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A DIFFUSIVE LOGISTIC EQUATION WITH MEMORY IN BESSEL POTENTIAL SPACES

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Dedicated to Agnaldo Maciel Viana on his 66th birthday

Abstract

This paper is devoted to the study of the local existence, uniqueness, regularity, and continuous dependence of solutions to a logistic equation with memory in the Bessel potential spaces.

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1. Introduction

The logistic equation that takes into account dispersal effects is given by

$$u_t(t,x) = D\Delta u(t,x) + au(t,x) - bu^2(t,x).$$

Here, $u(t,x)$ is the concentration of the population at the location x and time $t > 0$, D is the diffusion coefficient and a and b are the growth rate and the crowding effect, respectively. This model has been widely utilised for many different purposes. See, for example, [5, 6, 8, 9, 11, 13, 14, 16, 17] and the references therein.

Logistic equations subject to memory effects have also been considered; see, for example, [7, 10, 12, 18, 24–26]. Cushing [7] gives a systematic analysis of memory effects in population dynamics. Gopalsamy [12] investigated the asymptotic behaviour of nonconstant solutions of delay logistic equations. In particular, he considered the logistic equation with continuously distributed delays

$$\frac{dx}{dt} = x(t)\Big[a - b\int_{-\infty}^{t} H(t-s)x(s)\,ds\Big],$$

where a and b are positive numbers and H is a delay kernel representing the manner in which the past history of the species influences the current growth rate. Continuously distributed delays are also known as the memory.

Motivated by these considerations, we study the following logistic equation with memory starting from the initial time:

$$u_t(t,x) = \Delta u(t,x) + u(t,x)\Big[a - b\int_0^t \lambda(t-s)u(s,x)\,ds\Big]. \tag{1.1}$$

We also consider a more general Cauchy–Dirichlet problem:

$$u_t(t,x)\Delta u(t,x) + u(t,x)\Big[a - b\int_0^t \lambda(t-s)(-\Delta)^{\beta}u(s,x)\,ds\Big] \quad \text{in } (0,\infty)\times\Omega, \tag{1.2}$$

$$u = 0 \quad \text{on } (0,\infty)\times\partial\Omega, \tag{1.3}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.4}$$

in a sufficiently regular domain $\Omega \subset \mathbb{R}^n$. Notice that (1.2) reduces to (1.1) whenever $\beta = 0$. Here, $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ functions as a delay kernel representing the manner in which the history of the species influences the current growth rate. Condition (1.3) means that the boundary of Ω is inhospitable.

Under certain conditions, the existence of solutions to the problem

$$\begin{gathered} u_t(t,x) = \Delta u(t,x) + u(t,x)\Big[a - bu - \int_0^t \lambda(t-s)u(s,x)\,ds\Big], \\ \partial u/\partial n = 0, \\ u(x,0) = u_0(x), \end{gathered}$$

for $(t,x) \in (0,\infty)\times\Omega$, was proved by Schiaffino [19] and Yamada [24]. In [19] the initial data was taken in $\{\varphi \in C^1(\overline{\Omega}) : \partial u/\partial n = 0 \text{ on } \partial\Omega\}$, whereas initial data in $\{\varphi \in W^{2,p}(\Omega) : \partial u/\partial n = 0 \text{ on } \partial\Omega\}$ was considered in [24].

The study of partial differential equations in low-regularity spaces has attracted much interest of late, which motivates us to take the initial data in the Bessel potential space $H_0^{\sigma,p} = \{\varphi \in H^{\sigma,p}(\Omega) : \varphi|_{\partial\Omega} = 0\}$, with $1 < p < \infty$ and $0 < \sigma < 2$. Thus we allow more irregular initial data than Schiaffino [19] and Yamada [24]. As examples of parabolic problems treated in Bessel potential spaces, we cite the study of the Navier–Stokes equations (see [3, 4, 20, 21] and references therein) and the models of population dynamics in low-regularity spaces in the recent paper by Viana [23].

Our main result gives a unique mild solution to the problem (1.2)–(1.4) which is spatially more regular after the starting point, provided the indices N, p, σ and β are suitably chosen, and depends continuously on the initial data. Moreover, the time of existence of this solution is uniform for initial data taken in balls of small radius in $H_0^{\sigma,p}$, and therefore uniform in precompact subsets of $H_0^{\sigma,p}$.

2. Preliminaries

Given a Banach space Y, as usual, $\|\cdot\|_Y$ denotes the norm associated to Y. The ball of radius r and centred at $x \in Y$ is denoted by $B_Y(x,r)$. If X and Y are Banach spaces, $X \hookrightarrow Y$ means that X is continuously and densely embedded in Y.

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DEFINITION 2.1. A continuous function $u : [0,\tau] \longrightarrow H_0^{\sigma,p}$ is said to be a mild solution for (1.2)–(1.4), if it is a solution of the following integral equation:

$$u(t) = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}u(s)\Big[a - b\int_0^s \lambda(s-r)(-\Delta)^\beta u(r)\,dr\Big]\,ds.$$

2.1. Bessel potential spaces and the heat semigroup. Let $1 < p < \infty$ and let $\sigma \in (0,2)\backslash\{1/p\}$. The Bessel potential space $H_0^{\sigma,p}$ coincides with the complex interpolation space $[W^{2,p} \cap W_0^{1,p}, L^p(\Omega)]_{\sigma/2}$ for $0 < \sigma < 2, \sigma \neq 1/p$ (see [22, Section 4.3.3]).

It is well known that the Dirichlet Laplacian Δ is a sectorial operator from $W^{2,p} \cap W_0^{1,p}$ into $L^p(\Omega)$ (see, for example, [15]). Therefore, by the theory developed in [2, Ch. V], the heat semigroup $e^{\Delta t} : L^p(\Omega) \longrightarrow L^p(\Omega)$ satisfies the estimate

$$t^{\sigma'/2-\sigma/2}\|e^{\Delta t}\varphi\|_{H_0^{\sigma',p}} \le M\|\varphi\|_{H_0^{\sigma,p}}, \tag{2.1}$$

for all $\varphi \in H_0^{\sigma,p}$ and $t > 0$, where $M \ge 1$. Here $0 \le \sigma \le \sigma' < 2$ and neither of σ, σ' is equal to $1/p$. In particular, if $\sigma \in (0,2)\backslash\{1/p\}$, then

$$t^{\sigma/2}\|e^{\Delta t}\varphi\|_{H_0^{\sigma,p}} \le M\|\varphi\|_{L_p(\Omega)}, \tag{2.2}$$

for all $\varphi \in L_p(\Omega)$ and $t > 0$.

REMARK 2.2. The following estimate is essential to treat the term which involves the fractional Laplacian:

$$\|(-\Delta)^{\sigma/2}\varphi\|_{L^p} \le c\|\varphi\|_{H^{\sigma,p}},$$

for all $\varphi \in H^{\sigma,p}$. This is a consequence of the equivalence between the norms $\|(-\Delta)^\sigma(\cdot)\|_{L^p}$ and $\|(I-(-\Delta)^\sigma)(\cdot)\|_{L^p}$, the fact that $\|(I-(-\Delta)^\sigma)(\cdot)\|_{L^p(\mathbb{R}^N)}$ is a norm on $H^{2\sigma,p}(\mathbb{R}^N)$, and that the extension operator $E : H^{s,p}(\Omega) \longrightarrow H^{s,p}(R^N)$ is continuous (see [1, Ch. 7]).

Henceforth, we assume that

$$1 < p < \infty, \quad \sigma \in (0,2)\setminus\left\{\frac{1}{p}\right\} \quad \text{and} \quad \sigma \ge 2\beta + \frac{N}{2p}. \tag{H}$$

Under these conditions, we use the results contained in [22, Section 4.6] to obtain the embeddings

$$H_0^{\sigma,p} \hookrightarrow H^{\sigma,p}(\Omega) \hookrightarrow H^{2\beta,2p}(\Omega) \hookrightarrow L^{2p}(\Omega) \hookrightarrow L^p(\Omega). \tag{2.3}$$

2.2. Nonlinear estimates.

LEMMA 2.3. *Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a locally integrable function. Assume that (H) holds and consider the function $g : [0,\infty) \times H_0^{\sigma,p} \to L^p(\Omega)$ defined by*

$$g(t,\varphi) = \varphi\Big[a - b\int_0^t \lambda(t-s)(-\Delta)^\beta\varphi\,ds\Big].$$

Then, given $\varphi, \psi \in H_0^{\sigma,p}$, there exists $C > 0$ such that

$$\|g(t,\varphi) - g(t,\psi)\|_{L^p(\Omega)} \le C\|\varphi - \psi\|_{H_0^{\sigma,p}}[\|\lambda\|_{L^1_{\mathrm{loc}}(0,t)}(\|\varphi\|_{H_0^{\sigma,p}} + \|\psi\|_{H_0^{\sigma,p}}) + 1]$$

and

$$\|g(t,\varphi)\|_{L^p(\Omega)} \le C\|\varphi\|_{H_0^{\sigma,p}}(\|\lambda\|_{L^1(0,t)}\|\varphi\|_{H_0^{\sigma,p}} + 1),$$

where the constant C depends on $a, b, |\Omega|$ (the Lebesgue volume of Ω) and the embeddings (2.3).

Proof. Let $\varphi, \psi \in H_0^{\sigma,p}$. First we write

$$\begin{aligned} &g(t,\varphi) - g(t,\psi) \\ &\quad = (\varphi - \psi)\Big[a - b\int_0^t \lambda(t-s)(-\Delta)^\beta\varphi\Big] + b\psi\int_0^t \lambda(t-s)[(-\Delta)^\beta\psi - (-\Delta)^\beta\varphi]\,ds. \end{aligned}$$

We now use the Minkowski and Hölder inequalities combined with Remark 2.2 and the embeddings (2.3) to obtain

$$\begin{aligned} &\|g(t,\varphi) - g(t,\psi)\|_{L^p} \\ &\quad \le \|\varphi - \psi\|_{L^{2p}}\Big\|a - b\int_0^t \lambda(t-s)(-\Delta)^\beta\varphi\,ds\Big\|_{L^{2p}} \\ &\qquad + b\|\psi\|_{L^{2p}}\int_0^t |\lambda(t-s)|\,\|(-\Delta)^\beta\psi - (-\Delta)^\beta\varphi\|_{L^{2p}}\,ds \\ &\quad \le C\|\varphi - \psi\|_{H_0^{\sigma,p}}(a|\Omega|^{1/2p} + b\|\lambda\|_{L^1(0,t)}\|\varphi\|_{H_0^{\sigma,p}} + b\|\lambda\|_{L^1(0,t)}\|\psi\|_{H_0^{\sigma,p}}) \\ &\quad \le C\|\varphi - \psi\|_{H_0^{\sigma,p}}[\|\lambda\|_{L^1(0,t)}(\|\varphi\|_{H_0^{\sigma,p}} + \|\psi\|_{H_0^{\sigma,p}}) + 1]. \end{aligned}$$

In particular, taking $\psi \equiv 0$,

$$\|g(t,\varphi)\|_{L^p} \le C\|\varphi\|_{H_0^{\sigma,p}}(\|\lambda\|_{L^1(0,t)}\|\varphi\|_{H_0^{\sigma,p}} + 1)$$

for all $\varphi \in H^{\sigma,p}$, because $\sigma \ge 2\beta + N/2p$. □

Lemma 2.4. *Let $\lambda : \mathbb{R} \to \mathbb{R}$ be a locally integrable function. Assume that (H) holds and consider functions $u_i : [0,\tau] \to H_0^{\sigma,p}$ such that*

$$\sup_{t\in[0,\tau]} \|u_i(t)\|_{H_0^{\sigma,p}} \le \mu', \quad i = 1, 2,$$

where $\mu' > 0$. Further, suppose that $\lambda : [0,\infty) \to [0,\infty)$ is locally integrable. Then

$$\begin{aligned} &\Big\|\int_0^t e^{\Delta(t-s)}[g(r,u_1(r)) - g(r,u_2(r))]\,dr\,ds\Big\|_{H_0^{\sigma,p}} \\ &\quad \le MC(2\|\lambda\|_{L^1(0,t)}\mu' + 1)\frac{2}{2-\sigma}t^{1-\sigma/2}\sup_{s\in[0,t]}\|u_1(s) - u_2(s)\|_{H_0^{\sigma,p}}, \end{aligned}$$

and

$$\Big\|\int_0^t e^{\Delta(t-s)}g(r,u(r))\,dr\,ds\Big\|_{H_0^{\sigma,p}} \le MC(\|\lambda\|_{L^1(0,t)}\mu' + 1)\mu'\frac{2}{2-\sigma}t^{1-\sigma/2}. \tag{2.4}$$

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PROOF. A combination of Remark 2.2 and Lemma 2.3 gives

$$\begin{aligned}\Big\|\int_0^t & e^{(t-s)\Delta}[g(r,u_1(r))-g(r,u_2(r))]\,dr\,ds\Big\|_{H_0^{\sigma,p}} \\ &\le MC\sup_{s\in[0,t]}\|u_1(s)-u_2(s)\|_{H_0^{\sigma,p}}\int_0^t (t-s)^{-\sigma/2}\,ds(2\|\lambda\|_{L^1(0,t)}\mu'+1) \\ &\le MC(2\|\lambda\|_{L^1_{\rm loc}(0,t)}\mu'+1)\frac{2}{2-\sigma}t^{1-\sigma/2}\sup_{s\in[0,t]}\|u_1(s)-u_2(s)\|_{H_0^{\sigma,p}},\end{aligned}$$

and in a similar way one obtains (2.4). □

3. Main result

We can now state and prove our main result.

THEOREM 3.1. *Let $\lambda:\mathbb{R}\to\mathbb{R}$ be a locally integrable function and assume that (H) holds. Given $v_0\in H_0^{\sigma,p}$, there exist $\tau>0$ and $r>0$ such that for every $u_0\in B_{H_0^{\sigma,p}}(v_0,r)$ the Cauchy–Dirichlet problem (1.2)–(1.4) possesses a unique mild solution $u:[0,\tau]\to H_0^{\sigma,p}$. Further, $u\in C((0,\tau];H_0^{\sigma',p})$ for every $\sigma'\in[\sigma,2)\backslash\{1/p\}$ and the solutions depend continuously on the initial data.*

PROOF. Let $0<\mu\le 1$. Choose $\tau>0$ small enough so that, for all $t\in[0,\tau]$,

$$\|e^{\Delta t}v_0-v_0\|_{H_0^{\sigma,p}}<\frac{\mu}{3}$$

and

$$MC(\|\lambda\|_{L^1(0,t)}\mu'+1)\mu'\frac{2}{2-\sigma}t^{1-\sigma/2}<\frac{\mu}{3},$$

where $\mu':=\mu+\|v_0\|_{H_0^{\sigma,p}}$. Set $r=\mu/3M$. It follows that $\|e^{\Delta t}u_0-v_0\|_{H_0^{\sigma,p}}<2\mu/3$.

Now define

$$\mathcal{B}=\Big\{u\in C([0,\tau];H_0^{\sigma,p}):\sup_{t\in[0,\tau]}\|u(t)-v_0\|_{H_0^{\sigma,p}}\le\mu\Big\}.$$

Consider the map $\Lambda:\mathcal{B}\to\mathcal{B}$ defined by

$$(\Lambda u)(t)=e^{\Delta t}u_0+\int_0^t e^{\Delta(t-s)}u(s)\Big[a-b\int_0^s\lambda(s-r)(-\Delta)^\beta u(r)\,dr\Big]\,ds.$$

First, we will show that we actually have $\Lambda\mathcal{B}\subset\mathcal{B}$. For $0\le t_1<t_2\le\tau$ and $u\in\mathcal{B}$,

$$\begin{aligned}\|(\Lambda u)(t_1)&-(\Lambda u)(t_2)\|_{H_0^{\sigma,p}} \\ &\le\|e^{\Delta t_1}u_0-e^{\Delta t_2}u_0\|_{H_0^{\sigma,p}} \\ &\quad+\int_0^{t_1}\Big\|[e^{\Delta(t_1-s)}-e^{\Delta(t_2-s)}]u(s)\Big[a-b\int_0^s\lambda(s-r)(-\Delta)^\beta u(r)\,dr\Big]\Big\|_{H_0^{\sigma,p}}\,ds \\ &\quad+\int_{t_1}^{t_2}\Big\|e^{\Delta(t_2-s)}u(s)\Big[a-b\int_0^s\lambda(s-r)(-\Delta)^\beta u(r)\,dr\Big]\Big\|_{H_0^{\sigma,p}}\,ds\end{aligned}$$

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$$
\begin{aligned}
&\leq \|e^{\Delta t_1}u_0 - e^{\Delta t_2}u_0\|_{H_0^{\sigma,p}} \\
&\quad + \|I - e^{\Delta(t_2-t_1)}\|_{\mathcal{L}(H_0^{\sigma,p})} MC(\|\lambda\|_{L^1(0,t_1)}\mu' + 1)\mu' \frac{2}{2-\sigma} t_1^{1-\sigma/2} \\
&\quad + MC\frac{2}{2-\sigma}\Big(1 - \Big(\frac{t_1}{t_2}\Big)^{1-\sigma/2}\Big)(\|\lambda\|_{L^1(0,t)}\mu' + 1)\mu',
\end{aligned}
$$

which converges to zero as either $t_1 \to t_2^-$ or $t_2 \to t_1^+$. Moreover, from Lemma 2.4,

$$
\begin{aligned}
&\|(\Lambda u)(t) - v_0\|_{H_0^{\sigma,p}} \\
&\quad \leq \|e^{\Delta t}u_0 - v_0\|_{H_0^{\sigma,p}} + \int_0^t \Big\| e^{\Delta(t-s)}\Big[a - b\int_0^s \lambda(s-r)(-\Delta)^{\beta}u(r)\,dr\Big]\,dr\Big\|_{H_0^{\sigma,p}}\,ds \\
&\quad < \frac{2\mu}{3} + MC\frac{2}{2-\sigma}t^{1-\sigma/2}(\|\lambda\|_{L^1(0,t)}\mu' + 1)\mu' \leq \frac{2\mu}{3} + \frac{\mu}{3} = \mu.
\end{aligned}
$$

Hence, Λ is well defined.

Next, we show that Λ is a contraction. For $u, v \in \mathcal{B}$, by Lemma 2.4,

$$
\begin{aligned}
&\|(\Lambda u)(t) - (\Lambda v)(t)\|_{H_0^{\sigma,p}} \\
&\quad \leq MC\frac{2}{2-\sigma}\tau^{1-\sigma/2}(2\|\lambda\|_{L^1(0,\tau)}\mu' + 1)\sup_{s\in[0,t]}\|u(s) - v(s)\|_{H_0^{\sigma,p}} \\
&\quad \leq \frac{\mu}{3}\sup_{s\in[0,t]}\|u(s) - v(s)\|_{H_0^{\sigma,p}} \\
&\quad \leq \frac{1}{3}\sup_{s\in[0,t]}\|u(s) - v(s)\|_{H_0^{\sigma,p}}.
\end{aligned}
$$

Hence, by the Banach fixed point theorem, Λ has a unique fixed point $u \in \mathcal{B}$. This is a mild solution for (1.2)–(1.4).

If we repeat these steps, but using (2.1) instead of (2.2), we find that

$$
\begin{aligned}
&\|u(t_1) - u(t_2)\|_{H_0^{\sigma',p}} \\
&\quad \leq Mt^{\sigma/2-\sigma'/2}\|I - e^{\Delta(t_2-t_1)}\|_{\mathcal{L}(H_0^{\sigma',p})}\|u_0\|_{H_0^{\sigma,p}} \\
&\qquad + \|I - e^{\Delta(t_2-t_1)}\|_{\mathcal{L}(H_0^{\sigma',p})} MC(\|\lambda\|_{L^1(0,t)}\mu' + 1)\mu'\frac{2}{2-\sigma}t^{1-\sigma'/2} \\
&\qquad + MC\frac{2}{2-\sigma'}\Big(1 - \Big(\frac{t_1}{t_2}\Big)^{1-\sigma'/2}\Big)(\|\lambda\|_{L^1(0,t)}\mu' + 1)\mu',
\end{aligned}
$$

for $0 < t_1 < t_2 \leq \tau$. Consequently, $u \in C((0,\tau]; H_0^{\sigma',p})$. This shows the existence and regularity of the mild solution. Let us prove that it is unique. Let $\tilde{u}$ be a mild solution of (1.2)–(1.4). Then, by Lemma 2.3,

$$
\begin{aligned}
&\|u(t) - \tilde{u}(t)\|_{H_0^{\sigma,p}} \\
&\quad \leq M\int_0^t (t-s)^{-\sigma/2}\|g(u(r)) - g(\tilde{u}(r))\|_{L^p}\,dr\,ds \\
&\quad \leq MC\int_0^t (t-s)^{-\sigma/2}\sup_{r\in[0,s]}\|u(r) - \tilde{u}(r)\|_{H_0^{\sigma,p}}(2\|\lambda\|_{L^1(0,t)}\eta + 1)\,ds,
\end{aligned}
$$

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where $\eta := \max\{\sup_{t\in[0,\tau]} \|u(t)\|_{H_0^{\sigma,p}}, \sup_{t\in[0,\tau]} \|\tilde{u}(t)\|_{H_0^{\sigma,p}}\}$. Thus,

$$\|u(t) - \tilde{u}(t)\|_{H_0^{\sigma,p}} \le MC(2\|\lambda\|_{L^1(0,\tau)}\eta + 1) \int_0^t (t-s)^{-\sigma/2} \sup_{r\in[0,s]} \|u(r) - \tilde{u}(r)\|_{H_0^{\sigma,p}}\, ds.$$

Put $f(t) := \sup_{s\in[0,t]} \|u(s) - \tilde{u}(s)\|_{H_0^{\sigma,p}}$ and $C = MC(2\|\lambda\|_{L^1(0,\tau)}\eta + 1)$. It follows that

$$f(t) \le C \int_0^t (t-s)^{-\sigma/2} f(s)\, ds,$$

for all $t \in [0,\tau]$. By the singular Gronwall inequality, $f(t) = 0$ for all $t \in [0,\tau]$ and uniqueness follows.

Finally, take $u_1, u_2 \in B_{H_0^{\sigma,p}}(v_0, r)$ and, for $i = 1, 2$, let $u_i(t)$ be the mild solution that starts at u_i, $i = 1, 2$. Then

$$\begin{aligned}
&\|u_1(t) - u_2(t)\|_{H_0^{\sigma,p}} \\
&\quad \le \|e^{\Delta t}u_1 - e^{\Delta t}u_2\|_{H_0^{\sigma,p}} \\
&\qquad + MC(2\|\lambda\|_{L^1}(0,\tau)\mu' + 1)\frac{2}{2-\sigma}\tau^{1-\sigma/2} \sup_{s\in[0,t]} \|u_1(t) - u_2(t)\|_{H_0^{\sigma,p}} \\
&\quad \le \|u_1 - u_2\|_{H_0^{\sigma,p}} + \frac{\mu}{3\mu'} \sup_{s\in[0,t]} \|u_1(s) - u_2(s)\|_{H_0^{\sigma,p}}.
\end{aligned}$$

Hence,

$$\sup_{s\in[0,t]} \|u_1(s) - u_2(s)\|_{H_0^{\sigma,p}} \le \tfrac{3}{2}\|u_1 - u_2\|_{H_0^{\sigma,p}}.$$

Thus the mild solution of (1.2)–(1.4) depends continuously on the initial data. □

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