Can. J. Math.Vol. 45 (1), 1993 pp. 117–131

ON THE MINIMAL CROSSING NUMBER AND THE BRAID INDEX OF LINKS

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ABSTRACT In this paper we prove an inequality that involves the minimal crossing number and the braid index of links by estimating Murasugi and Przytycki's index for a planar bipartite graph

1. **Introduction.** This paper is concerned with classical links, that is, closed 1-manifolds embedded piecewise-linearly in the oriented 3-sphere S^3. Every oriented link L in S^3 is represented as a closed braid with a finite number of strings[1]. The braid index of L, denoted by $b(L)$, is defined as the minimal number of strings needed for L. Yamada[10] proves that the number of Seifert circles, denoted by $S(D)$, of a link diagram D of L is more than or equal to the braid index of L. Moreover, Murasugi and Przytycki[8] determine the deficit $S(D) - b(L)$ for some kind of links by making use of a new concept, called an index of a graph. In this paper, we prove the following theorem by making use of Murasugi and Przytycki's results.

THEOREM 3.8. *Let L be a nonsplit link and c(L) the minimal crossing number of L, that is, the minimal number of double points among all projections of L, then we have*

$$c(L) \geq 2\{b(L) - 1\}.$$

In [4], Fox conjectured an inequality that involves the bridge index and the minimal crossing number of a knot. In §4, we prove Fox's conjecture for a certain kind of special alternating links by using the braid index of L. We refer to Berge[2], Burde and Zieschang[3] and Rolfsen[9] for standard definitions and results in graph theory and knot theory. The author would like to express his appreciation to the referees for their valuable suggestions.

2. **Definitions and results.** Throughout this paper what is meant by a graph is frequently a geometric realization of a graph as a finite 1-dimensional CW-complex in R^3. A vertex and an edge correspond to a 0-simplex and a 1-simplex, respectively.

A graph G is said to be *separable* if there are two subgraphs H and K such that $G = H \cup K$ and $H \cap K = \{v_0\}$, where H and K both have at least one edge and v_0 is a vertex.

Received by the editors January 22, 1991, revised June 19, 1991

AMS subject classification 57M25

Key words and phrases index of a graph, knot, link, the braid index, the minimal crossing number

Otherwise G is *nonseparable*. The vertex v_0 is called *a cut vertex*. A *block* is a maximal nonseparable connected subgraph of G. A connected graph is decomposed into finitely many blocks. If $G_1, G_2, \ldots, G_k$ are blocks of G, we write $G = G_1 * G_2 * \cdots * G_k$ and G is called *the block sum* of $G_1, G_2, \ldots, G_k$.

If two or more than two edges have the same ends, these edges are called *multiple-edges*. On the other hand, if two distinct vertices are joined by exactly one edge e, then e is called a *singular* edge of G.

A *cut edge* of G is an edge whose removal increases the number of connected components.

For a vertex v, *star* v denotes the smallest subgraph of G that contains all edges and vertices of G which are incident to v. If X is a connected subgraph of G, then G/X is defined as the graph obtained from G by identifying all points in X to one point.

The *valency* of a vertex v, $\mathrm{val}(v)$, is the number of edges incident to v.

Murasugi and Przytycki[8] define a new concept called 'an index of a graph' and prove Theorem 2.2 below.

DEFINITION 2.1 ([8] DEFINITION 2.1). Let G be a graph.

(1) A family $F = \{e_1, e_2, \ldots, e_k\}$ of edges of G is said to be *independent* if (i) all e_j $(j = 1, 2, \ldots, k)$ are singular and (ii) there is an edge e_i in F and a vertex v, one of the ends of e_i, such that $\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_k\}$ is independent in the graph $G/$ star v. We assume that the empty set of edges is independent.

(2) $\mathrm{ind}(G)$ is defined to be the maximal number of independent edges in G.

THEOREM 2.2 ([8] THEOREM 2.4). *Let G be a bipartite graph. If G consists of blocks $G_1, G_2, \ldots, G_k$ then*

$$\mathrm{ind}(G) = \sum_{i=1}^{k} \mathrm{ind}(G_i).$$

By $|E(G)|$ and $|V(G)|$, we denote the number of egdes and vertices in G, respectively. In §3, we prove the following theorem.

THEOREM 3.7. *Let G be a connected, planar, bipartite graph without a cut edge, then*

$$|E(G)| \geq 2\{|V(G)| - \mathrm{ind}(G) - 1\}.$$

Let L be a link and D a diagram of L. If we split D at each crossing of D according to the orientation of D, then D is decomposed into finitely many simple closed curves on a plane, called *Seifert circles* of D. Let $S(D)$ be the number of Seifert circles of D and $c(D)$ the number of crossings in D. The *Seifert graph* $\Gamma(D)$ *(associated with D)* is a graph with $S(D)$ vertices $v_1, v_2, \ldots, v_{S(D)}$ and $c(D)$ edges $e_1, e_2, \ldots, e_{c(D)}$. Each vertex corresponds to a Seifert circle and each edge corresponds to a crossing. Two distinct vertices v_i and v_j are connected by e_k if two Seifert circles S_i and S_j(corresponding to v_i and v_j) are joined by a crossing c_k (corresponding to e_k). Therefore the Seifert graph is a planar bipartite graph and $\left|E\big(\Gamma(D)\big)\right| = c(D)$, $\left|V\big(\Gamma(D)\big)\right| = S(D)$. The graph in Figure 2-1 is the Seifert graph of the figure eight knot.

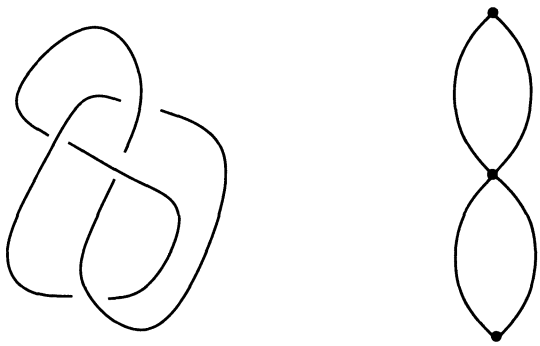


FIGURE 2-1

DEFINITION 2.3 ([8] DEFINITION 7.1). For an oriented diagram D of an oriented link L, we define

$$\operatorname{ind}(D) = \operatorname{ind}\Gamma(D).$$

From Definition 2.3, we see immediately that Theorem 3.7 is equivalent to the following:

THEOREM 3.7A. *If D is a diagram of a link L with no nugatory crossings, where a nugatory crossing is a removable crossing as is shown in Figure 2-2, then*

$$c(D) \geq 2\{S(D) - \operatorname{ind}(D) - 1\}.$$

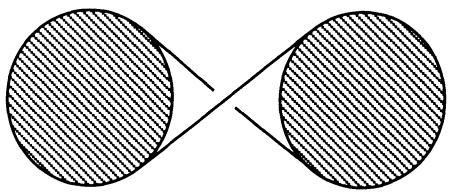


FIGURE 2-2

Murasugi and Przytycki[8] show the following theorem by making use of Yamada's result [10].

THEOREM 2.4 ([8] LEMMA 8.6). *For any diagram D of a link L,*

$$b(L) \leq S(D) - \operatorname{ind}(D).$$

3. **Proofs.** Let Γ be a connected planar bipartite graph without a cut vertex or a cut edge. Obviously Γ has no loops. We call a vertex v as is shown in Figure 3-1 a stump of order n where n is the valency of v. A vertex v with $\mathrm{val}(v) = 3$ in Figure 3-2(a) and (b) will be called a vertex of type (3.1) and type (3.2), respectively.

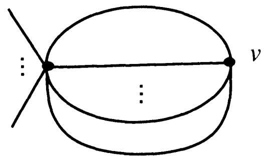
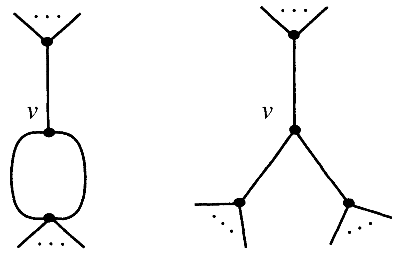


FIGURE 3-1



(a) (b)

FIGURE 3-2

We define the following operations for a planar graph G without a loop and a cut edge.

OPERATION A. For a graph G, if there is a vertex v with $\mathrm{val}(v) = 2$ as is shown in Figure 3-3 or of type (3.1), we form the quotient graph $G/$ star v and whenever stumps occur in $G/$ star v delete these stumps and edges incident to them.

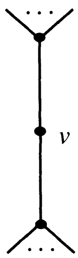


FIGURE 3-3

OPERATION B. (1) For a graph G without a vertex whose valency is two or of type (3.1), if there is a vertex v of type (3.2) we form the quotient graph $G' = G/$ star v. (2) If a vertex v of type (3.1) occurs in G', we form the graph $G'' = G'/\operatorname{star} v$. (3) If a vertex v of type (3.1) ocuurs in G'', we form the graph $G''/\operatorname{star} v$.

First we perform Operation A on Γ repeatedly till all vertices with valencies two or of type (3.1) vanish. We obtain a sequence of graphs $\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_l$ where Γ_i is obtained from Γ_{i-1} by Operation A. Γ_l does not contain a vertex v with $\operatorname{val}(v) = 2$ or of type (3.1).

Next we perform Operation B on Γ_l repeatedly till all vertices of type (3.2) or of type (3.1) vanish. Then we obtain a sequence of graphs $\Gamma_l = \Gamma_l^0, \Gamma_l^1, \ldots, \Gamma_l^{m_1+m_2}$, where m_1 is the number of times we perform Operation B(1) and m_2 is the total number of times we perform Operation B(2) and B(3). In $\Gamma_l^{m_1+m_2}$, a vertex with valency three is the stump of order 3 that are created by Operation B.

Now we perform Operation A l times. Let p be the number of deleted stumps in Operation A. And let s_1 and s_2 be the numbers of stumps of order 3 which occur in Operation B(1), and B(2) or B(3), respectively.

The following Proposition 3.1 is immediate from Definition 2.1.

PROPOSITION 3.1.

$$\operatorname{ind}(\Gamma) \geq l + m_1 + m_2.$$

When we perform Operation A, the number of edges decreases by two or three and the number of vertices decreases by two. The order of deleted stumps created in Operation A is at least two. When we perform Operation B, the number of edges decreases by three and the number of vertices decreases by three and two depending to Operation B(1), and B(2) or B(3). Therefore we can estimate the numbers of edges and vertices in $\Gamma_l^{m_1+m_2}$ as follows.

PROPOSITION 3.2.

$$|E(\Gamma_l^{m_1+m_2})| \leq |E(\Gamma)| - 2l - 2p - 3m_1 - 3m_2,$$
$$|V(\Gamma_l^{m_1+m_2})| = |V(\Gamma)| - 2l - p - 3m_1 - 2m_2.$$

Let $g(G) = \sum_{v \in V(G), \operatorname{val}(v) \geq 4} \{\operatorname{val}(v) - 4\}$ for a graph G. Then we prove the following.

LEMMA 3.3.

$$g(\Gamma_l^{m_1+m_2}) \geq 2m_1 + s_1.$$

PROOF. Let $h_i = g(\Gamma_l^i) - g(\Gamma_l^{i-1})$ $(i = 1, 2, \ldots, m_1 + m_2)$. By v_1, v_2 and v_3, we denote the vertices which are incident to the vertex v of type (3.2) and they are identified to v'. Suppose that Γ_l^i is obtained from Γ_l^{i-1} by Operation B(1). Then to estimate h_i, it is enough to consider the following four cases.

CASE 1. $\mathrm{val}(v_i) \geq 4$ $(i = 1,2,3)$. Let $\mathrm{val}(v_i) = 4 + t_i$ $(i = 1,2,3)$, then we have

$$g(\Gamma_l^i) - g(\Gamma_l^{i-1}) = \{\mathrm{val}(v') - 4\} - \sum_{i=1}^{3}\{\mathrm{val}(v_i) - 4\} = (9 + t_1 + t_2 + t_3) - 4 - (t_1 + t_2 + t_3) = 5.$$

CASE 2. $\mathrm{val}(v_1) = 3$, $\mathrm{val}(v_i) \geq 4$ $(i = 2,3)$. Then $g(\Gamma_l^i) - g(\Gamma_l^{i-1}) = 4$.

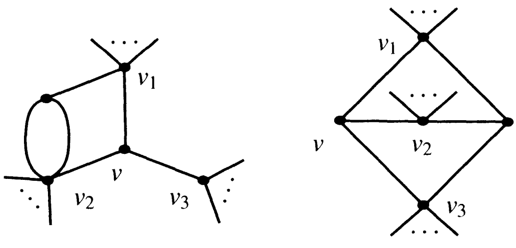
CASE 3. $\mathrm{val}(v_1) = \mathrm{val}(v_2) = 3$, $\mathrm{val}(v_3) \geq 4$. Then $g(\Gamma_l^i) - g(\Gamma_l^{i-1}) = 3$.

CASE 4. $\mathrm{val}(v_i) = 3$ $(i = 1,2,3)$. Then $g(\Gamma_l^i) - g(\Gamma_l^{i-1}) = 2$.

Therefore we have

$$h_i = g(\Gamma_l^i) - g(\Gamma_l^{i-1}) \geq 2. \tag{3-1}$$

Moreover we consider the case where stumps of order 3 occur after Operation B(1). When we perform Operation B on $G = \Gamma_l^{i-1}$, since G has no vertex with valency two the orders of stumps in $G' = \Gamma_l^i$ are at least three. If a stump of order 3 occurs in G', then G has either the subgraph as is shown in Figure 3-4 (a) or (b). However, since G in Operation B(1) has no vertex of type (3.1), we only need to consider a graph G in Figure 3-4 (b).



(a) (b)

FIGURE 3-4

If the valency of each v_i $(i = 1,2,3)$ is three, two of these v_i are joined with an edge and the third vertex is incident to a cut edge in G as is shown in Figure 3-5 since G is planar.

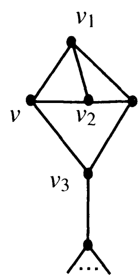


FIGURE 3-5

Since Γ has no cut edges and cut edges cannot be created by Operation A or B, a graph G has no cut edges. Therefore the valency of one of v_i $(i = 1,2,3)$ is more than three.

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Namely, Case 4 cannot occur when stumps of order 3 occur after Operation B(1). If two stumps of order 3 are created by Operation B(1) a graph G has the subgraph as is shown in Figure 3-6 which is nonplanar. Therefore at most one stump of order 3 can be created by Operation B(1).

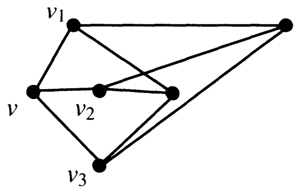

FIGURE 3-6

Then we have by Case 1, 2 and 3

$$g(\Gamma_l^i) - g(\Gamma_l^{i-1}) \geq 3. \tag{3-2}$$

When Γ_l^i is obtained by Operation B(2) or B(3), the value $g(\Gamma_l^i)$ does not decrease. Since $g(\Gamma) \geq 0$, it follows from (3-1) and (3-2)

$$g(\Gamma_l^{m_1+m_2}) \geq 2(m_1 - s_1) + 3s_1 = 2m_1 + s_1.$$

This completes the proof of Lemma 3.3.

Next we estimate the number of stumps of order 3 that are created by Operation B(2) and B(3).

LEMMA 3.4.

$$2m_2 \geq s_2.$$

PROOF. The graph G' which is obtained by Operation B(1) is decomposed into finitely many blocks. It is enough to consider one of these blocks H which contains a vertex of type (3.1), since Operation B(2) and B(3) have no influence on the other blocks. Operation B cannot decrease the valency of any vertex. A vertex of type (3.1) which are created by Operation B(1) is obtained from a vertex of type (3.2) by identifying two vertices incident to the vertex of type (3.2) as is shown in Figure 3-7.

Therefore all the vertices of type (3.1) which are created by Operation B(1) are connected to the identified vertex v' by multiple edges in G' depicted in Figure 3-8.

If four vertices of type (3.1) occur in the block H, then H has the subdivision of the graph in Figure 3-9.

Using the inverse operation as is shown in Figure 3-7, we consider a graph G before Operation B(1). G has the subdivision of one of four graphs in Figure 3-10. None of these four graphs is planar. Therefore at most three vertices of type (3.1) occur in H.

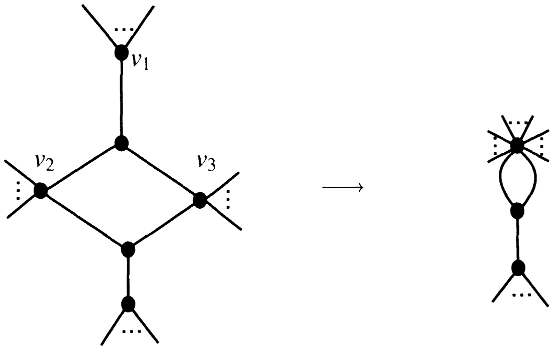

FIGURE 3-7

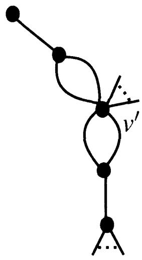

FIGURE 3-8

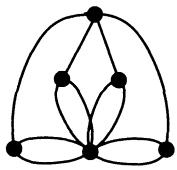

FIGURE 3-9

Next we consider the number of vertices of type (3.1) which are created by Operation B(2) and B(3). When we perform Operation B(2) or B(3) at a vertex v of type (3.1) once, the block H' which contains the vertex v may be decomposed into the blocks $H'_1, H'_2, \ldots, H'_q$ some of which contain vertices of type (3.1). A vertex of type (3.1) which are created by Operation B(2) or B(3) is obtained from a vertex of type (3.2) as is shown in Figure 3-11. In each block H'_j $(j = 1, 2, \ldots, q)$, at most two vertices of type (3.1) occur. In fact if three vertices of type (3.1) occur in the block H'_j, then H' has the subdivision of ' ·aph in Figure 3-12 which is nonplanar.

Let m be the number of times we perform Operation B(2) or B(3) on the block H till all vertices of type (3.1) vanish, and s the number of stumps of order 3 that are created by Operation B(2) and B(3) on the block H. After Operation B(2) and B(3) on the block H,

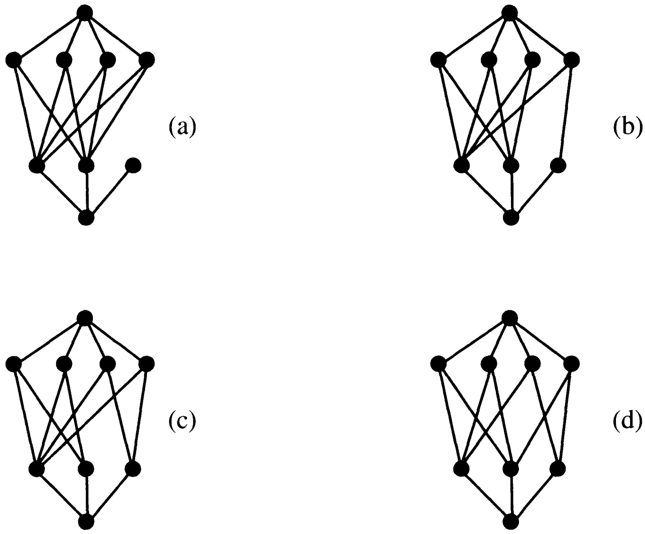

FIGURE 3-10

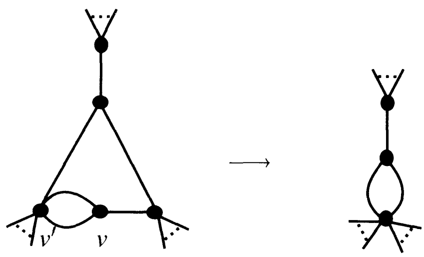

FIGURE 3-11

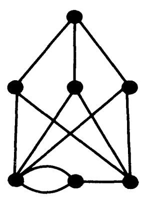

FIGURE 3-12

finally H is decomposed into k blocks $H_1, H_2, \ldots, H_k$. In each block H_i $(i = 1, 2, \ldots, k)$, no vertex of type (3.1) are created by the last operation. Therefore if V_3 denotes the number of vertices of type (3.1) that are in the block H and are created by Operation B(2) and B(3) on the block H, we have

(3-3) $$V_3 \leq 3 + 2m + 2(k-1) - 2k = 2m + 1.$$

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All vertices of type (3.1) are identified with one vertex or become to stumps of order 3 in Operation B(2) or B(3). Since we perform Operation B(2) and B(3) m times, we have by (3-3)

$$s \leq 2m + 1 - m = m + 1 \leq 2m.$$

This completes the proof of Lemma 3.4.

LEMMA 3.5.

$$|E(\Gamma_l^{m_1+m_2})| \geq 2\{|V(\Gamma_l^{m_1+m_2})| - 1\} + m_1 - \frac{1}{2}s_2.$$

PROOF. First we consider the case where we cannot perform Operation B, that is, $m_1 = m_2 = s_1 = s_2 = 0$. It is divided into two cases whether there is a vertex whose valency is less than four or not. If the graph Γ_l has no edges, since Γ_l is connected, Γ_l has only one vertex. If Γ_l has a vertex with valency two, since there is not a vertex in Figure 3-3 or a stump of order n in Γ_l, Γ_l is the graph as is shown in Figure 3-13 (a). Similarly if Γ_l has a vertex with valency three, Γ_l is the graph as is shown in Figure 3-13 (b). Therefore we have

$$|E(\Gamma_l)| \geq 2\{|V(\Gamma_l)| - 1\}.$$

(a) (b)

FIGURE 3-13

If there is not a vertex whose valency is less than four, then we have

$$|E(\Gamma_l)| = \frac{1}{2}\sum_{v\in\Gamma_l} \mathrm{val}(v) \geq \frac{4}{2}|V(\Gamma_l)| = 2|V(\Gamma_l)|,$$

where we prove Lemma 3.5.

Next we consider the case $m_1 \geq 1$. Let k_3 and k_4 be respectively the numbers of vertices in $\Gamma_l^{m_1+m_2}$ whose valencies are equal to three and more than three, then we have

$$|E(\Gamma_l^{m_1+m_2})| = \frac{1}{2}\Big\{3k_3 + \sum_{v\in\Gamma_l^{m_1+m_2},\mathrm{val}(v)\geq 4} \mathrm{val}(v)\Big\}. \tag{3-4}$$

Since the graph $\Gamma_l^{m_1+m_2}$ has no vertex of type (3.1) and (3.2), a vertex with valency three is a stump of order 3. Thus we have

$$k_3 = s_1 + s_2. \tag{3-5}$$

By Lemma 3.3, it follows that

$$g(\Gamma_l^{m_1+m_2}) \geq 2m_1 + s_1,$$

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and hence

$$\sum_{v\in\Gamma_l^{m_1+m_2}, \mathrm{val}(v)\geq 4} \{\mathrm{val}(v) - 4\} \geq 2m_1 + s_1.$$

We have now

$$\sum_{v\in\Gamma_l^{m_1+m_2}, \mathrm{val}(v)\geq 4} \mathrm{val}(v) \geq 4k_4 + 2m_1 + s_1. \tag{3-6}$$

From (3-4), (3-5) and (3-6), we see

$$\begin{aligned}|E(\Gamma_l^{m_1+m_2})| &\geq \frac{1}{2}\{3(s_1+s_2) + 4k_4 + 2m_1 + s_1\} = \frac{4}{2}(s_1+s_2+k_4) + m_1 - \frac{1}{2}s_2 \\ &= 2(k_3+k_4) + m_1 - \frac{1}{2}s_2 = 2|V(\Gamma_l^{m_1+m_2})| + m_1 - \frac{1}{2}s_2.\end{aligned}$$

This completes the proof of Lemma 3.5.

LEMMA 3.6.

$$|E(\Gamma)| \geq 2\{|V(\Gamma)| - \mathrm{ind}(\Gamma) - 1\}.$$

PROOF. By Proposition 3.2 and Lemma 3.5, we have

$$|E(\Gamma)| - 2l - 2p - 3m_1 - 3m_2 \geq 2\{|V(\Gamma)| - 2l - p - 3m_1 - 2m_2 - 1\} + m_1 - \frac{1}{2}s_2,$$

and thus

$$|E(\Gamma)| \geq 2\{|V(\Gamma)| - l - m_1 - m_2 - 1\} + m_2 - \frac{1}{2}s_2.$$

From Proposition 3.1 and Lemma 3.4, it follows

$$|E(\Gamma)| \geq 2\{|V(\Gamma)| - \mathrm{ind}(\Gamma) - 1\}.$$

THEOREM 3.7. *Let G be a connected, planar, bipartite graph without a cut edge, then*

$$|E(G)| \geq 2\{|V(G)| - \mathrm{ind}(G) - 1\}.$$

PROOF. Let G be the block sum of $G_1, G_2, \ldots, G_n$, then we have

$$|E(G)| = \sum_{i=1}^{n} |E(G_i)|, \ |V(G)| = \sum_{i=1}^{n} |V(G_i)| - (n-1). \tag{3-7}$$

From Theorem 2.2, it follows

$$\mathrm{ind}(G) = \sum_{i=1}^{n} \mathrm{ind}(G_i). \tag{3-8}$$

By Lemma 3.6, we have

$$|E(G_i)| \geq 2\{|V(G_i)| - \operatorname{ind}(G_i) - 1\}. \tag{3-9}$$

By (3-7), (3-8) and (3-9), we have

$$\begin{aligned}|E(G)| &= \sum_{i=1}^{n} |E(G_i)| \geq 2\Big\{\sum_{i=1}^{n} |V(G_i)| - \sum_{i=1}^{n} \operatorname{ind}(G_i) - n\Big\} \\ &= 2\Big\{\sum_{i=1}^{n} |V(G_i)| - (n-1) - \sum_{i=1}^{n} \operatorname{ind}(G_i) - 1\Big\} = 2\{|V(G)| - \operatorname{ind}(G) - 1\}.\end{aligned}$$

This completes the proof of Theorem 3.7.

Theorem 3.7 is rewritten to the following theorem in terms of a link diagram.

THEOREM 3.7A. *If D is a diagram of a link L with no nugatory crossings, then*

$$c(D) \geq 2\{S(D) - \operatorname{ind}(D) - 1\}.$$

Let D be a minimal crossing diagram of a link L, then $c(D)$ is equal to the minimal crossing number of L. Therefore by Theorem 2.4 and Theorem 3.7A, we obtain Theorem 3.8.

THEOREM 3.8. *Let L be a nonsplit link, $c(L)$ the minimal crossing number of L and $b(L)$ the braid index of L, then we have*

$$c(L) \geq 2\{b(L) - 1\}.$$

REMARK. Of all prime knots up to 10 crossings, equality in Theorem 3.8 holds for each of the following knots; 4_1, 6_1, 8_1, 8_3, 8_{12}, 10_1, 10_3, 10_{13}, 10_{35}, 10_{58}.

4. **On Fox's conjecture.** In 1950, R. H. Fox conjectured an estimate of the bridge index, denoted by $\operatorname{bg}(K)$, in terms of the minimal crossing number $c(K)$ for a knot, and Murasugi[6] proposes the following conjecture for a $\mu(L)$ components link. If $\mu(L) = 1$, that is reduced to Fox's conjecture.

CONJECTURE. *Let $\mu(L)$ be the number of components of L. Then for any nonsplit link L,*

$$3\{\operatorname{bg}(L) - 1\} \leq c(L) + \mu(L) - 1. \tag{4-1}$$

Murasugi shows that (4-1) holds for alternating algebraic links. In this section we prove the conjecture for a certain kind of special alternating links by making use of their braid indices.

Let L be a special alternating link, D a reduced special alternating diagram of L and Γ the Seifert graph of D. A special diagram is said to be *nice* if a disk in R^2 bounded by a 2-cycle $c = \{v_0, e_1, v_1, e_2, v_0\}$ in Γ has only edges connecting two vertices v_0 and v_1. By the following lemma, we may assume that any special diagram is always nice.

https://doi.org/10.4153/CJM-1993-007-x Published online by Cambridge University Press

LEMMA 4.1 ([8] LEMMA 10.2). *Any special diagram can be transformed into a nice special diagram by an ambient isotopy of a link.*

We defined the Seifert graph using a reduced special alternating diagram D. Conversely given a Seifert graph Γ, we can construct a special alternating diagram D of a link L. For convenience, L will be called the link associated with Γ and it will be denoted by $l(\Gamma)$. Precisely speaking, two different links $l(\Gamma)$ can be associated with Γ, however one is the mirror image of the other. Since we are only concerned about $c(L)$ and $\mathrm{bg}(L)$, this ambiguity will not cause any confusion.

We define the order of multiplicity for a reduced special alternating diagram D.

DEFINITION 4.2. Let Γ be the Seifert graph of a special alternating diagram D, and Γ' the graph obtained from Γ by deleting all singular edges. The *order of multiplicity of* D, $\lambda(D)$, is the number of components of Γ'.

THEOREM 4.3. *If a link L has a reduced special alternating diagram D with* $\lambda(D)=1$ *or 2, then we have*

$$3\{b(L)-1\} \leq c(L)+\mu(L)-1, \tag{4-2}$$

where $b(L)$ *is the braid index of L.*

Since the bridge index $\mathrm{bg}(L)$ is equal to or less than the braid index $b(L)$, we have the following corollary.

COROLLARY 4.4. *If a link L has a reduced special alternating diagram D with* $\lambda(D)=1$ *or 2, then we have*

$$3\{\mathrm{bg}(L)-1\} \leq c(L)+\mu(L)-1.$$

PROOF OF THEOREM 4.3. At first we show that (4-2) holds for the case $\lambda=1$. Let T be a spanning tree of Γ' and T' the graph obtained from T by making all edges double as is shown in Figure 4-1.

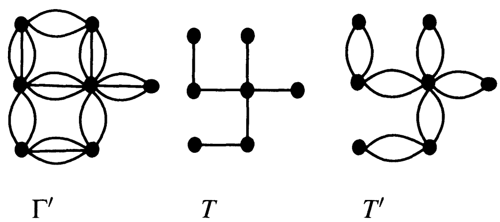

FIGURE 4-1

Since T' is the subgraph of Γ which contains all vertices in Γ, we have

$$|E(T')| = 2\{|V(T')|-1\} = 2\{|V(\Gamma)|-1\}. \tag{4-3}$$

Since Γ is obtained from T' by adding some edges, $l(\Gamma)$ is obtained from $l(T')$ by adding some crossing points. Every time that we add an edge to T', the number of components of $l(T')$ increases or decreases by exactly one. Since $\mu(l(T')) = |V(T')| \geq \mu(l(\Gamma))$, we must add at least $\mu(l(T')) - \mu(l(\Gamma))$ edges. Therefore we have

$$|E(\Gamma)| \geq |E(T')| + \mu(l(T')) - \mu(l(\Gamma)) = |E(T')| + |V(\Gamma)| - \mu(l(\Gamma)). \tag{4-4}$$

From (4-3) and (4-4), it follows

$$\begin{aligned} |E(\Gamma)| &\geq 2\{|V(\Gamma)| - 1\} + |V(\Gamma)| - \mu(l(\Gamma)) \\ &= 3\{|V(\Gamma)| - 1\} + 1 - \mu(l(\Gamma)). \end{aligned} \tag{4-5}$$

Since D is a reduced alternating diagram of L, $|E(\Gamma)| = c(L)$ by Murasugi[5] and $|V(\Gamma)| = S(D) \geq b(L)$. Therefore we have

$$c(L) \geq 3\{b(L) - 1\} + 1 - \mu(L).$$

Next we show (4-2) for the case $\lambda = 2$. Let Γ' be the disjoint union of Γ_1' and Γ_2'. As we did above, we form the subgraph T_1' and T_2' of Γ_1' and Γ_2', respectively. Then we have

$$|E(T_i')| = 2\{|V(T_i')| - 1\} \quad (i = 1, 2), \tag{4-6}$$

$$|E(\Gamma)| \geq |E(T_1')| + |E(T_2')| + \mu(l(T_1' \cup T_2')) - \mu(l(\Gamma)). \tag{4-7}$$

From (4-6) and (4-7), it follows

$$\begin{aligned} |E(\Gamma)| &\geq 2\{|V(T_1')| - 1\} + 2\{|V(T_2')| - 1\} + |V(T_1')| + |V(T_2')| - \mu(l(\Gamma)) \\ &= 3\{|V(T_1')| + |V(T_2')| - 2\} + 2 - \mu(l(\Gamma)). \end{aligned} \tag{4-8}$$

Since Γ has a singular edge, $\operatorname{ind}(\Gamma) \geq 1$ and since $|V(T_1')| + |V(T_2')| = |V(\Gamma)| = S(D)$, we have

$$|V(T_1')| + |V(T_2')| - 1 \geq S(D) - \operatorname{ind}(\Gamma). \tag{4-9}$$

By (4-8) and (4-9), we have

$$|E(\Gamma)| \geq 3\{S(D) - \operatorname{ind}(\Gamma) - 1\} + 2 - \mu(l(\Gamma)).$$

By Theorem 2.4, we finally obtain

$$c(L) \geq 3\{b(L) - 1\} + 1 - \mu(L).$$

This completes the proof of Theorem 4.3.

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https://doi.org/10.4153/CJM-1993-007-x Published online by Cambridge University Press