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DECOMPOSITION OF K_n INTO DRAGONS

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ABSTRACT. It is shown that if $1 < n \equiv 0$ or $1 \pmod{2m}$, then the edges of K_n may be partitioned into isomorphic copies of a graph $D_3(m)$ and also of a graph $D_4(m)$, graphs consisting respectively of a triangle with an attached path of $m-3$ edges or a quadrilateral with an attached path of $m-4$ edges. If m is a power of 2 then the above condition is shown to be necessary and sufficient for the existence of such a partition.

1. **Introduction.** The complete graph K_n is said to have a G-decomposition, if it is the union of edge disjoint subgraphs each isomorphic to G.

An immediate and well-known necessary condition for the existence of a G-decomposition of K_n, if G has m edges, is

$$(1) \qquad n(n-1) \equiv 0 \pmod{2m}.$$

The problem of determining the set of integers $N(G)$ for which K_n has a G-decomposition has been solved completely or partially only for some particular graphs G namely for stars, paths, circles and also for all graphs having no more than four vertices. For more detailed references see [1].

DEFINITION 1. A *dragon* $D_3(m)$ respectively $D_4(m)$ is a graph having m edges and consisting of a triangle or a quadrilateral respectively and an attached path, called tail.

In this paper G will always denote a dragon.

As a first result of this paper we will prove in Theorem 1 the sufficiency of each of the conditions

$$(2) \qquad n \equiv 1 \pmod{2m},$$

$$(3) \qquad n \equiv 0 \pmod{2m},$$

if G is a dragon $D_3(m)$ or $D_4(m)$. Consequently as formulated in Theorem 2 condition (1) for dragons appears to be necessary and sufficient if m is a power of 2. This establishes a complete solution of the G-decomposition problem for $D_3(2^\alpha)$ and $D_4(2^\alpha)$ namely: for

$$i = 3, 4 \qquad 2^\alpha > i$$

$$(4) \qquad N(D_i(2^\alpha)) = \{n \mid 1 < n \equiv 0 \quad \text{or} \quad 1 \pmod{2^{\alpha+1}}\}.$$

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This generalizes

$$N(D_3(4)) = \{n \mid 1 < n \equiv 0 \quad \text{or} \quad 1 \pmod 8\},$$

a result in [1].

2. **Notation and definitions.** The vertex set of K_n will be either Z_n or $Z_{n-1} \cup \infty$ depending on whether $n \equiv 1$ or $n \equiv 0 \pmod{2m}$.

A dragon $D_3(m)$ consisting of the triangle $\{a, b, c\}$ with tail attached to the vertex c will be denoted by $(a, b, c; x_1, x_2, \ldots, x_{m-3})$. Similarly $(a, b, c, d; x_1, x_2, \ldots, x_{m-4})$ is a dragon consisting of the quadrilateral with edges $\{ab, bc, cd, da\}$ and with the tail attached to the vertex d.

It seems to be useful to denote by $(a, b, c; x_1, x_2, \ldots, x_{m-3}) \pmod n$ the set of graphs

$$(a+j, b+j, c+j; x_1+j, x_2+j, \ldots, x_{m-3}+j) \qquad j = 0, 1, 2, \ldots, n-1,$$

where all the vertices are in Z_n.

When the vertex set is $Z_{n-1} \cup \infty$, denote by

$$(a, b, c; x_1, x_2, \ldots, x_{m-4}, \infty) \pmod{n-1}$$

the set of graphs

$$(a+j, b+j, c+j; x_1+j, x_2+j, \ldots, x_{m-4}+j, \infty) \qquad j = 0, 1, 2, \ldots, n-2.$$

A similar notation will be used also for $D_4(m)$.

Two partitions of integers, which are known [2, 3] to exist will be used in our construction and are as follows:

DEFINITION 2

(i) Let $t \equiv 0, 1 \pmod 4$. A partition of the integers $\{1, 2, \ldots, 2t\}$ into t pairs (p_i, q_i) such that $q_i - p_i = i$ for $i = 1, 2, \ldots, t$ will be called partition A.

(ii) Let $t \equiv 2, 3 \pmod 4$. A partition of the integers $\{1, 2, \ldots, 2t-1, 2t+1\}$ into t pairs (p_i, q_i) such that $q_i - p_i = i$ for $i = 1, 2, \ldots, t$ will be called partition B.

The set of pairs $\{(p_i, q_i), i = 1, 2, \ldots, t\}$ will denote a partition A or a partition B, depending on whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3 \pmod 4$.

3. **Four lemmas.**

LEMMA 1. *Let*

$$y_i = \begin{cases} \dfrac{m-3}{2}t & \text{if } m \text{ is odd} \\[2ex] \dfrac{m+2}{2}t + i & \text{if } m \text{ is even} \end{cases} \tag{5}$$

$$i = 2, 3, \ldots, t$$

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and let

$$y_1=\begin{cases}\text{as defined in (5), if } t\equiv 0,1 \pmod 4 \text{ i.e.}\\ \frac{m-3}{2}t, \text{ for } m \text{ odd, } \frac{m+2}{2}t+1, \text{ for } m \text{ even,}\\ \frac{m-3}{2}t+1, \text{ for } m \text{ odd, } \frac{m+2}{2}t \text{ for } m \text{ even, if}\\ t\equiv 2,3 \pmod 4\end{cases}$$

then the following $t(2mt+1)$ *graphs form a* $D_3(m)$*-decomposition of* K_{2mt+1}:

$$(v-p_i-t, v-q_i-t, 0; (m-1)t+i, t, (m-2)t+i, 2t, \ldots, y_i) \pmod v$$

where $i=1,2,\ldots,t$, $v=2mt+1$.

LEMMA 2. *Let*

$$(6)\qquad y_i=\begin{cases}\frac{m-3}{2}t & \text{for } m \text{ odd}\\ \frac{m+2}{2}t+i-1 & \text{for } m \text{ even}\end{cases}$$

$$i=3,4,\ldots,t$$

$$y_1=\begin{cases}\text{as defined in (6), with } i=1, \text{ if}\\ t\equiv 2,3 \pmod 4\\ \infty \qquad \text{if } t\equiv 0,1 \pmod 4\end{cases}$$

$$y_2=\begin{cases}\text{as defined in (6), with } i=2, \text{ if}\\ t\equiv 0,1 \pmod 4\\ \infty \qquad \text{if } t\equiv 2,3 \pmod 4\end{cases}$$

then the following $t(2mt-1)$ *graphs form a* $D_3(m)$*-decomposition of* K_{2mt}:

$$(v-p_i-t, v-q_i-t, 0; (m-1)t+i-1, t, (m-2)t+i-1, 2t, \ldots, y_i) \pmod v$$

where $i=1,2,\ldots,t$, $v=2mt-1$.

LEMMA 3. *Let*

$$y_i=\begin{cases}\frac{m-4}{2}t & \text{if } m \text{ is even}\\ \frac{m+3}{2}t+i & \text{if } m \text{ is odd.}\end{cases}$$

then the following $t(2mt+1)$ *graphs form a* $D_4(m)$*-decomposition of* K_{2mt+1}:

$$(v-2i, 2mt, v-(4t-2i+2), 0; (m-1)t+i, t, (m-2)t+i, 2t, \ldots, y_i) \pmod v$$

where $i=1,2,\ldots,t$, $v=2mt+1$.

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LEMMA 4. *Let*

$$y_i = \begin{cases} \dfrac{m-4}{2}t & \text{if } m \text{ is even} \\ \dfrac{m+3}{2}t+i-1 & \text{if } m \text{ is odd} \end{cases}$$

for $i=1,2,\ldots,t-1$, *while* $y_t=\infty$, *then the following* $t(2mt-1)$ *graphs form a* $D_4(m)$*-decomposition of* K_{2mt}:

$$(v-2i, 2mt, v-(4t-2i+2), 0; (m-1)t+i-1, t, (m-2)t+i-1, 2t, \ldots, y_i) \times (\text{mod } v)$$

where $i=1,2,\ldots,t$, $v=2mt-1$.

Proof of Lemmas 1–4. The direct construction exhibited in Lemmas 1–4 may be checked as follows. Every edge (x, y) of K_n occurs in some graph of the claimed decomposition. Indeed, in the case of Lemma 1, $t \equiv 0, 1 \pmod 4$ for instance, if $\min\{|x-y|, n-|x-y|\} \leq 3t$ then (x, y) occurs in some triangle, otherwise in the tail. For $t \equiv 2, 3 \pmod 4$ the minimum $3t$ does not occur in the triangle, but this is compensated by the change in y_1. The unicity follows from the fact that the total number of edges in the decomposition is precisely the number of edges of K_n.

The argument is similar in the other lemmas. Edges (∞, x) present no difficulty.

Notice that the labels used in any graph of the decomposition are different.

4. Results.

THEOREM 1. *Let* $i=3$ *and* 4 *if* $1<n\equiv 0$ *or* $1 \pmod{2m}$ *then* K_n *has a* $D_i(m)$*-decomposition.*

THEOREM 2. *Let* $i=3$ *or* 4, $2^\alpha > i$, *then* K_n *has a* $D_i(2^\alpha)$ *decomposition if and only if*

$$1<n\equiv 0 \quad or \quad 1 \pmod{2^{\alpha+1}}.$$

Proof. Lemmas 1–4 give direct constructions for all decompositions claimed in Theorem 1.

Theorem 2 follows from condition (1) and Theorem 1 since m is a power of 2.

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