularity of the white noise. One can easily check that, under the scaling (1.16), the $\dot{\mathcal{F}L}^{-(d/p)-,p}$-norm also scales by the factor of $\lambda^{-(d/2)-1}$ (just like the $H^{-(d/2)-}$-norm), confirming the criticality of SNLS (1.18) with an additive space–time white noise when $d = 1$. The main point is that, in discussing a notion of criticality, we need to specify a function space suitable for studying a given equation and to incorporate the effect of the scaling on the size of the scaled function (measured in the relevant norm). This point was not clearly mentioned in Hairer's presentation since the relevant Hölder norm is preserved under the scaling (1.16).

We can also argue in terms of the more standard scaling analysis. It is well known that the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$ remains invariant under the following scaling symmetry:

$$u^\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x). \tag{1.20}$$

A direct computation shows that the homogeneous $\dot{W}^{s_{\text{crit}}(p),p}$-norm is preserved under the scaling (1.20), where $s_{\text{crit}}(p)$ is given by

$$s_{\text{crit}}(p) = \frac{d}{p} - 1. \tag{1.21}$$

In studying the NLS, we need to use the L^2-based Sobolev space since the linear Schrödinger propagator $e^{-it\Delta}$ is bounded on L^2 but is unbounded on L^p, $p \neq 2$. This gives the (usual) scaling-critical Sobolev regularity

$$s_{\text{crit}}(2) = \frac{d}{2} - 1.$$

Now, consider SNLS (1.18) with an additive space–time white noise forcing. Then

$$s_{\text{crit}}(2) = \frac{d}{2} - 1 > -\frac{d}{2}- = \text{spatial regularity of the space–time white noise}$$

with almost an equality when $d = 1$. This shows that the equation (1.18) is critical when $d = 1$. We point out that, when $d = 1$, the (deterministic) Wick ordered NLS (1.6) is known to be ill-posed at the critical regularity $s_{\text{crit}}(2) = (d/2) - 1$ [30, 35, 42], which shows the difficulty of the SNLS problem (1.8) with $\phi = \text{Id}$.

Similarly, noting that the scaling-critical regularity (with respect to the scaling symmetry (1.20)) for the homogeneous Fourier–Lebesgue spaces $\mathcal{F}\dot{L}^{s,p}$ is given by

$$\widehat{s}_{\text{crit}}(p) = s_{\text{crit}}(p') = d - 1 - \frac{d}{p},$$

we have

$$\begin{aligned}\widehat{s}_{\text{crit}}(p) = d - 1 - \frac{d}{p} &> -\frac{d}{p}- \\ &= \text{spatial regularity of the space–time white noise measured in } \mathcal{F}\dot{L}^{s,p}\end{aligned}$$

with almost an equality when $d = 1$. Hence, we also conclude that the equation (1.18) is critical when $d = 1$ in terms of the Fourier–Lebesgue spaces.

Lastly, let us consider SQE (1.14). In this case, the stochastic convolution Ψ_{heat} gains one spatial derivative and has spatial regularity

$$-\frac{d}{2} + 1 - . \tag{1.22}$$

On the other hand, we know that the linear heat propagator $e^{t\Delta}$ is bounded in the L^∞-based Sobolev spaces and the Hölder spaces $B^s_{\infty,\infty}$. Namely, we can set $p = \infty$ in (1.21), yielding

$$s_{\text{crit}}(\infty) = -1. \tag{1.23}$$

By comparing (1.22) and (1.23),

$$s_{\text{crit}}(\infty) = -1 < -\frac{d}{2} + 1-$$

for $d \leq 3$ and an almost equality holds when $d = 4$, showing that the SQE problem (1.14) is critical when $d = 4$.

This paper is organized as follows. In Section 2, we recall the definition and basic properties of the $X^{s,b}$-spaces adapted to the Fourier–Lebesgue spaces. We then study the regularity properties of the stochastic convolution in Section 3. In Section 4, we prove the crucial trilinear estimate, which is then used to prove Theorem 1.1 in Section 5. In Appendix A, we go over basic definitions and properties of γ-radonifying operators.

2. Function spaces and their properties

Let $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ be the vector space of C^∞-functions $u : \mathbb{R}^2 \to \mathbb{C}$ such that

$$u(t, x) = u(t, x + 1) \quad \text{and} \quad \sup_{(t,x)\in\mathbb{R}^2} |t^\alpha \partial_t^\beta \partial_x^\gamma u(t, x)| < \infty$$

for any $\alpha, \beta, \gamma \in \mathbb{N} \cup \{0\}$. In the seminal paper [6], Bourgain introduced the $X^{s,b}$-spaces defined by the norm

$$\|u\|_{X^{s,b}(\mathbb{R}\times\mathbb{T})} = \|\langle n\rangle^s \langle \tau - n^2\rangle^b \widehat{u}(\tau, n)\|_{\ell^2_n L^2_\tau(\mathbb{Z}\times\mathbb{R})}.$$

We now recall the definition of the $X^{s,b}$-spaces adapted to the Fourier–Lebesgue spaces; see Grünrock–Herr [24].

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Definition 2.1. Let $s, b \in \mathbb{R}$, $1 \le p, q \le \infty$. We define the space $X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})$ as the completion of $\mathcal{S}(\mathbb{R}\times\mathbb{T})$ with respect to the norm

$$\|u\|_{X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})} = \|\langle n\rangle^s \langle \tau - n^2\rangle^b \widehat{u}(\tau, n)\|_{\ell^p_n L^q_\tau(\mathbb{Z}\times\mathbb{R})}.$$

For brevity, we simply denote $X^{s,b}_{p,q}(\mathbb{R}\times\mathbb{T})$ by $X^{s,b}_{p,q}$. When $p = q$, we set $X^{s,b}_p = X^{s,b}_{p,p}$. Recall the following characterization of the $X^{s,b}_{p,q}$-norm in terms of the interaction representation $S(-t)u(t)$:

$$\|u\|_{X^{s,b}_{p,q}} = \|S(-t)u(t)\|_{\mathcal{F}L^{s,p}_x \mathcal{F}L^{b,q}_t}, \tag{2.1}$$

where the iterated norm is to be understood in the following sense:

$$\|u\|_{\mathcal{F}L^{s,p}_x \mathcal{F}L^{b,q}_t} := \|\langle n\rangle^s \langle \tau\rangle^b \widehat{u}(\tau, n)\|_{\ell^p_n L^q_\tau} = \| \, \|\langle n\rangle^s \widehat{u}(t, n)\|_{\mathcal{F}L^{b,q}_t} \|_{\ell^p_n}. \tag{2.2}$$

Note that these spaces are separable when $p, q < \infty$. The identity (2.1) follows from $\widehat{S(t)u}(\tau, n) = \widehat{u}(\tau + n^2, n)$, the first equality in (2.2) and a change of variable: $\tau \mapsto \tau + n^2$.

For any $1 \le p < \infty$, $s \in \mathbb{R}$,

$$X^{s,b}_{p,q} \hookrightarrow C(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T})) \quad \text{if } b > \frac{1}{q'} = 1 - \frac{1}{q}. \tag{2.3}$$

This is a consequence of the dominated convergence theorem along with the following embedding relation: $\mathcal{F}L^{b,q}_t \hookrightarrow \mathcal{F}L^1_t \hookrightarrow C_t$, where the second embedding is the Riemann–Lebesgue lemma.

Given $T > 0$, we also define the local-in-time version $X^{s,b}_{p,q}([0, T])$ of the $X^{s,b}_{p,q}$-space as the collection of functions u such that

$$\|u\|_{X^{s,b}_{p,q}([0,T])} := \inf\{\|v\|_{X^{s,b}_{p,q}} : v|_{[0,T]} = u\} \tag{2.4}$$

is finite. For simplicity, we often denote $X^{s,b}_{p,q}([0, T])$ by $X^{s,b}_{p,q;T}$.

Lastly, we recall the following linear estimates.

Lemma 2.2. (i) (Homogeneous linear estimate). *Given* $1 \le p, q \le \infty$ *and* $s, b \in \mathbb{R}$,

$$\|S(t) f\|_{X^{s,b}_{p,q;T}} \lesssim \|f\|_{\mathcal{F}L^{s,p}}$$

for any $0 < T \le 1$.

(ii) (Nonhomogeneous linear estimate). *Let* $s \in \mathbb{R}$, $1 \le p \le \infty$, $1 < q < \infty$ *and* $-(1/q) < b' \le 0 \le b \le 1 + b'$. *Then*

$$\left\| \int_0^t S(t - t')F(t')\, dt' \right\|_{X^{s,b}_{p,q;T}} \lesssim T^{1+b'-b} \|F\|_{X^{s,b'}_{p,q;T}} \tag{2.5}$$

for any $0 < T \le 1$.

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PROOF. The proof of (i) is straightforward from (2.1). More precisely, by letting $\eta \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function such that $\eta(t) \equiv 1$ for $t \in [0,1]$ and $\operatorname{supp}\eta \subset [-2,2]$, it follows from (2.4) and then (2.1) that

$$\begin{aligned}\|S(t)f\|_{X^{s,b}_{p,q;T}} &\le \|\eta(t)S(t)f\|_{X^{s,b}_{p,q}} = \|\langle n\rangle^s\langle\tau\rangle^b\widehat{\eta}(\tau)\widehat{f}(n)\|_{\ell^p_n L^q_\tau}\\ &= \|\eta\|_{H^b_t}\|f\|_{\mathcal{F}L^{s,p}}.\end{aligned}$$

This proves (i). The proof of the nonhomogeneous estimate (2.5) is also standard. From [23, (2.21)],

$$\left\|\eta\Big(\frac{t}{T}\Big)\int_0^t f(t')\,dt'\right\|_{\mathcal{F}L^{b,q}_t} \lesssim T^{1+b'-b}\|f\|_{\mathcal{F}L^{b',q}_t}. \tag{2.6}$$

Given a function F on $[0,T]\times\mathbb{T}$, let $\widetilde{F}$ be an extension of F onto $\mathbb{R}\times\mathbb{T}$. Then it follows from (2.1) and (2.6) that

$$\begin{aligned}&\left\|\int_0^t S(t-t')F(t')\,dt'\right\|_{X^{s,b}_{p,q;T}}\\ &\quad\le \left\|\eta\Big(\frac{t}{T}\Big)\int_0^t S(t-t')\widetilde{F}(t')\,dt'\right\|_{X^{s,b}_{p,q}}\\ &\quad= \left\|\langle n\rangle^s\left\|\eta\Big(\frac{t}{T}\Big)\int_0^t \mathcal{F}_x(S(-t')\widetilde{F})(t',n)\,dt'\right\|_{\mathcal{F}L^{b,q}_t}\right\|_{\ell^p_n}\\ &\quad\lesssim T^{1+b'-b}\|\langle n\rangle^s\|\mathcal{F}_x(S(-t)\widetilde{F}(t))(n)\|_{\mathcal{F}L^{b',q}_t}\|_{\ell^p_n}\\ &\quad= T^{1+b'-b}\|\widetilde{F}\|_{X^{s,b'}_{p,q}}.\end{aligned}$$

Here, $\mathcal{F}_x$ denotes the spatial Fourier transform. Taking the infimum over all extensions $\widetilde{F}$ of F, we obtain (2.5). □

3. On the stochastic convolution

In this section, we study the regularity properties of the stochastic convolution Ψ defined in (1.3). For this purpose, let us first recall the definition of a cylindrical Wiener process W on $L^2(\mathbb{T})$; a cylindrical Wiener process W on $L^2(\mathbb{T})$ is defined by the following random Fourier series:

$$W(t) = \sum_{n\in\mathbb{Z}}\beta_n(t)e_n,$$

where $e_n(x) = e^{2\pi i n x}$ and $\{\beta_n\}_{n\in\mathbb{Z}}$ is a family of mutually independent complex-valued Brownian motions. In terms of the cylindrical Wiener process W, we can express the stochastic convolution Ψ in (1.3) as

$$\Psi(t) = \int_0^t S(t-t')\phi\,dW(t') = \sum_{n\in\mathbb{Z}}\int_0^t S(t-t')\phi(e_n)\,d\beta_n(t'). \tag{3.1}$$

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It is easy to see that the space–time white noise almost surely belongs to $\mathcal{F}L_x^{s,p}\mathcal{F}L_{t,\mathrm{loc}}^{b,q}$ if and only if $sp < -1$ and $bq < -1$. The stochastic convolution, when measured in terms of its interaction representation $S(-t)\Psi(t)$, then gains one temporal regularity and thus has temporal regularity $b < 1 - (1/q)$. This is precisely the regularity of the Brownian motion measured in the Fourier–Lebesgue spaces; see [3]. The next lemma tells us the regularity of the stochastic convolution with respect to the $X_{p,q}^{s,b}$-spaces.

Lemma 3.1. *Let $s \in \mathbb{R}$, $1 \le p < \infty$ and $1 < q < \infty$ be such that $b < 1 - (1/q)$. Given $\phi \in \gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$, there exist $C, c > 0$ such that*

$$P(\|\Psi\|_{X^{s,b}_{p,q;T}} > \lambda) \le C \exp\bigg(-\frac{c\lambda^2}{T^{3-2b-(2/q)}\|\phi\|^2_{\gamma(L^2;\mathcal{F}L^{s,p})}}\bigg) \tag{3.2}$$

for any $\lambda > 0$ and $0 < T \le 1$. In particular, $\Psi \in X^{s,b}_{p,q}([0,T])$ almost surely.

We first recall the following equivalence of moments for Gaussian random variables.

Lemma 3.2. *Let $\{g_n\}_{n\in\mathbb{Z}}$ be a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$. Then, for $p \ge 2$,*

$$\bigg\|\sum_{n\in\mathbb{Z}} a_n g_n\bigg\|_{L^p(\Omega)} \lesssim \sqrt{p}\|a_n\|_{\ell^2_n}$$

for any sequence $\{a_n\}_{n\in\mathbb{Z}} \in \ell^2_n(\mathbb{Z}; \mathbb{C})$.

Recall that a mean-zero complex-valued Gaussian random variable g with variance σ^2 satisfies

$$\mathbb{E}[|g|^{2k}] = k! \cdot \sigma^{2k} \tag{3.3}$$

for any $k \in \mathbb{N}$. This identity follows from a simple computation involving the moment generating function for $|g|^2$. Suppose that $\sigma^2 = 1$. On the one hand, we have

$$\mathbb{E}[e^{t|g|^2}] = \mathbb{E}[e^{t(\mathrm{Re}\, g)^2}]\mathbb{E}[e^{t(\mathrm{Im}\, g)^2}] = \frac{1}{1-t} = \sum_{j=0}^\infty t^j$$

for $|t| < 1$. On the other hand, the Taylor expansion gives

$$\mathbb{E}[e^{t|g|^2}] = \sum_{k=0}^\infty \frac{t^k}{k!}\mathbb{E}[|g|^{2k}].$$

By comparing the coefficients, we obtain $\mathbb{E}[|g|^{2k}] = k!$ for any $k \in \mathbb{N}$ when $\sigma^2 = 1$.

In particular, from (3.3) and Stirling's formula, we obtain $\|g\|_{L^p(\Omega)} \lesssim \sqrt{p}\cdot\sigma$ for any $p \ge 2$. Then Lemma 3.2 follows easily once we observe that $\sum_{n\in\mathbb{Z}} a_n g_n$ is a mean-zero complex-valued Gaussian variable with variance $\|a_n\|^2_{\ell^2_n}$. See also [47, Theorem I.22].

PROOF OF LEMMA 3.1. The proof follows along the lines in [19, 34]. From (3.1),

$$\Psi(t) = \sum_{n\in\mathbb{Z}} e_n \sum_{k\in\mathbb{Z}} \widehat{(\phi e_k)}(n) \int_0^t e^{i(t-t')|n|^2}\, d\beta_k(t').$$

Then

$$\begin{aligned} S(-t)\mathbf{1}_{[0,T]}(t)\Psi(t) &= \sum_{n\in\mathbb{Z}} e_n \sum_{k\in\mathbb{Z}} \widehat{(\phi e_k)}(n)\mathbf{1}_{[0,T]}(t) \int_0^t \mathbf{1}_{[0,T]}(t')e^{-it'|n|^2}\, d\beta_k(t') \\ &= \sum_{n\in\mathbb{Z}} \psi_n(t)e_n, \end{aligned} \tag{3.4}$$

where

$$\psi_n(t) := \sum_{k\in\mathbb{Z}} \widehat{(\phi e_k)}(n)\mathbf{1}_{[0,T]}(t) \int_0^t \mathbf{1}_{[0,T]}(t')e^{-it'|n|^2}\, d\beta_k(t').$$

By the stochastic Fubini theorem,

$$\mathcal{F}_t(\psi_n)(\tau) = \sum_{k\in\mathbb{Z}} \int_0^T e^{-it'|n|^2}\widehat{(\phi e_k)}(n)\Big(\int_{t'}^T \mathbf{1}_{[0,T]}(t)e^{-it\tau}\, dt\Big)\, d\beta_k(t'). \tag{3.5}$$

For fixed T, τ and n, the summands in (3.5) are mean-zero independent complex-valued Gaussian random variables. Then, for fixed T, τ and n, it follows from Lemma 3.2 and the properties of the Wiener integral that, for $\sigma \geq 2$,

$$\begin{aligned} &\|\mathcal{F}_t(\psi_n)(\tau)\|_{L^\sigma(\Omega)} \\ &\quad\sim \sigma^{1/2}\Big(\sum_{k\in\mathbb{Z}} \mathbb{E}\Big[\Big|\int_0^T e^{-it'|n|^2}\widehat{(\phi e_k)}(n)\Big(\int_{t'}^T \mathbf{1}_{[0,T]}(t)e^{-it\tau}\, dt\Big) d\beta_k(t')\Big|^2\Big]\Big)^{1/2} \\ &\quad\sim \sigma^{1/2}\Big(\sum_{k\in\mathbb{Z}} \int_0^T |\widehat{(\phi e_k)}(n)|^2 \Big|\int_{t'}^T \mathbf{1}_{[0,T]}(t)e^{-it\tau}\, dt\Big|^2\, dt'\Big)^{1/2} \\ &\quad\lesssim \sigma^{1/2}T^{1/2}\min(T, |\tau|^{-1})\Big(\sum_{k\in\mathbb{Z}} |\widehat{(\phi e_k)}(n)|^2\Big)^{1/2}. \end{aligned} \tag{3.6}$$

Hence, from (2.1), (3.4) and (3.6) with Minkowski's integral inequality and (1.11),

$$\begin{aligned} \big\| \|\mathbf{1}_{[0,T]}(t)\Psi(t)\|_{X^{s,b}_{p,q}}\big\|_{L^\sigma(\Omega)} &= \big\|\langle n\rangle^s\langle\tau\rangle^b \|\mathcal{F}_t(\psi_n)(\tau)\|_{L^\sigma(\Omega)}\big\|_{\ell^p_n L^q_\tau} \\ &\lesssim \sigma^{1/2}T^{1/2}\|\langle\tau\rangle^b\min(T, |\tau|^{-1})\|_{L^q_\tau} \\ &\qquad \times \Big\|\Big(\sum_{k\in\mathbb{Z}} |\langle n\rangle^s\widehat{(\phi e_k)}(n)|^2\Big)^{1/2}\Big\|_{\ell^p_n} \\ &\lesssim \sigma^{1/2}T^{(3/2)-b-(1/q)}\|\phi\|_{\gamma(L^2;\mathcal{F}L^{s,p})} \end{aligned} \tag{3.7}$$

for any $\sigma \geq \max(p, q)$, provided that $b < 1 - (1/q)$. Therefore, (3.2) follows from (3.7) and Chebyshev's inequality. □

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Next, we show the continuity in time of the stochastic convolution. This is necessary as Lemma 3.1, while useful for the fixed point argument, does not imply any continuity in time since $b < 1 - (1/q)$; see (2.3).

Lemma 3.3. *Let $s \in \mathbb{R}$ and $1 \le p < \infty$. Then, given $\phi \in \gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$, we have $\Psi \in C(\mathbb{R}_+; \mathcal{F}L^{s,p}(\mathbb{T}))$ almost surely.*

Proof. Let $\widetilde{\Psi}(t) := S(-t)\Psi(t)$. Note that $\|\widetilde{\Psi}(t)\|_{\mathcal{F}L^{s,p}} = \|\Psi(t)\|_{\mathcal{F}L^{s,p}}$ for all $t \ge 0$. Then, for $t_2 > t_1 \ge 0$ and $\sigma \ge p$, Lemma 3.2 and (1.11) imply that

$$\begin{aligned}
&\mathbb{E}[\|\widetilde{\Psi}(t_2) - \widetilde{\Psi}(t_1)\|^{\sigma}_{\mathcal{F}L^{s,p}}] \\
&\quad \lesssim \bigg(\sum_{n\in\mathbb{Z}} \langle n\rangle^{sp} \bigg\| \sum_{k\in\mathbb{Z}} \widehat{(\phi e_k)}(n) \int_{t_1}^{t_2} e^{-it'|n|^2}\, d\beta_k(t') \bigg\|^p_{L^{\sigma}(\Omega)}\bigg)^{\sigma/p} \\
&\quad \sim \sigma^{\sigma/2} \bigg(\sum_{n\in\mathbb{Z}} \langle n\rangle^{sp} \bigg(\sum_{k\in\mathbb{Z}} |\widehat{(\phi e_k)}(n)|^2 \int_{t_1}^{t_2} 1\, dt'\bigg)^{p/2}\bigg)^{\sigma/p} \\
&\quad \lesssim \sigma^{\sigma/2} \|\phi\|^{\sigma}_{\gamma(L^2; \mathcal{F}L^{s,p})} |t_2 - t_1|^{\sigma/2}.
\end{aligned}$$

By applying Kolmogorov's continuity criterion [2, Exercise 8.2], we see that $\widetilde{\Psi} \in C(\mathbb{R}; \mathcal{F}L^{s,p}(\mathbb{T}))$ almost surely. By continuity of the semigroup $S(t)$ on $\mathcal{F}L^{s,p}(\mathbb{T})$ for $1 \le p < \infty$, we conclude that $\Psi(t) \in C(\mathbb{R}_+; \mathcal{F}L^{s,p}(\mathbb{T}))$ almost surely. □

4. Nonlinear estimates

In this section, we establish crucial nonlinear estimates for proving Theorem 1.1. First, write the Wick ordered nonlinearity in (1.8) as follows:

$$\begin{aligned}
\mathcal{N}(u) &= \bigg(|u|^2 - 2 \int_{\mathbb{T}} |u|^2\, dx\bigg) u \\
&= \sum_{n\in\mathbb{Z}} e^{inx} \sum_{\substack{n = n_1 - n_2 + n_3 \\ n \ne n_1, n_3}} \widehat{u}(n_1)\overline{\widehat{u}(n_2)}\widehat{u}(n_3) - \sum_{n\in\mathbb{Z}} e^{inx} |\widehat{u}(n)|^2 \widehat{u}(n) \\
&=: \mathcal{N}_1(u) + \mathcal{N}_2(u).
\end{aligned}$$

Here, $\mathcal{N}_1(u)$ and $\mathcal{N}_2(u)$ correspond to the nonresonant and resonant parts of the nonlinearity, respectively. By viewing $\mathcal{N}_1(u)$ and $\mathcal{N}_2(u)$ as trilinear operators, we write

$$\mathcal{N}_1(u_1, u_2, u_3) := \sum_{n\in\mathbb{Z}} e^{inx} \sum_{\substack{n = n_1 - n_2 + n_3 \\ n \ne n_1, n_3}} \widehat{u_1}(n_1)\overline{\widehat{u_2}(n_2)}\widehat{u_3}(n_3), \tag{4.1}$$

$$\mathcal{N}_2(u_1, u_2, u_3) := -\sum_{n\in\mathbb{Z}} e^{inx} \widehat{u_1}(n)\overline{\widehat{u_2}(n)}\widehat{u_3}(n). \tag{4.2}$$

Then we have the following nonlinear estimates.

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Proposition 4.1. *Let $s > 0$ and $1 < p < \infty$. Then there exist $b, b' \in \mathbb{R}$ with $-(1/p) < b' < 0 < b < 1 - (1/p)$ such that*

$$\|\mathcal{N}_1(u_1, u_2, u_3)\|_{X^{s,b'}_{p;T}} + \|\mathcal{N}_2(u_1, u_2, u_3)\|_{X^{s,b'}_{p;T}} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}_{p;T}}. \tag{4.3}$$

Grünrock–Herr [24] proved an analogous trilinear estimate but with higher temporal regularity $b > 1 - (1/p)$ (and $s = 0$). See [24, Proposition 1.3]. In our case, the temporal regularity of the stochastic convolution (Lemma 3.1) forces us to work with lower temporal regularity: $b < 1 - (1/p)$. We compensate this loss of temporal regularity by assuming a slightly higher spatial regularity $s > 0$.

Before proceeding to the proof of Proposition 4.1, recall the following elementary calculus lemma. See, for example, [22, Lemma 4.2].

Lemma 4.2. *Let $\beta \geq \gamma \geq 0$ and $\beta + \gamma > 1$. Then*

$$\int_{\mathbb{R}} \frac{1}{\langle x - a_1\rangle^{\beta}\langle x - a_2\rangle^{\gamma}} \lesssim \frac{1}{\langle a_1 - a_2\rangle^{\alpha}},$$
$$\sum_{n\in\mathbb{Z}} \frac{1}{\langle n - k_1\rangle^{\beta}\langle n - k_2\rangle^{\gamma}} \lesssim \frac{1}{\langle k_1 - k_2\rangle^{\alpha}},$$

where α is given by

$$\alpha = \begin{cases} \gamma, & \beta > 1, \\ \gamma - \varepsilon, & \beta = 1, \\ \beta + \gamma - 1, & \beta < 1 \end{cases}$$

for any $\varepsilon > 0$.

Recall also the following arithmetic fact [27]. Given $n \in \mathbb{N}$, the number $d(n)$ of the divisors of n satisfies

$$d(n) \leq C_\delta n^{\delta} \tag{4.4}$$

for any $\delta > 0$.

Proof. Let $\widetilde{u}_j$ be an extension of u_j, $j = 1, 2, 3$. Then it suffices to prove that

$$\|\mathcal{N}_k(\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3)\|_{X^{s,b'}_{p}} \lesssim \prod_{j=1}^{3} \|\widetilde{u}_j\|_{X^{s,b}_{p}} \tag{4.5}$$

for $k = 1, 2$ since the desired estimate (4.3) then follows from taking an infimum over all extensions. For simplicity, we denote $\widetilde{u}_j$ by u_j in the following.

We first estimate the nonresonant part $\mathcal{N}_1(u_1, u_2, u_3)$. Let $f_j(\tau, n) := \langle n\rangle^s\langle \tau - n^2\rangle^b|\widehat{u}_j(\tau, n)|$ for $j = 1, 2, 3$. By noting that $\|f_j\|_{\ell^p_n L^p_\tau} = \|u_j\|_{X^{s,b}_p}$, we see that (4.5) follows once we prove that

$$\left\| \frac{\langle n\rangle^s}{\langle \sigma_0\rangle^a} \sum_{\substack{n = n_1 - n_2 + n_3 \\ n \neq n_1, n_3}} \int_{\tau = \tau_1 - \tau_2 + \tau_3} \prod_{j=1}^{3} \frac{f_j(\tau_j, n_j)}{\langle n_j\rangle^s\langle \sigma_j\rangle^b}\, d\tau_1\, d\tau_2 \right\|_{\ell^p_n L^p_\tau} \lesssim \prod_{j=1}^{3} \|f_j\|_{\ell^p_n L^p_\tau}, \tag{4.6}$$

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where $a := -b' > 0$, $\sigma_0 := \tau - n^2$ and $\sigma_j := \tau_j - n_j^2$, $j = 1, 2, 3$. By Hölder's inequality,

$$\text{LHS of (4.6)} \le \Big(\sup_{n,\tau} M_{n,\tau}\Big)^{1/p'} \prod_{j=1}^{3} \|f_j\|_{\ell^p_n L^p_\tau},$$

where $M_{n,\tau}$ is defined by

$$M_{n,\tau} := \frac{\langle n\rangle^{sp'}}{\langle\sigma_0\rangle^{ap'}} \sum_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3}} \int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{3} \frac{1}{\langle n_j\rangle^{sp'}\langle\sigma_j\rangle^{bp'}}\, d\tau_1\, d\tau_2.$$

Hence, it suffices to show that

$$\sup_{n,\tau} M_{n,\tau} < \infty.$$

By the triangle inequality,

$$\langle n-n_1\rangle\langle n-n_3\rangle \sim \langle n^2 - n_1^2 + n_2^2 - n_3^2\rangle \le \sum_{j=0}^{3}\langle\sigma_j\rangle \le \prod_{j=0}^{3}\langle\sigma_j\rangle \tag{4.7}$$

under $n = n_1 - n_2 + n_3$ and $\tau = \tau_1 - \tau_2 + \tau_3$. By symmetry, assume that $|n_1| \ge |n_3|$. By the triangle inequality once again,

$$\frac{\langle n\rangle}{\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle} \lesssim \frac{1}{\langle n_1\rangle\langle n_3\rangle} \quad \text{or} \quad \lesssim \frac{1}{\langle n_2\rangle\langle n_3\rangle}. \tag{4.8}$$

We assume that the latter holds. The situation for the former is essentially identical to what we detail for the latter. From (4.7) and (4.8),

$$\begin{aligned} M_{n,\tau} \lesssim \sum_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3}} &\frac{1}{\langle n_2\rangle^{sp'}\langle n_3\rangle^{sp'}} \frac{1}{\langle n-n_1\rangle^{ap'}\langle n-n_3\rangle^{ap'}} \\ &\times \int_{\tau=\tau_1-\tau_2+\tau_3} \prod_{j=1}^{3} \frac{1}{\langle\sigma_j\rangle^{(b-a)p'}}\, d\tau_1\, d\tau_2. \end{aligned}$$

By choosing $b < 1/p'$ sufficiently close to $1/p'$ and $-(1/p) < b' < 0$ sufficiently close to 0,

$$\tfrac{2}{3} < (b-a)p' = (b+b')p' < 1. \tag{4.9}$$

Then, applying Lemma 4.2 twice,

$$\begin{aligned} M_{n,\tau} \lesssim \sum_{\substack{n=n_1-n_2+n_3\\ n\neq n_1,n_3}} &\frac{1}{\langle n_2\rangle^{sp'}\langle n_3\rangle^{sp'}} \frac{1}{\langle n-n_1\rangle^{ap'}\langle n-n_3\rangle^{ap'}} \\ &\times \frac{1}{\langle\sigma_0 + 2(n-n_1)(n-n_3)\rangle^{3(b-a)p'-2}}. \end{aligned}$$

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Fix $q \in [1, \infty]$ to be chosen later. Then, by Hölder's inequality,

$$M_{n,\tau} \lesssim \bigg(\sum_{n_2,n_3} \frac{1}{\langle n_2\rangle^{sp'q'}\langle n_3\rangle^{sp'q'}}\bigg)^{1/q'} \bigg(\sum_{\substack{n_2,n_3\\ n_3\neq n_2,n}} \frac{1}{\langle n_2-n_3\rangle^{ap'q}\langle n-n_3\rangle^{ap'q}} \times \frac{1}{\langle \sigma_0 - 2(n_2-n_3)(n-n_3)\rangle^{(3bp'-3ap'-2)q}}\bigg)^{1/q}.$$

The first factor on the right-hand side is finite, provided that

$$sp'q' > 1. \tag{4.10}$$

As for the second factor, we set $k_1 = n_2 - n_3$, $k_2 = n - n_3$ and $h = k_1k_2$. By the divisor estimate (4.4),

$$\begin{aligned} &\sum_{h\neq 0} \frac{1}{\langle \sigma_0 - 2h\rangle^{(3bp'-3ap'-2)q}} \frac{1}{\langle h\rangle^{ap'q}} \sum_{\substack{k_1,k_2\neq 0\\ h=k_1k_2}} 1 \\ &\quad\lesssim \sum_{h\neq 0} \frac{1}{\langle \sigma_0 - 2h\rangle^{(3bp'-3ap'-2)q}} \frac{1}{\langle h\rangle^{(ap'-\theta)q}} \end{aligned} \tag{4.11}$$

for any $\theta > 0$. By applying Lemma 4.2 with (4.9), we see that the sum (4.11) is finite with a bound independent of $\sigma_0 \in \mathbb{R}$, provided that

$$(3bp' - 2ap' - 2 - \theta)q > 1. \tag{4.12}$$

Putting (4.10) and (4.12) together, we obtain the restriction

$$3bp' - 2ap' - 2 - \theta > 1 - sp',$$

which holds true for any $s > 0$ and $1 < p < \infty$ by choosing (i) $b < 1/p'$ sufficiently close to $1/p'$, (ii) $-(1/p) < b' = -a < 0$ sufficiently close to 0 and (iii) $\theta > 0$ sufficiently small.

Next, we consider the resonant part $\mathcal{N}_2(u_1, u_2, u_3)$. In this case, we prove the desired estimate with $s = 0$ and $b' = 0$. A general case $s > 0$ then follows from the triangle inequality: $\langle n\rangle^s \lesssim \langle n_1\rangle^s\langle n_2\rangle^s\langle n_3\rangle^s$ for $s \geq 0$. By Young's inequality, Hölder's inequality $((2p + 1)/3p = (1/p) + (2p - 2)/3p)$ and $\ell^p_n \subset \ell^{3p}_n$,

$$\begin{aligned} \|\mathcal{N}_2(u_1,u_2,u_3)\|_{X^{0,0}_p} &\leq \bigg\|\prod_{j=1}^{3} \|\widehat{u}_j(t,n)\|_{\mathcal{F}L_t^{3p/(2p+1)}}\bigg\|_{\ell^p_n} \lesssim \bigg\|\prod_{j=1}^{3} \|\widehat{u}_j(t,n)\|_{\mathcal{F}L_t^{b,p}}\bigg\|_{\ell^p_n} \\ &\lesssim \prod_{j=1}^{3} \|u_j\|_{X^{0,b}_p}, \end{aligned}$$

provided that $b > (2p - 2)/3p$. The last condition obviously holds by taking $b < 1/p'$ sufficiently close to $1/p'$ as long as $p > 1$. □

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5. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. Given $s > 0$ and $1 < p < \infty$, fix $\phi \in \gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$. Given $u_0 \in \mathcal{F}L^{s,p}(\mathbb{T})$, define the operator Γ_{u_0} by

$$\Gamma_{u_0} u(t) := S(t)u_0 + i\int_0^t S(t-t')\mathcal{N}(u)(t')\,dt' - i\Psi.$$

Let $b = (1/p') - \delta$ for some $\delta > 0$ sufficiently small be given by Proposition 4.1. Then, by Lemma 2.2 with $b' = -(1/p) + \delta$ and Proposition 4.1,

$$\|\Gamma_{u_0}(u)\|_{X^{s,b}_{p;T}} \le C_1\|u_0\|_{\mathcal{F}L^{s,p}} + C_2T^{2\delta}\|u\|^3_{X^{s,b}_{p;T}} + \|\Psi\|_{X^{s,b}_{p;T}} \tag{5.1}$$

for $0 < T \le 1$. Similarly,

$$\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X^{s,b}_{p;T}} \le C_3T^{2\delta}(\|u\|^2_{X^{s,b}_{p;T}} + \|v\|^2_{X^{s,b}_{p;T}})\|u - v\|_{X^{s,b}_{p;T}}. \tag{5.2}$$

It follows from Lemma 3.1 that there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) = 1$ such that $\Psi = \Psi^\omega \in X^{s,b}_p([0,1])$ for each $\omega \in \Sigma$. Now, choose $R^\omega := 2C_1\|u_0\|_{\mathcal{F}L^{s,p}} + 2\|\Psi^\omega\|_{X^{s,b}_p([0,1])}$ and positive $T = T^\omega = T(R^\omega) \ll 1$ such that

$$C_2T^{2\delta}R^2 \le \tfrac{1}{2} \quad \text{and} \quad C_3T^{2\delta}R^2 \le \tfrac{1}{4}.$$

(Here, we used the $X^{s,b}_p$-norm of Ψ^ω on the interval $[0,1]$ so that it does not depend on T, since it is used to determine R^ω, which in turn determines T^ω.) Then it follows from (5.1) and (5.2) that Γ_{u_0} is a contraction on the closed ball $B_R \subset X^{s,b}_p([0,T])$ of radius R and thus has a unique fixed point $u = \Gamma_{u_0}(u) \in X^{s,b}_p([0,T])$ for any $\omega \in \Sigma$.

Our choice of $b = (1/p') - \delta$ does not allow us to conclude the continuity of u in time in a direct manner. From Lemma 2.2(i) with (2.3), we see that the linear solution $S(t)u_0$ belongs to $C([0,T]; \mathcal{F}L^{s,p}(\mathbb{T}))$. From Lemma 3.3, we also have $\Psi \in C([0,T]; \mathcal{F}L^{s,p}(\mathbb{T}))$ almost surely. Finally, by applying Lemma 2.2(ii) with $b' = -(1/p) + 2\delta > -(1/p)$ and Proposition 4.1,

$$\bigg\|\int_0^t S(t-t')\mathcal{N}(u)(t')\,dt'\bigg\|_{X^{s,b+2\delta}_{p;T}} \lesssim T^\delta\|\mathcal{N}(u)\|_{X^{s,-(1/p)+2\delta}_{p;T}} \lesssim \|u\|^3_{X^{s,b}_{p;T}} < \infty.$$

Since $b + 2\delta = (1/p') + \delta > 1/p'$, we conclude from (2.3) that the nonlinear part $u - S(t)u_0 + i\Psi$ is also continuous in time with values in $\mathcal{F}L^{s,p}(\mathbb{T})$. Putting all this together, we conclude that $u \in C([0,T]; \mathcal{F}L^{s,p}(\mathbb{T}))$ almost surely. This completes the proof of Theorem 1.1.

Appendix A. On the γ-radonifying operators

In this appendix, we go over the basic definitions and properties of γ-radonifying operators. These operators appear as natural extensions of Hilbert–Schmidt operators to a Banach space setting. A practical example is the theory of stochastic integrations in the Banach space setting, where γ-radonifying operators suitably generalize the role played by Hilbert–Schmidt operators in the more ordinary Hilbert space setting [9, 48]. The content of this section is mostly taken from the textbook [28, Ch. 9].

Let H be a separable Hilbert space and B be a Banach space. We denote the space of bounded linear operators from H into B by $\mathcal{L}(H;B)$. An operator $T \in \mathcal{L}(H;B)$ is said to be of finite rank if it can be represented as

$$T(\cdot) = \sum_{n=1}^{N} \langle \cdot, e_n \rangle x_n,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H, $\{e_n\}_{n=1}^N$ is orthonormal in H and $\{x_n\}_{n=1}^N \subset B$.

Definition A.1. We define the space $\gamma(H;B)$ as the closure of the set of finite-rank operators in $\mathcal{L}(H;B)$ under the norm:

$$\|T\|_{\gamma(H;B)} := \sup_{N\in\mathbb{N}} \bigg(\mathbb{E}\bigg[\bigg\| \sum_{n=1}^{N} g_n T(e_n) \bigg\|_B^2 \bigg] \bigg)^{1/2}, \tag{A.1}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for H and $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of independent standard complex-valued Gaussian random variables. An operator $T \in \gamma(H;B)$ is called a γ-radonifying operator.

The definition (A.1) is independent of a choice of orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for H. In the following, it is understood that the results are independent of this choice. Note that $(\gamma(H;B), \|\cdot\|_{\gamma(H;B)})$ is a Banach space and is separable whenever B is separable.

Remark A.2. The $L^2(\Omega)$-norm appearing in (A.1) can be replaced with $L^p(\Omega)$ for any $1 \le p < \infty$. Strictly speaking, this creates a family of spaces $\gamma_p(H;B)$; however, by the Kahane–Khintchine inequality (namely, the Banach-valued extension of Lemma 3.2), all these norms are equivalent. Hence, it suffices to consider the most natural choice $p = 2$ and set $\gamma(H;B) := \gamma_2(H;B)$.

The property of being γ-radonifying is stable under transformations by bounded linear operators in either direction.

Lemma A.3 (Ideal property [28, Theorem 9.1.10]). *Let H and H' be Hilbert spaces and B and B' be Banach spaces. Let $U \in \mathcal{L}(H';H)$, $S \in \mathcal{L}(B;B')$ and $T \in \gamma(H;B)$. Then $STU \in \gamma(H';B')$ and*

$$\|STU\|_{\gamma(H';B')} \le \|S\| \, \|T\|_{\gamma(H;B)} \|U\|.$$

A very useful characterization of γ-radonifying operators is the following proposition.

Proposition A.4 [28, Theorem 9.1.17]. *An operator $T \in \mathcal{L}(H;B)$ is γ-radonifying if and only if the sum $\sum_{n\in\mathbb{N}} g_n T(e_n)$ converges in $L^p(\Omega;B)$ for some $1 \le p < \infty$ and some orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for H. If $T \in \gamma(H;B)$, then this sum also converges almost*

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surely and defines a B-valued Gaussian random variable with covariance operator TT^*. Furthermore,

$$\|T\|_{\gamma(H;B)} = \Big(\mathbb{E}\Big[\Big\|\sum_{n\in\mathbb{N}} g_n T(e_n)\Big\|_B^2\Big]\Big)^{1/2} \tag{A.2}$$

for any orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for H.

Recall that $T \in \mathcal{L}(H; H')$ is Hilbert–Schmidt from H to Hilbert H' if

$$\|T\|^2_{HS(H;H')} := \sum_{n\in\mathbb{N}} \|T(e_n)\|^2_{H'} < \infty$$

for some (and hence any) orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for H. It is then clear from (A.2) that when B is a Hilbert space, then we have $\gamma(H; B) = HS(H; B)$ and $\|T\|_{\gamma(H;B)} = \|T\|_{HS(H;B)}$.

Note that when B is a Hilbert space, we can express the $\gamma(H; B)$-norm *without* the use of probability. When B is a Lebesgue space, we can also characterize the $\gamma(H; B)$-norm without the use of probability.

Proposition A.5 [28, Proposition 9.3.1 and (9.21) on page 285]. *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $1 \le p < \infty$ and $T \in \mathcal{L}(H; L^p(X))$.*

Then $T \in \gamma(H; L^p(X))$ if and only if the function $(\sum_{n\in\mathbb{N}} |T(e_n)|^2)^{1/2}$ lies in $L^p(X)$ for some orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for H. In this case,

$$\|T\|_{\gamma(H;L^p(X))} \sim \Big\|\Big(\sum_{n\in\mathbb{N}} |T(e_n)|^2\Big)^{1/2}\Big\|_{L^p(X)}.$$

In Section 1, we defined the $\gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$-norm by (1.11) for $s \in \mathbb{R}$ and $1 \le p < \infty$. Let us see how this definition (1.11) appears from the theory discussed above. Let $\phi \in \mathcal{L}(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$. Then, from the definition (1.10) of the Fourier–Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$, we have $\langle\nabla\rangle^s\phi \in \mathcal{L}(L^2(\mathbb{T}); \mathcal{F}L^{0,p}(\mathbb{T}))$ and $\mathcal{F}\langle\nabla\rangle^s\phi \in \mathcal{L}(L^2(\mathbb{T}); \ell^p(\mathbb{Z}))$. From the ideal property (Lemma A.3) and the invertibility of $\langle\nabla\rangle^s$ and $\mathcal{F}$, we see that $\phi \in \gamma(L^2(\mathbb{T}); \mathcal{F}L^{s,p}(\mathbb{T}))$ if and only if $\mathcal{F}\langle\nabla\rangle^s\phi \in \gamma(L^2(\mathbb{T}); \ell^p(\mathbb{Z}))$. Furthermore, from Proposition A.5,

$$\begin{aligned}\|\phi\|_{\gamma(L^2(\mathbb{T});\mathcal{F}L^{s,p}(\mathbb{T}))} &= \|\mathcal{F}\langle\nabla\rangle^s\phi\|_{\gamma(L^2(\mathbb{T});\ell^p(\mathbb{Z}))}\\ &\sim \Big\|\Big(\sum_{k\in\mathbb{Z}} |\mathcal{F}(\langle\nabla\rangle^s\phi(e_k))(n)|^2\Big)^{1/2}\Big\|_{\ell^p_n(\mathbb{Z})},\end{aligned}$$

where $e_k(x) = e^{2\pi ikx}$, $k \in \mathbb{Z}$. This justifies the definition (1.11).

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