J. Austral. Math. Soc. **72** (2002), 199–208

ERGODIC PATH PROPERTIES OF PROCESSES WITH STATIONARY INCREMENTS

OFFER KELLA and WOLFGANG STADJE

(Received 14 July 1999; revised 7 January 2001)

Communicated by V. Stefanov

Abstract

For a real-valued ergodic process X with strictly stationary increments satisfying some measurability and continuity assumptions it is proved that the long-run 'average behaviour' of all its increments over finite intervals replicates the distribution of the corresponding increments of X in a strong sense. Moreover, every Lévy process has a version that possesses this ergodic path property.

2000 *Mathematics subject classification*: primary 60G17, 60G51, 60F15.

1. Introduction

Let $X = (X(t))_{t\geq 0}$ be a real-valued process with strictly stationary increments, that is, the distribution of $(X(s+t) - X(s))_{t\geq 0}$ is the same for every $s \geq 0$. All strictly stationary processes and all Lévy processes have this property. We assume that the underlying probability space is complete and that X is separable, measurable, and ergodic. For example, every separable centered Lévy process satisfies these conditions [3, pages 422 and 511–512]. We will show that under certain regularity conditions almost all sample paths are connected to the distribution of X in the following strong sense: Call a function $x : [0, \infty) \to \mathbb{R}$ an X*-function* if for every $n \in \mathbb{N}$, disjoint finite intervals $I_1, \dots, I_n \subset [0, \infty)$ that are open from the left and closed from the right, and real numbers $u_1, \dots, u_n$ the following asymptotic relation holds:

$$(1.1)\qquad \lim_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta x(I_j + t) \leq u_j \text{ for } j = 1, \dots, n\} = P\big(\Delta X(I_j) \leq u_j \text{ for } j = 1, \dots, n\big),$$

© 2002 Australian Mathematical Society 1446-7887/2002 $A2.00 + 0.00

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

where λ denotes Lebesgue measure, $I + t = \{s + t \mid s \in I\}$ is the interval I shifted by t, and $\Delta x(I) = x(b) - x(a)$ is the increment of $x(\cdot)$ in $I = (a, b]$.

THEOREM 1. *Assume that*

$$\overline{X}(I) = \sup\{|\Delta X(I')| \mid I' \subset I\} \xrightarrow{P} 0, \quad \text{as } \lambda(I) \to 0 \tag{1.2}$$

and that

$$P(\Delta X(I) = u) = 0 \text{ for all intervals } I \subset [0, \infty) \text{ and } u \in \mathbb{R}. \tag{1.3}$$

Then almost every sample path of X is an X-function.

This theorem is proved in Section 2. In Section 3, X is taken to be an arbitrary Lévy process. In this case we show that there is always a version of X for which almost all sample paths are X-functions (even without the conditions (1.2) and (1.3)).

It is not difficult to prove that for any fixed $n \in \mathbb{N}$, any prespecified $u_1, \dots, u_n \in \mathbb{N}$ and any intervals $I_1, \dots, I_n$ as above the limiting relation

$$\begin{aligned}\lim_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta X(I_j + t) \le u_j \text{ for } j = 1, \dots, n\} \\ = P(\Delta X(I_j) \le u_j \text{ for } j = 1, \dots, n)\end{aligned} \tag{1.4}$$

holds *almost surely*; but the exceptional null set on which (1.4) is not valid depends on $n, u_1, \dots, u_n$ and $I_1, \dots, I_n$. In order to show that there is a 'universal' null set on whose complement (1.4) holds for all n, u_i and I_i, we need that the increments of X are 'locally small' uniformly in probability (that is, assumption (1.2)) and pointwise convergence of the distribution function of the random vector $(\Delta X(J_1), \dots, \Delta X(J_n))$ to that of $(\Delta X(I_1), \dots, \Delta X(I_n))$, if the intervals J_i increase or decrease to the bounded intervals I_i, $i = 1, \dots, n$, which requires assumption (1.3).

For Lévy processes, there is always, after suitably centering, a version satisfying (1.2). Moreover, by a classical theorem due to Hartman and Wintner [5], $X(t)$ can have an atom for some $t > 0$ only if X is a compound Poisson process with drift. Hence, (1.3) holds except for this special case. If X is compound Poisson with drift zero, the set of discontinuity points of the distribution function (d.f.) of $X(t)$ is the same for every $t > 0$. Using this observation and the explicit form of the d.f. of $\Delta X(I)$ as a Poisson sum of convolutions, we will show in Section 3 that the assertion of Theorem 1 remains true for compound Poisson processes (also with nonzero drift) and thus for Lévy processes in general. We conclude the paper with a new short and elementary proof of the Hartman-Wintner theorem, which is seen to be an immediate consequence of a neat inequality, of independent interest, for sums of i.i.d. symmetric random variables.

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

An interesting consequence of our results is that *there exist càdlàg functions* $x : [0, \infty) \to \mathbb{R}$ *for which*

$$\lim_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta x(I_j + t) \le u_j,\ j = 1, \dots n\} = \prod_{j=1}^{n} \lim_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta x(I_j + t) \le u_j\} \tag{1.5}$$

for all $n \in \mathbb{N}$ *and all* $I_1, \dots, I_n, u_1, \dots, u_n$ *as above, and these limits are strictly between* 0 *and* 1. Indeed, the set of these functions has probability 1 under any distribution on the space of càdlàg functions generated by some non-deterministic Lévy process. *Constructing* such a function seems to be quite difficult; we know *no explicit example* of a function with this property. Note that the limits on the right-hand side of (1.5) can be specified to be given by

$$\lim_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta x(I + t) \le u\} = P\big(\Delta X(I) \le u\big) = P\big(X(b - a) \le u\big)$$

for all $I = (a, b]$ and $u \in \mathbb{R}$, where X is an arbitrary Lévy process. By suitably choosing the underlying Lévy process we can also achieve various additional properties of $x(\cdot)$ besides (1.5), such as continuity, monotonicity, having only positive jumps, etc.

It is one of the fundamental ideas in probability theory that the average behaviour of a single realization of a stochastic process over a long time horizon should replicate the underlying distribution of the process. Property (1.1) is a strong version of this principle. For sample properties of Lévy processes see Fristedt [4] and Bertoin [1]. Recently, a sample path approach has been frequently used to analyze stochastic systems by studying a fixed 'typical' realization (see for example Stidham and El-Taha [7]). For the Poisson process, relation (1.1) and related questions were studied in [6].

2. Proof of Theorem 1

All intervals below are open from the left and closed from the right. Fix the intervals $I, I_1, \dots, I_n$ and the real numbers $u, u_1, \dots, u_n$. Define the auxiliary processes

$$Y_t = 1_{\{\Delta X(I_j + t) \le u_j,\ j = 1, \dots, n\}}, \qquad Z_t = 1_{\{\overline{X}(I + t) \le u\}}.$$

Obviously, Y_t and Z_t can be written in the form

$$Y_t = f\left((X(s + t) - X(t))_{s \ge 0}\right), \quad Z_t = g\left((X(s + t) - X(t))_{s \ge 0}\right),$$

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

so that the processes $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are stationary and ergodic. They are also measurable and bounded. Thus, the ergodic theorem yields

$$T^{-1}\int_0^T Y_t(\omega)\,dt \xrightarrow{a.s.} E(Y_0) = P\big(\Delta X(I_j) \leq u_j,\ j=1,\dots,n\big), \tag{2.1}$$

$$T^{-1}\int_0^T Z_t(\omega)\,dt \xrightarrow{a.s.} E(Z_0) = P\big(\overline{X}(I) \leq u\big) \tag{2.2}$$

(see for example [8, pages 315–316]). The exceptional null sets on which convergence in (2.1) or in (2.2) does not hold depend on $n, I_1, \dots, I_n, u_1, \dots, u_n$ or on I, u, respectively. Taking the (denumerable) union of all these null sets over $n \in \mathbb{N}$, intervals $I, I_1, \dots, I_n \subset [0,\infty)$ with rational endpoints and $u, u_1, \dots, u_n \in \mathbb{Q}$ we get a set of probability 0. On its complement C (a set of probability 1) relations (2.1) and (2.2) hold for all $n \in \mathbb{N}$, $u, u_1, \dots, u_n \in \mathbb{Q}$ and $I, I_1, \dots, I_n \subset [0,\infty)$ with rational endpoints. From now on we only consider sample paths corresponding to points in C.

Now let $I_1, \dots, I_n, u_1, \dots, u_n$ be arbitrary. Let $(u_j^{(m)})_{m\in\mathbb{N}}$, $j = 1, \dots, n$, be sequences of rational numbers such that $u_j^{(m)} \uparrow u_j$, as $m \uparrow \infty$, and $u_j - u_j^{(m)} \geq 2\varepsilon_m > 0$, where ε_m is rational and $\varepsilon_m \to 0$. Furthermore, choose intervals $J_j^{(k)}$, $L_j^{(k)}$, $R_j^{(k)}$ with rational endpoints such that $J_j^{(k)} \subset I_j$ approximates I_j from inside and the $L_j^{(k)}$ ($R_j^{(k)}$) cover the left (right) endpoint of I_j and satisfy

$$I_j \subset J_j^{(k)} \cup L_j^{(k)} \cup R_j^{(k)}, \quad j = 1, \dots, n,$$
$$\lambda(L_j^{(k)}) \downarrow 0 \text{ and } \lambda(R_j^{(k)}) \downarrow 0, \quad \text{as } k \uparrow \infty.$$

Clearly, the following inclusion holds for every j, m, k and t:

$$\big\{\Delta X(J_j^{(k)}+t) \leq u_j^{(m)}\big\} \subset \big\{\Delta X(I_j+t) \leq u_j \text{ or } \overline{X}(L_j^{(k)}+t) \geq \big(u_j - u_j^{(m)}\big)/2 \text{ or } \overline{X}(R_j^{(k)}+t) \geq \big(u_j - u_j^{(m)}\big)/2\big\}.$$

Hence, for every sample path of X and for every $T > 0$ we have

$$\begin{aligned}
&\big\{t \in [0,T] \mid \Delta X(J_j^{(k)}+t) \leq u_j^{(m)},\ j=1,\dots,n\big\} \\
&\quad\subset \big\{t \in [0,T] \mid \Delta X(I_j+t) \leq u_j,\ j=1,\dots,n\big\} \\
&\quad\cup \bigcup_{j=1}^{n} \big\{t \in [0,T] \mid \overline{X}(L_j^{(k)}+t) \geq \big(u_j - u_j^{(m)}\big)/2,\ j=1,\dots,n\big\} \\
&\quad\cup \bigcup_{j=1}^{n} \big\{t \in [0,T] \mid \overline{X}(R_j^{(k)}+t) \geq \big(u_j - u_j^{(m)}\big)/2,\ j=1,\dots,n\big\}.
\end{aligned}$$

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

It follows that

$$\begin{aligned}\lambda\{t \in [0,T] \mid \Delta X(I_j+t) \le u_j,\ j=1,\dots,n\}\\ \ge \lambda\{t \in [0,T] \mid \Delta X(J_j^{(k)}+t) \le u_j^{(m)},\ j=1,\dots,n\}\\ -\sum_{j=1}^{n}\lambda\{t\in[0,T] \mid \overline{X}(L_j^{(k)}+t)>\varepsilon_m\}-\sum_{j=1}^{n}\lambda\{t\in[0,T] \mid \overline{X}(R_j^{(k)}+t)>\varepsilon_m\}.\end{aligned}$$

From (2.1) and (2.2) we can now conclude that

$$\text{(2.3)}\qquad \begin{aligned}&\liminf_{T\to\infty} T^{-1}\lambda\{t \in [0,T] \mid \Delta X(I_j+t) \le u_j,\ j=1,\dots,n\}\\ &\quad\ge P\big(\Delta X(J_j^{(k)}) \le u_j^{(m)},\ j=1,\dots,n\big)-\sum_{j=1}^{n}P\big(\overline{X}(L_j^{(k)})>\varepsilon_m\big)\\ &\quad\quad-\sum_{j=1}^{n}P\big(\overline{X}(R_j^{(k)})>\varepsilon_m\big).\end{aligned}$$

Now let $k \to \infty$ in (2.3). Condition (1.2) clearly implies that X is stochastically continuous so that $(\Delta X(J_1^{(k)}),\dots,\Delta X(J_n^{(k)})) \xrightarrow{P} (\Delta X(I_1),\dots,\Delta X(I_n))$ with respect to the Euclidean metric. It follows that

$$P\big((\Delta X(J_1^{(k)}),\dots,\Delta X(J_n^{(k)}))\in B\big)\to P\big((\Delta X(I_1),\dots,\Delta X(I_n))\in B\big)$$

for all Borel sets in $\mathbb{R}^n$ satisfying $P\big((\Delta X(I_1),\dots,\Delta X(I_n))\in\partial B\big)=0$. We can take $B=\prod_{j=1}^{n}(-\infty,u_j^{(m)}]$ because $\partial B\subset\{y\in\mathbb{R}^n \mid y_j=u_j^{(m)}$ for some $j\}$ and the increments $\Delta X(I_j)$ have continuous distributions (by condition (1.3)). Therefore, we obtain

$$\lim_{k\to\infty}P\big(\Delta X(J_j^{(k)})\le u_j^{(m)},\ j=1,\dots,n\big)=P\big(\Delta X(I_j)\le u_j^{(m)},\ j=1,\dots,n\big).$$

The other probabilities on the right-hand side of (2.3) all tend to zero because $\varepsilon_m>0$ and m is still fixed. Thus, letting first $k\to\infty$ and then $m\to\infty$, inequality (2.3) yields

$$\text{(2.4)}\qquad \begin{aligned}&\liminf_{T\to\infty} T^{-1}\lambda\{t \in [0,T] \mid \Delta X(I_j+t) \le u_j,\quad j=1,\dots,n\}\\ &\quad\ge P\big(\Delta X(I_j))\le u_j,\quad j=1,\dots,n\big).\end{aligned}$$

For the reverse direction, take sequences $v_j^{(m)}\downarrow u_j$ as $m\to\infty$, $v_j^{(m)}\in\mathbb{Q}$ and $v_j^{(m)}-v_j>2\varepsilon_m>0$, $\varepsilon_m\in\mathbb{Q}$. Then

$$\begin{aligned}\{\Delta X(I_j+t)\le u_j\}\subset\{\Delta X(J_j^{(k)}+t)\le v_j^{(m)} \text{ or } \overline{X}(L_j^{(k)}+t)>\varepsilon_m\\ \text{or } \overline{X}(R_j^{(k)}+t)>\varepsilon_m\}.\end{aligned}$$

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

As above, (2.1) and (2.2) imply that

$$\limsup_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta X(I_j + t) \le u_j,\ j = 1, \dots, n\}$$
$$\le P\big(\Delta X(J_j^{(k)}) \le v_j^{(m)}\big) + \sum_{j=1}^{n} P\big(\overline{X}(L_j^{(k)}) > \varepsilon_m\big) + \sum_{j=1}^{n} P\big(\overline{X}(R_j^{(k)}) > \varepsilon_m\big).$$

Letting first $k \to \infty$, then $m \to \infty$ and reasoning as above we obtain

$$\limsup_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta X(I_j + t) \le u_j,\ j = 1, \dots, n\} \le P\big(\Delta X(I_j) \le u_j,\ j = 1, \dots, n\big). \tag{2.5}$$

The assertion follows from (2.4) and (2.5). □

3. Lévy processes

We now consider the result for Lévy processes. Every Lévy process has, after a suitable deterministic centering, a version with càdlàg paths, and we will take such a version X from now on. Furthermore, we assume that $X(0) = 0$. Note that for this X we have $\overline{X}(I) \stackrel{D}{=} \sup_{0\le t\le\lambda(I)} |X(t)| \xrightarrow{a.s.} 0$. Thus, Theorem 1 implies that almost every sample path of X is an X-function if the distribution of $X(t)$ is continuous for every $t > 0$. What happens if (1.3) does not hold, that is, if $P\big(X(t) = u\big) > 0$ for some $t > 0$ and some $u \in \mathbb{R}$? Then a classical result by Hartman and Wintner [5] states that X must be a compound Poisson process with drift (see also [1, pages 30–31]). The next theorem covers this case.

THEOREM 2. *If X is a compound Poisson process with drift, then almost all its paths are X-functions.*

PROOF. If (1.1) holds for the function x and the process X, it is also valid for $(x(t) + \beta t)_{t\ge 0}$ and $(X(t) + \beta t)_{t\ge 0}$. Thus, we can assume that the drift of X is zero, so that X is piecewise constant between jumps distributed according to some d.f. F, and

$$P\big(\overline{X}(I) > 0\big) = e^{-b\lambda(I)}, \tag{3.1}$$

where $b > 0$ is the intensity of the underlying Poisson process. We have

$$P\big(\Delta X(I) \le u\big) = e^{-b\lambda(I)} \sum_{i=0}^{\infty} \frac{(b\lambda(I))^i}{i!} F^{*i}(u),$$

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

where F^{*i} is the i-fold convolution of F. Let D be the union of the sets of discontinuity points of F^{*i}, $i \in \mathbb{Z}_+$. Then D is denumerable and it is the set of atoms of $\Delta X(I)$ for any I.

Now we repeat the construction of Theorem 1 but take C to be the set of all points ω for which (2.1) and (2.2) hold for all $n \in \mathbb{N}$, all intervals $I, I_1, \dots, I_n \subset [0, \infty)$ with rational endpoints and all $u, u_1, \dots, u_n \in \mathbb{Q} \cup D$. Then $P(C) = 1$ and it suffices to consider paths corresponding to points in C.

For arbitrary $I_1, \dots, I_n$ and $u_1, \dots, u_n$ take sequences $J_j^{(k)}$, $L_j^{(k)}$, $R_j^{(k)}$ as in Theorem 1, but set

$$\tilde{u}_j^{(m)} = \begin{cases} u_j^{(m)} & \text{if } u_j \notin \mathbb{Q} \cup D; \\ u_j & \text{if } u_j \in \mathbb{Q} \cup D. \end{cases}$$

Then for all j, k, m, t

$$\left\{\Delta X(J_j^{(k)} + t) \le \tilde{u}_j^{(m)}\right\} \subset \left\{\Delta X(I_j + t) \le u_j \text{ or } \overline{X}(L_j^{(k)}) > 0 \text{ or } \overline{X}(R_j^{(k)}) > 0\right\}$$

so that we obtain, for all k, m,

$$\text{(3.2)} \qquad \begin{aligned} \liminf_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta X(I_j + t) \le u_j,\ j = 1, \dots, n\} \\ \ge P\big(\Delta X(J_j^{(k)}) \le \tilde{u}_j^{(m)},\ j = 1, \dots, n\big) \\ - \sum_{j=1}^{n} P\big(\overline{X}(L_j^{(k)}) > 0\big) - \sum_{j=1}^{n} P\big(\overline{X}(R_j^{(k)}) > 0\big). \end{aligned}$$

Let $k \to \infty$. Then, by (3.1), the two sums in (3.2) converge to 0. The first term on the right-hand side is equal to $\prod_{j=1}^{n} P\big(\Delta X(J_j^{(k)}) \le \tilde{u}_j^{(m)}\big)$ and

$$\text{(3.3)} \qquad \begin{aligned} P\big(\Delta X(J_j^{(k)}) \le \tilde{u}_j^{(m)}\big) &= e^{-b\lambda(J_j^{(k)})} \sum_{i=0}^{\infty} \frac{[b\lambda(J_j^{(k)})]^i}{i!} F^{*i}(\tilde{u}_j^{(m)}) \\ &\to e^{-b\lambda(I_j)} \sum_{i=0}^{\infty} \frac{[b\lambda(I_j)]^i}{i!} F^{*i}(\tilde{u}_j^{(m)}), \quad \text{as } k \to \infty \end{aligned}$$

by bounded convergence, since $\lambda(J_j^{(k)}) \to \lambda(I_j)$ as $k \to \infty$, and the renewal function $\sum_{i=0}^{\infty} F^{*i}(t)$ is finite for all $t \ge 0$. Now let $m \to \infty$. If $u_j \notin \mathbb{Q} \cup D$, then $F^{*i}(\tilde{u}_j^{(m)}) \to F^{*i}(u_j)$ because every F^{*i} is continuous in u_j. But if $u_j \in \mathbb{Q} \cup D$, then $F^{*i}(\tilde{u}_j^{(m)}) = F^{*i}(u_j)$ for all $i \in \mathbb{Z}_+$. Hence, the limit in (3.3) tends to

$$e^{-b\lambda(I_j)} \sum_{i=0}^{\infty} \frac{[b\lambda(I_j)]^i}{i!} F^{*i}(u_j) = P\big(\Delta X(I_j) \le u_j\big).$$

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

We have proved that

$$\liminf_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta X(I_j + t) \le u_j, j = 1, \dots, n\} \ge \prod_{j=1}^{n} P(\Delta X(I_j) \le u_j). \tag{3.4}$$

For the other direction we follow the proof of Theorem 1 and obtain, for every k and m,

$$\limsup_{T\to\infty} T^{-1}\lambda\{t \in [0, T] \mid \Delta X(I_j + t) \le u_j,\ j = 1, \dots, n\} \le \prod_{j=1}^{n} P(\Delta X(J_j^{(k)}) \le v_j^{(m)}) + \sum_{j=1}^{n} P(\overline{X}(L_j^{(k)}) > 0) + \sum_{j=1}^{n} P(\overline{X}(R_j^{(k)}) > 0). \tag{3.5}$$

As $k \to \infty$, the two sums in (3.5) converge to zero by (3.1), while the product tends to $\prod_{j=1}^{n} P(\Delta X(I_j) \le v_j^{(m)})$. By right-continuity, this latter product converges to $\prod_{j=1}^{n} P(\Delta X(I_j) \le u_j)$ as $m \to \infty$, since $v_j^{(m)} \downarrow u_j, j = 1, \dots, n$. □

We have shown that for every centered Lévy process there is a version X for which almost all paths are X-functions. But the centering function, say f, can be chosen to be additive, that is, to satisfy the equation $f(t + s) = f(t) + f(s)$ for all $t, s \ge 0$. Now note that if a function $x : [0, \infty) \to \mathbb{R}$ satisfies (1.1) for some process X, then $x + f$ satisfies (1.1) for the process $X + f$. Hence, *for every (not necessarily centered) Lévy process there is a version with almost all paths being X-functions.*

Finally, we remark that the Hartman-Wintner theorem on which Theorem 2 relies is a straightforward consequence of the following interesting inequality.

THEOREM 3. *Let $U_1, U_2, \dots$ be a sequence of i.i.d. symmetric random variables satisfying $P(U_1 = 0) = 0$. Then for every $j \in \mathbb{N}$,*

$$P(U_1 + \cdots + U_{2j} = 0) \le 2^{-2j}\binom{2j}{j} \tag{3.6}$$

and

$$P(U_1 + \cdots + U_{2j+1} = 0) \le P(U_1 + \cdots + U_{2j} = 0). \tag{3.7}$$

PROOF. Let $\rho(\alpha) = E(e^{i\alpha U_1})$. By Fourier inversion ([2, pages 144–145]),

$$P(U_1 + \cdots + U_{2j} = 0) = \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} \rho(\alpha)^{2j}\, d\alpha = \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{-T} [E(\cos\alpha U_1)]^{2j}\, d\alpha \tag{3.8}$$

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

$$\begin{aligned}
&\leq \limsup_{T\to\infty}(2T)^{-1}\int_{-T}^{T} E(\cos^{2j}\alpha U_1)\,d\alpha \\
&= \limsup_{T\to\infty}(2T)^{-1}E\left(\int_{-T}^{T}\cos^{2j}(\alpha U_1)\,d\alpha\right) \\
&= \limsup_{T\to\infty} E\left((2TU_1)^{-1}\int_{-TU_1}^{TU_1}\cos^{2j}x\,dx\right) = 2^{-2j}\binom{2j}{j}.
\end{aligned}$$

The inequality in (3.8) follows from $(E(\cos\alpha U_1))^{2j} \leq E(\cos^{2j}(\alpha U_1))$, which is a consequence of Jensen's inequality, and for the second-last equality we have used the substitution $x = \alpha U_1$, which is possible as $P(U_1 = 0) = 0$. Finally, since $\rho(\alpha) \in [-1, 1]$ by symmetry, (3.7) follows from

$$\begin{aligned}
P(U_1 + \cdots + U_{2j+1} = 0) &= \lim_{T\to\infty}\int_{-T}^{T}\rho(\alpha)^{2j+1}\,d\alpha \leq \lim_{T\to\infty}\int_{-T}^{T}\rho(\alpha)^{2j}\,d\alpha \\
&= P(U_1 + \cdots + U_{2j} = 0). \qquad \square
\end{aligned}$$

REMARK. Inequality (3.6) states that in the considered class of random walks the probability $P(U_1 + \cdots + U_{2j} = 0)$ is maximal for the simple ± 1-walk for which $P(U_1 = 1) = P(U_1 = -1) = 1/2$.

Now assume that Y is a Lévy process which is not a compound Poisson process with drift. Let Y' be an independent copy of Y. Then $X = Y - Y'$ is a symmetric Lévy process, and since $P(Y(t) = a)^2 \leq P(X(t) = 0)$ for all $a \in \mathbb{R}$ and $t > 0$, we have to prove $P(X(t) = 0) = 0$. For arbitrary $\delta > 0$ let X_1^δ be the process obtained from X by deleting all jumps that are smaller than δ in absolute value and set $X_2^\delta = X - X_1^\delta$. Then X_1^δ is a symmetric compound Poisson process of intensity ν_δ, say, and $\lim_{\delta\downarrow 0}\nu_\delta = \infty$. Let $\psi_{t,1}^\delta$ ($\psi_{t,1}^\delta$) be the characteristic function of $X_1^\delta(t)$ ($X_2^\delta(t)$). By Fourier inversion,

$$\begin{aligned}
(3.9)\qquad P(X(t) = 0) &= \lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\psi_{t,1}^\delta(\alpha)\psi_{t,2}^\delta(\alpha)\,d\alpha \\
&\leq \lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\psi_{t,1}^\delta(\alpha)\,d\alpha \\
&= P(X_1^\delta(t) = 0) = \sum_{j=0}^{\infty} e^{-\nu_\delta t}\frac{[\nu_\delta t]^j}{j!}P(U_1^\delta + \cdots + U_j^\delta = 0) \\
&\leq e^{-\nu_\delta t}\sum_{j=0}^{\infty}\binom{2j}{j}2^{-2j}\left(\frac{[\nu_\delta t]^{2j}}{(2j)!} + \frac{[\nu_\delta t]^{2j+1}}{(2j+1)!}\right),
\end{aligned}$$

where the U_i^δ are the jump sizes of X_1^δ, which are certain i.i.d. symmetric random variables satisfying $P(U_i^\delta = 0) = 0$. The first inequality in (3.9) follows from

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press

$\psi_{t,1}(\alpha) \geq 0$ and $|\psi_{t,2}(\alpha)| \leq 1$ for all $\alpha \in \mathbb{R}$ and the second from the Lemma. But as $\delta \downarrow 0$, we have $\nu_\delta \to \infty$ and thus the right-hand side of (3.9) tends to zero. Hence, $P\big(X(t) = 0\big) = 0$.

References

[1] J. Bertoin, *Lévy processes* (Cambridge University Press, Cambridge, 1996).

[2] K. L. Chung, *A course in probability theory* (Harcourt, Brace & World, New York, 1968).

[3] J. L. Doob, *Stochastic processes* (Wiley, New York, 1953).

[4] B. Fristedt, 'Sample functions of stochastic processes with stationary independent increments', in: *Advances in probability and related topics*, Vol. 3 (eds. P. Ney and S. Port) (Dekker, New York, 1974) pp. 241–396.

[5] P. Hartman and A. Wintner, 'On the infinitesimal generator of integral convolutions', *Amer. J. Math.* **64** (1942), 273–298.

[6] W. Stadje, 'Two ergodic sample-path properties of the Poisson process', *J. Theor. Probab.* **11** (1998), 197–208.

[7] S. Stidham and M. El-Taha, 'Sample-path techniques in queueing theory', in: *Advances in queueing* (ed. J. W. Dshalalow) (CRC Press, Boca Raton, 1998) pp. 119–166.

[8] D. W. Stroock, *Probability theory: an analytic view* (Cambridge University Press, Cambridge, 1992).

Department of Statistics
The Hebrew University of Jerusalem
Mount Scopus, Jerusalem 91905
Israel
e-mail: offer.kella@huji.ac.il

Department of Mathematics
and Computer Science
University of Osnabrück
49069 Osnabrück
Germany
e-mail: wolfgang@mathematik.uni-osnabrueck.de

https://doi.org/10.1017/S1446788700003840 Published online by Cambridge University Press