A WEIGHT THEORY FOR UNITARY REPRESENTATIONS

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Over a field of characteristic 0 certain of the simple Lie algebras have a root theory, namely those called "split" in Jacobson's book **(3)**. We shall assume some familiarity with the subject matter of this book. Then the finite-dimensional representations of these Lie algebras have a weight theory. Our purpose here is to present a kind of weight theory for the representations of these Lie algebras when their ground field is the real numbers, and when the representation comes from a unitary group representation.

To summarize our results we let $\mathfrak{G}$ be a real simple split Lie algebra and $\mathfrak{H}$ a splitting Cartan subalgebra with real dual space $\mathfrak{H}'$. A strongly continuous unitary representation (of a Lie group) will go by the name "representation" in this paper. Let π be a representation of G, a Lie group with the Lie algebra $\mathfrak{G}$. Then for every $\psi \in \mathfrak{H}'$, $i\psi$ is a "weight" of π, the "weights" have constant multiplicity (assuming that the identity representation does not occur in π), and the representation space may be regarded as the direct integral over $\mathfrak{H}'$ (with respect to Lebesgue measure) of the "infinitesimal weight spaces." In other words the representation space may be regarded as all square-integrable functions on $\mathfrak{H}'$ with values in some fixed Hilbert space. Then for x in $\mathfrak{H}$, $d\pi(x)$ is multiplication by $i(\psi, x)$ ($\psi \in \mathfrak{H}'$). One biproduct of this study, useful for further application, is the fact that if e_ϕ is a root vector, then $d\pi(e_\phi)$ annihilates no vector. (For more discussion of $d\pi(e_\phi)$ see §3.)

These results were obtained in the author's doctoral dissertation at the Massachusetts Institute of Technology.

1. We begin by developing the theory for three-dimensional groups with split simple Lie algebra. These are all locally isomorphic to SL(2, **R**). Groups of larger dimension are in a sense "pieced together" from these three-dimensional ones. We obtain the general theorem by "piecing it together" from the three-dimensional theorem.

Let $\mathfrak{G}$ denote the three-dimensional real split simple Lie algebra until further notice. Let G be a fixed connected Lie group corresponding to $\mathfrak{G}$. $\mathfrak{G}$ has a basis $\{e_+, x, e_-\}$ such that $[e_+, e_-] = x$ and $[x, e_\pm] = \pm e_\pm$. Let $\mathfrak{S}$ denote the solvable subalgebra of $\mathfrak{G}$ spanned by x and e_+. Let S denote the connected subgroup of G with Lie algebra $\mathfrak{S}$. It is known that there is (up to isomorphism) only one connected Lie group with Lie algebra $\mathfrak{S}$. It is (isomorphic to) the subgroup of SL(2, **R**) consisting of upper-triangular matrices

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with positive diagonal entries. Observe that this group has trivial centre and is simply connected, and is thus unique with Lie algebra $\mathfrak{S}$. Let E and X be the subgroups of S generated by e_+ and x respectively. Then E is normal and S is the semi-direct product of E and X. The representation theory of S is known; see (**1** or **5**, p. 132, Example I). There are two faithful irreducible representations σ_+ and σ_- of S. All other irreducible representations of S are the identity on E. Every representation σ of S may be written

$$\sigma = C_+ \sigma_+ \oplus C_- \sigma_- \oplus \sigma_0,$$

where $C_+ \sigma_+$ (or $C_- \sigma_-$) denotes the direct sum of σ_+ (or σ_-) a cardinal number C_+ (or C_-) times, and σ_0 is the identity on the subgroup E. σ_+ and σ_- act on $L_2(\mathbf{R})$ as follows:

$$\sigma_+(\exp(tx))f(t') = \sigma_-(\exp(tx))f(t') = f(t + t'), \tag{1.1}$$

$$\sigma_+(\exp(te_+))f(t') = \sigma_-(\exp(-te_+))f(t') = \exp(it\exp(t'))f(t'). \tag{1.2}$$

We need some facts about the differential of a representation. So if π is a representation of a real Lie group L on a Hilbert space H, let $C^\infty(\pi)$ denote the set of vectors v in H such that $\pi(\cdot)\, v$ is a C^∞ function on L. $C^\infty(\pi)$ is a linear subset of H. It is dense and in fact contains the analytic vectors which are dense (**6**). For any $y \in \mathfrak{L}$, the Lie algebra of L, the one-parameter unitary group $\pi(\exp(\mathbf{R}y))$ is generated by a skew-adjoint operator, which we denote by $d\pi(y)$, so that

$$\pi(\exp(ty)) = \exp(td\pi(y)), \qquad t \text{ in } \mathbf{R}.$$

For all v in $C^\infty(\pi)$, v is in the domain of $d\pi(y)$ and

$$d\pi(y)v = d\pi(\exp(ty))v/dt \qquad (\text{at } t = 0).$$

$C^\infty(\pi)$ is stable under $d\pi(y)$ for all y in $\mathfrak{L}$ and $y \to d\pi(y)|C^\infty(\pi)$ defines a representation of $\mathfrak{L}$. $d\pi(y)$ is essentially skew-adjoint on $C^\infty(\pi)$ (**6**, Lemma 5.1). $d\pi$ extends to a representation, also denoted $d\pi$, of the universal enveloping algebra U of $\mathfrak{L}$. Also if c is a central element of U, fixed under the anti-automorphism $u \to u'$ of U, where $y' = -y$ for $y \in \mathfrak{L}$, then $d\pi(c)$ is essentially self-adjoint, and the spectral resolution of its self-adjoint closure commutes with $\pi(L)$ (**7**).

For the group S and the representations σ_+ and σ_- we have

$$C^\infty(\sigma_\pm) = C^\infty(\mathbf{R}) \cap \bigcap_{n=0}^{\infty} L^2(\mathbf{R}, e^{nt}\,dt),$$

$$d\sigma_+(e_+) = -d\sigma_-(e_+) = \text{multiplication by } ie^t; \tag{1.3}$$

$$d\sigma_+(x) = d\sigma_-(x) = d/dt. \tag{1.4}$$

LEMMA 1. *Let σ be an arbitrary representation of S on a Hilbert space H. Let*

$$H(y) = \{v \in H | \sigma(\exp(y))v = v\},$$

for any y in $\mathfrak{S}$. Then for any $t \neq 0$, $H(te_+)$ reduces σ and $\sigma(E)|H(te_+) = I$. If $t \neq 0$, $H(tx)$ reduces σ and in fact $\sigma(S)|H(tx) = I$.

Proof. Write $\sigma = C_+ \sigma_+ \oplus C_- \sigma_- \oplus \sigma_0$. If $t \neq 0$, the representations σ_+ and σ_- leave no vector of $L^2(\mathbf{R})$ fixed under $\exp te_+$ by (1.2). Thus the same is true of $C_+ \sigma_+$ and $C_- \sigma_-$ and their direct sum. Hence $H(te_+)$ is exactly the representation subspace of σ_0 and thus reduces σ. The same argument shows that

$$H(tx) \subseteq \text{the representation subspace of } \sigma_0 = H(te_+).$$

Thus

$$\pi(S)|H(tx) = \pi(XE)|H(tx) \cap H(te_+) = I.$$

LEMMA 2. *Let π be a representation of G and suppose that the identity representation of G does not occur in π. Then for any vector v in the representation space H, and any $t \neq 0$, $\pi(\exp te_+)v = v$ implies $v = 0$.*

Proof. In the universal enveloping algebra U of $\mathfrak{G}$ consider the element $c = e_+ e_- + e_- e_+ + x^2$. c is central, as a calculation easily shows. (It suffices to check that c commutes with e_+, x, and e_-.) It is fixed under the anti-automorphism $u \to u'$ of U, which on $\mathfrak{G}$ is $y' = -y$. Thus by Segal's theorem (7), the closure of $d\pi(c)$ is self-adjoint and has a spectral resolution that commutes with π. The representation π is consequently the direct integral over the spectrum of $d\pi(c)$ of representations π^r for which $d\pi^r(c)$ is the real scalar r. If for some $v \neq 0$ in H and some $t \neq 0$ (fixed for the rest of the proof) we have $\pi(\exp te_+)v = v$, then writing $v = \int \oplus v^r$, we get

$$\pi^r(\exp te_+)v^r - v^r = 0$$

for almost all r. It therefore suffices to show that for any real number r, the lemma holds under the added assumption that $d\pi(c) = r$.

Let σ denote the restriction of π to S. Let

$$H_0 = \{v \in H|\pi(\exp(te_+))\, v = v\} = H(te_+).$$

By Lemma 1, H_0 reduces σ, and $\pi(E)|H = I$. Thus by the spectral theorem,

$$H_0 = \{v \in H|d\pi(e_+)v = 0\}.$$

Here $d\pi(e_+)$ is regarded as a skew-adjoint operator. Also since H_0 reduces σ, H_0 reduces the skew-adjoint operator $d\pi(x)$. In particular, $d\pi(x)$ and $(d\pi(x))^2$ are densely defined in H_0.

Now for all $v \in C^\infty(\pi)$,

$$d\pi(e_+)d\pi(e_-)v = d\pi(e_-)d\pi(e_+)v + d\pi(x)v.$$

Choose v_0 in H_0 in the dense intersection of the domains of $d\pi(x)$ and $(d\pi(x))^2$. Then

$$\begin{aligned}\langle d\pi(e_-)d\pi(e_+)v, v_0\rangle &= \langle d\pi(e_+)d\pi(e_-)v, v_0\rangle - \langle d\pi(x)v, v_0\rangle \\ &= \langle d\pi(e_-)v, -d\pi(e_+)v_0\rangle + \langle v, d\pi(x)v_0\rangle = \langle v, d\pi(x)v_0\rangle.\end{aligned}$$

On the other hand,

$$rI = d\pi(c) \supseteq (2d\pi(e_-)d\pi(e_+) + d\pi(x) + (d\pi(x))^2)|C^\infty(\pi).$$

This implies that

$$2d\pi(e_-)d\pi(e_+)v = rv - d\pi(x)v - (d\pi(x))^2 v.$$

Hence

$$\langle v, d\pi(x)v_0\rangle = \tfrac{1}{2}\langle (r - d\pi(x) - (d\pi(x))^2)v, v_0\rangle$$
$$= \tfrac{1}{2}\langle v, (r + d\pi(x) - (d\pi(x))^2)v_0\rangle.$$

Therefore

$$\langle v, (r - d\pi(x) - (d\pi(x))^2)v_0\rangle = 0.$$

Since v was arbitrary in the dense set $C^\infty(\pi)$, we have

$$(r - d\pi(x) - (d\pi(x))^2)v_0 = 0.$$

Since $d\pi(x)$ is skew-adjoint,

$$\langle d\pi(x)v_0, v_0\rangle = 0.$$

Thus

$$0 = \langle (r - d\pi(x) - (d\pi(x))^2)v_0, v_0\rangle = \langle (r - (d\pi(x))^2)v_0, v_0\rangle.$$

Since v_0 was chosen arbitrarily from a dense set in H_0, and since $r - (d\pi(x))^2$ is self-adjoint on H_0, it is 0 on H_0. Thus on H_0 we have

$$0 = r - (d\pi(x))^2 - d\pi(x) = -d\pi(x).$$

Hence $\pi(x)|H_0 = I$.

Let $\mathfrak{S}_-$ denote the subalgebra of $\mathfrak{G}$ spanned by e_- and x. Then $\mathfrak{S}_-$ is isomorphic to $\mathfrak{S}$ by $e_+ \to e_-$, $x \to -x$. Let S_- denote the connected subgroup of G with Lie algebra $\mathfrak{S}_-$. Then S_- is isomorphic to S and consequently has the same representation theory. We may therefore apply Lemma 1 to the restriction of the representation π to the group S_-. Since $\pi(\exp(X\mathbf{R}))|H_0 = I$, we conclude that $\pi(S_-)|H_0 = I$. Since the subgroups S and S_- generate G, and since $\pi(S_-)|H_0 = \pi(S)|H_0 = I$, we have $\pi(G)|H_0 = I$. This contradicts our assumption that the identity representation does not occur in π, unless $H_0 = 0$.

The significance of this lemma may be seen if we again write

$$\pi|S = \sigma = C_+\sigma_+ \oplus C_-\sigma_- \oplus \sigma_0.$$

The lemma then states that the piece σ_0 does not occur, so we have simply $\sigma = C_+\sigma_+ \oplus C_-\sigma_-$. In particular, $\pi|X$ is just $(C_+ + C_-)$ copies of translation in $L^2(\mathbf{R})$ (see 1.1), or if one wishes, of the regular representation of X. Let us replace σ_+ and σ_- by their conjugates under the Fourier transform on $L^2(\mathbf{R})$. Then for f in $L^2(\mathbf{R})$, t, t' in $\mathbf{R}$, we have

$$(1.5) \qquad (\sigma_+(\exp(tx))f)(t') = (\sigma_-(\exp(tx))f)(t') = \exp(itt')f(t').$$

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($\sigma_{\pm}(\exp te_+)$ is difficult to describe explicitly and this is why we did not originally use this form.) In differential terms (1.5) reads

$$d\sigma_+(x) = d\sigma_-(x) = \text{multiplication by } it. \tag{1.6}$$

$\mathfrak{H} = \{\mathbf{R}x\}$ is a splitting Cartan subalgebra of $\mathfrak{G}$. If we identify $\mathbf{R}$ with $\mathfrak{H}'$, the real dual of $\mathfrak{H}$, then the representation space of π is $C_+ + C_-$ copies of $L^2(\mathfrak{H}')$ and for $f \in (C_+ + C_-)L^2(\mathfrak{H}')$, $\phi \in \mathfrak{H}'$ we have

$$(d\pi(x)f)(\phi) = i\phi(x)f(\phi). \tag{1.7}$$

Here we have regarded $(C_+ + C_-)\, L^2(\mathfrak{H}')$ as the set of all measurable functions f from $\mathfrak{H}'$ to a fixed Hilbert space $H\cdot$ of dimension $C_+ + C_-$ such that the $H\cdot$ norm of f as a real function on $\mathfrak{H}'$ is square-integrable. This is a standard identification; see **(4)**. (1.7) may be interpreted as saying that each point of $i\mathfrak{H}'$ is an infinitesimal weight of multiplicity $C_+ + C_-$.

Before going on to establish these results for an arbitrary real split simple Lie algebra we need one more observation about the representation σ. Regard σ_+ and σ_- as operating on $L^2(\mathbf{R})$ with (1.5) giving σ_+ and σ_- on X. Let ρ denote the regular representation of $\mathbf{R}$ on $L^2(\mathbf{R})$:

$$\rho(t)f(t') = f(t' - t) \qquad (f \in L^2(\mathbf{R});\, t, t' \in \mathbf{R}).$$

The following three sets of operators act irreducibly on $L^2(\mathbf{R})$: $\sigma_+(S)$, $\sigma_-(S)$, and $\sigma_{\pm}(X) \cup \rho(\mathbf{R})$. We now have

LEMMA 3. *Let $\sigma = C_+\sigma_+ \oplus C_-\sigma_-$ be a representation of S on a Hilbert space H. Let P be the projection-valued measure on $\mathbf{R}$ such that*

$$\sigma(\exp t'x) = \int_{\mathbf{R}} \exp(itt')P(dt).$$

Then there is a representation τ of $\mathbf{R}$ on H such that for any real t, and measurable subset M of $\mathbf{R}$,

$$\tau(t)P(M)\tau(-t) = P(M + t)$$

and τ is such that if a normal operator commutes with σ, then it commutes with τ (and, of course, with P).

Proof. Let $C = C_+ + C_-$. Then $H = CL^2(\mathbf{R})$ consists of all square-integrable functions from $\mathbf{R}$ to a Hilbert space $H\cdot$ of dimension C. In this representation $P(M)$ is just multiplication by the characteristic function K_M of M, and $\tau = C\rho$. So

$$\begin{aligned}(\tau(t)P(M)\tau(-t)f)(t') &= (P(M)\tau(-t)f)(t' - t) \\ &= K_M(t' - t)f(t') = (P(M + t)f)(t').\end{aligned}$$

Now suppose the operator N commutes with σ. Since $C_+\sigma_+$ and $C_-\sigma_-$ are primary, N is completely reduced by the representation subspace of $C_+\sigma_+$ (and of $C_-\sigma_-$). $C\rho$ restricted to this space is $C_+\rho$ $(C_-\rho)$. Since σ_+ is irreducible, any operator on $L^2(\mathbf{R})$ may be strongly approximated by finite sums of ele-

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ments in $\sigma_+(S)$ and, in particular, $\rho(t)$ may be so approximated. Thus if N commutes with $C_+ \sigma_+$, it commutes with all finite sums in $C_+ \sigma_+(S)$ and hence with all strong limits of such sums including $C_+ \rho(t)$. Similarly N commutes with $C_- \rho(t)$, so N commutes with $C\rho(t) = \tau(t)$ for all t.

2. Now let $\mathfrak{G}$ denote any real simple split Lie algebra and $\mathfrak{H}$ a splitting Cartan subalgebra with real dual space $\mathfrak{H}'$. Let Φ be a fundamental system of roots for $\mathfrak{H}$. Then Φ is a basis of $\mathfrak{H}'$. For any root ψ let e_ψ be a root vector for ψ. Let $x_\psi = [e_\psi, e_{-\psi}]$ and assume e_ψ and $e_{-\psi}$ to be so normalized that $\psi(x_\psi) = 1$. The set of vectors $F = \{x_\phi | \phi \in \Phi\}$ is a basis of $\mathfrak{H}$. Let F' denote the basis of $\mathfrak{H}$ dual to Φ. We shall denote the elements of F' by x'_ϕ in such a way that $\phi_1(x'_\phi) = 1$ if and only if $\phi_1 = \phi$. Thus for all x in $\mathfrak{H}$, $x = \sum \phi(x) x'_\phi$ $(\phi \in \Phi)$.

Now let G be any connected Lie group with Lie algebra $\mathfrak{G}$. The connected subgroup corresponding to $\mathfrak{H}$ is isomorphic as a Lie group with the additive vector group $\mathfrak{H}$ by way of the exponential map. Indeed, since $\mathfrak{H}$ is abelian, exp is a locally isomorphic epimorphism. It is a monomorphism when G is the adjoint group, since for each x in $\mathfrak{H}$, ad x is diagonalizable over $\mathbf{R}$. Since every other group G covers the adjoint group, it is a monomorphism in general.

Now the character group of $\exp(\mathfrak{H})$ may be identified with $\mathfrak{H}'$ by $(\exp x, \psi) = \exp(i\psi(x))$ for $x \in \mathfrak{H}$, $\psi \in \mathfrak{H}'$. If η is any representation of the group $\exp \mathfrak{H}$, there is a projection valued measure $\mathfrak{P}_\eta$ on $\mathfrak{H}'$ such that

$$\eta(\exp x) = \int_{\mathfrak{H}'} \exp(i\phi(x)) \mathfrak{P}_\eta(d\phi) \qquad \text{for all } x \text{ in } \mathfrak{H}.$$

The following theorem asserts that when η is the restriction to $\exp(\mathfrak{H})$ of a representation of G, then $\mathfrak{P}_\eta$ is distributed over $\mathfrak{H}'$ as evenly as possible.

THEOREM 1. *Let π be a representation of G on the Hilbert space H. Assume that the identity representation does not occur in π. Then H consists of C copies of $L^2(\mathfrak{H}')$ for some cardinal number C, and π restricted to the subgroup* $\exp(\mathfrak{H})$ *consists of C copies of the representation η^0 on $L^2(\mathfrak{H}')$*:

$$(\eta^0(\exp x) f)(\phi) = \exp(i\phi(x)) f(\phi).$$

Preliminaries to the proof. Let $\mathfrak{P} = \mathfrak{P}_\eta$ be the projection valued measure on $\mathfrak{H}'$ for the representation $\eta = \pi | \exp(\mathfrak{H})$. We shall show that for every $\psi \in \mathfrak{H}'$ there is a unitary operator $\tau(\psi)$ on H such that if $\mathfrak{M}$ is a measurable subset of $\mathfrak{H}'$, then

$$\tau(\psi) \mathfrak{P}(\mathfrak{M}) \tau(\psi)^{-1} = \mathfrak{P}(\mathfrak{M} + \psi).$$

We shall do this by applying Lemmas 2 and 3 to the connected three-dimensional subgroups G_ϕ of G which correspond to the Lie albegras $\mathfrak{G}_\phi$ spanned by e_ϕ, x_ϕ, and $e_{-\phi}$, ϕ a root. But in order to apply Lemma 2, we must show that the restriction of π to G_ϕ does not contain the identity representation of G_ϕ.

Now a vector v in H is fixed under $\pi(G_\phi)$ if and only if it is fixed under $\pi(\exp(\mathbf{R}x))$. The necessity of this condition is clear. The sufficiency follows from Lemma 1 applied to the subgroups S_ϕ and $S_{-\phi}$ spanned by $\{e_\phi, x_\phi\}$ and

$\{e_{-\phi}, -x_\phi\}$, showing that v is fixed under $\pi(S_\phi)$ and $\pi(S_{-\phi})$, which generate $\pi(G_\phi)$. Let ψ be another root. Then either $[x_\phi, e_\psi] = 0$ or $\psi(x_\phi) \neq 0$. In the first case, $\pi(\exp(\mathbf{R}e_\psi))$ commutes with $\pi(\exp(\mathbf{R}x_\phi))$. In the second case, we may apply Lemma 1 to the connected subgroup of G whose Lie algebra is spanned by $\{x_\phi/\psi(x_\phi), e_\psi\}$. In either case we conclude that $\pi(\exp(te_\psi))$ maps the space H_0 of fixed vectors of $\pi(\exp(\mathbf{R}x_\phi))$ onto itself. Since the root vectors generate $\mathfrak{G}$, we have that a generating set of one-parameter subgroups of $\pi(G)$ leave H_0 fixed. So H_0 reduces π. Restrict π to H_0. We have already observed that $\pi(G_\phi)|H_0 = I$. Since $\mathfrak{G}$ is simple, it follows that $\pi(G)|H_0 = I$. Since we are assuming that the identity representation does not occur in π, we have proved

LEMMA 4. *Let π be a representation of G in which the identity representation does not occur. Then the identity representation does not occur in the restriction of π to G_ϕ for any root ϕ.*

COROLLARY 1. *Let π be as in Lemma* 4. *Let e_ϕ be a root vector and $t \neq 0$. Then $\pi(\exp(te_\phi))$ leaves no non-zero vector fixed.*

Proof. Apply Lemma 2 to the restriction of π to G_ϕ.

COROLLARY 2. *Let $\phi \in \Phi$ and let $\mathfrak{S}'_\phi$ denote the subalgebra of $\mathfrak{G}$ spanned by $\{x'_\phi, e_\phi\}$. Let S'_ϕ be the corresponding connected subgroup of G. S'_ϕ is isomorphic to the subgroup S of* $\mathrm{SL}(2, \mathbf{R})$. *If $\sigma = \pi|S'_\phi$, then $\sigma = C_+\sigma_+ \oplus C_-\sigma_-$, i.e., σ_0 does not occur in σ.*

Proof. S'_ϕ is isomorphic to S since $\mathfrak{S}'_\phi$ is isomorphic to $\mathfrak{S}$ by $x'_\phi \to x$, $e_\phi \to e_+$. So the representation theory of S'_ϕ is identical with that of S. In particular, we may write $\sigma = C_+\sigma_+ \oplus C_-\sigma_- \oplus \sigma_0$ for any representation σ of S'_ϕ. If $\sigma = \pi|S'_\phi$, however, it follows immediately from Corollary 1 that σ_0 does not occur.

Proof of Theorem 1. Throughout this proof M, M_1, etc. will denote Lebesgue measurable subsets of $\mathbf{R}$. So for such a set M and $\phi_1 \in \Phi$ we define (M, ϕ_1) to be the subset of $\mathfrak{H}'$:

$$\{\textstyle\sum_{\phi\in\Phi} t_\phi\,\phi | t_{\phi_1} \in M,\ t_\phi \in \mathbf{R} \text{ for } \phi \neq \phi_1\}.$$

So if we were to co-ordinatize $\mathfrak{H}'$ with the basis $\{\phi_1, \ldots\} = \Phi$, then

$$(M, \phi_1) = M \times \mathbf{R} \times \ldots \times \mathbf{R}.$$

For the next two paragraphs fix $\phi \in \Phi$. Consider π restricted to the subgroup S'_ϕ of Corollary 2. By Corollary 2 and Lemma 3, there exists a projection valued measure P on $\mathbf{R}$ such that

$$\pi(\exp(t'x'_\phi)) = \int_{\mathbf{R}} (\exp(it't)P_\phi(dt)$$

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and a representation τ_ϕ of $\mathbf{R}$ on H such that

$$\tau_\phi(t)P_\phi(M)\tau_\phi(-t) = P_\phi(M + t).$$

Now on the other hand we have the projection valued measure $\mathfrak{P}$ on $\mathfrak{H}'$ such that

$$\pi(\exp x) = \int_{\mathfrak{H}'} \exp(i\psi(x))\mathfrak{P}(d\psi).$$

Let P'_ϕ be defined on the measurable sets M of $\mathbf{R}$ by $P'_\phi(M) = \mathfrak{P}((M, \phi))$. Thus $P'_\phi(dt) = \mathfrak{P}((dt, \phi))$. Then

$$\int_{\mathbf{R}} \exp(it't)P'_\phi(dt) = \int_{\mathbf{R}} \exp(i\psi(t'x'_\phi))\mathfrak{P}((dt, \phi))$$
$$= \int_{\mathbf{R}} \exp(i\psi(t'x'_\phi))\mathfrak{P}(d\psi) = \pi(\exp t'x'_\phi) = \int_{\mathbf{R}} \exp(it't)P_\phi(dt),$$

where in the second expression on the left we take $\psi = t\phi + \psi^\sim$, where $\psi^\sim(x'_\phi) = 0$ and otherwise $\psi^\sim$ is arbitrary. We conclude, by the uniqueness of the measure P_ϕ, that $P'_\phi = P_\phi$, i.e. $P_\phi(M) = \mathfrak{P}((M, \phi))$. So we have

$$\tau_\phi(t)\mathfrak{P}((M, \phi))\tau_\phi(t)^{-1} = \tau_\phi(t)P_\phi(M)\tau_\phi(t)^{-1} = P_\phi(M + t) = \mathfrak{P}((M, \phi) + t\phi).$$

Now pick $\psi \in \Phi$, $\psi \neq \phi$. Then for $x'_\psi \in F'$ we have $[x'_\psi, x'_\phi] = 0$ and $[x'_\psi, e_\phi] = \phi(x'_\psi)e_\phi = 0$. So $\exp(\mathbf{R}x'_\psi)$ commutes with S'_ϕ and $\pi(\exp(\mathbf{R}x'_\psi))$ commutes with the representation τ_ϕ of $\mathbf{R}$ by Lemma 3. Now, as with ϕ, we define P_ψ and prove that $P_\psi(M) = \mathfrak{P}((M, \psi))$. Then since τ_ϕ commutes with

$$\pi(\exp t'x'_\psi) = \int_{\mathbf{R}} \exp(it't)P_\psi\,(dt),$$

τ_ϕ also commutes with $P_\psi(M) = \mathfrak{P}((M, \psi))$ for all measurable sets M in $\mathbf{R}$. But since $\psi \neq \phi$, we have $(M, \psi) = (M, \psi) - t\phi$. Thus

$$\tau_\phi(t)\mathfrak{P}((M, \psi))\tau_\phi(t)^{-1} = \mathfrak{P}((M, \psi)) = \mathfrak{P}((M, \psi) - t\phi).$$

Thus for all $\psi \in \Phi$, whether $\psi = \phi$ or not, we have

$$\tau_\phi(t)\mathfrak{P}((M, \psi))\tau_\phi(t)^{-1} = \mathfrak{P}((M, \psi) - t\phi).$$

The projection valued measure $\mathfrak{P}$ is known to be regular (**2**, §§38 and 39) and is therefore determined by its values on the rectangles $(M_1, \phi_1) \cap \ldots \cap (M_n, \phi_n)$, where $\phi_1, \ldots, \phi_n \in \Phi$ and $M_1, \ldots, M_n$ are measurable subsets of $\mathbf{R}$. But

$$\tau_\phi(t)\mathfrak{P}((M_1, \phi_1) \cap \ldots \cap (M_n, \phi_n))\tau_\phi(t)^{-1}$$
$$= (\tau_\phi(t)\mathfrak{P}((M_1, \phi_1))\tau_\phi(t)^{-1}) \cdot \ldots \cdot (\tau_\phi(t)\mathfrak{P}((M_n, \phi_n))\tau_\phi(t)^{-1})$$
$$= \mathfrak{P}((M_1, \phi_1) + t\phi) \cdot \ldots \cdot \mathfrak{P}((M_n, \phi_n) + t\phi)$$
$$= \mathfrak{P}((M_1, \phi_1) \cap \ldots \cap (M_n, \phi_n) + t\phi).$$

So for any measurable subset $\mathfrak{M}$ of $\mathfrak{H}'$ and t real and $\phi \in \phi$ we have

$$\tau_\phi(t)\mathfrak{P}(\mathfrak{M})\tau_\phi(t)^{-1} = \mathfrak{P}(\mathfrak{M} + t\phi).$$

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Now for each ψ in $\mathfrak{H}'$ write

$$\psi = \sum_{j=1}^{n} t_j \phi_j \qquad (\phi_j \in \Phi)$$

and let $\tau(\psi) = \tau_{\phi_1}(t_1)\cdot \ldots \cdot\tau_{\phi_n}(t_n)$. $\tau(\psi)$ is not uniquely defined and τ is not a representation of $\mathfrak{H}'$. However, it is unitary and

$$\begin{aligned}\tau(\psi)\mathfrak{P}(\mathfrak{M})\tau(\psi)^{-1} &= \tau_{\phi_1}(t_1)\cdot\ldots\cdot\tau_{\phi_n}(t_n)\mathfrak{P}(\mathfrak{M})\tau_{\phi_n}(t_n)^{-1}\cdot\ldots\cdot\tau_{\phi_1}(t_1)^{-1}\\ &= \tau_{\phi_1}(t_1)\cdot\ldots\cdot\tau_{\phi_{n-1}}(t_{n-1})\mathfrak{P}(\mathfrak{M} + t_n\,\phi_n)\tau_{\phi_{n-1}}(t_{n-1})^{-1}\cdot\ldots\cdot\tau_{\phi_1}(t_1)^{-1}\\ &= \ldots = \mathfrak{P}(\mathfrak{M} + t_1\phi_1 + \ldots + t_n\,\phi_n) = \mathfrak{P}(\mathfrak{M} + \psi).\end{aligned}$$

We may now apply the second and third paragraphs of **(4,** §6**)**. There Mackey proves exactly what we want. In his notation, $\mathfrak{H}'$ is an abelian locally compact group G, ψ is σ, $\mathfrak{P}$ is P, $\mathfrak{M}$ is E, $\tau(-\psi)$ is U_σ. For him, U is a representation, but this fact is not used in the paragraphs in question or in the results invoked there. His conclusion stated in our notation is that H is some cardinal number of copies of $L^2(\mathfrak{H}')$ and that $\mathfrak{P}(\mathfrak{M})$ is multiplication by the characteristic function of $\mathfrak{M}$ on each copy. Since

$$\pi(\exp x) = \int_{\mathfrak{H}'} \exp(i\psi(x))\mathfrak{P}(d\psi),$$

this completes our proof.

3. We conclude with some heuristic remarks intended to strengthen the impression that we have here a weight theory. H will be a fixed Hilbert space and π a representation of G in which the identity representation does not occur. Then, by Theorem 1, we may regard H as the set of all square-integrable functions from $\mathfrak{H}'$ to some fixed Hilbert space $H^{\cdot}$, and $\pi(\exp x)$ is multiplication by the function $(\psi \to \exp(i\psi(x)) \ (\psi \in \mathfrak{H}'))$. Let Ω denote the set of all functions f in H (from $\mathfrak{H}'$ to $H^{\cdot}$) which are the restriction to $\mathfrak{H}'$ of entire (vector-valued) functions, again denoted by f, on the complexification of $\mathfrak{H}'$; assume further that the function f_ψ defined by $f_\psi(\cdot) = f(\cdot + i\psi)$ is in H for each $\psi \in \mathfrak{H}'$. Ω may easily be seen to be dense in H. For any root ϕ define the operator T_ϕ on Ω by $(T_\phi f)(\psi) = f(\psi + i\phi) = f_\phi(\psi)$. Now for any x in $\mathfrak{H}$ and f in Ω we have

$$\begin{aligned}[d\pi(x), T_\phi]f(\psi) &= (d\pi(x)T_\phi - T_\phi d\pi(x))f(\psi)\\ &= i\psi(x)f(\psi + i\phi) - i(\psi + i\phi)(x)f(\psi + i\phi) = \phi(x)T_\phi f(\psi)\end{aligned}$$

or $[d\pi(x), T_\phi] = \phi(x)T_\phi$. Thus T_ϕ interacts with $d\pi(\mathfrak{H})$ on Ω in the same way $d\pi(e_\phi)$ does on $C^\infty(\pi)$. Were Ω and $C^\infty(\pi)$ to coincide, this would imply that $d\pi(e_\phi) = AT_\phi$, where A is some unbounded operator commuting with $d\pi(\mathfrak{H})$. The actual situation is more complicated, but one can show that $d\pi(e_\phi) = UiT_\phi U^{-1}$, where the unitary operator U commutes with $d\pi(\mathfrak{H})$ and may therefore be regarded as a function on $\mathfrak{H}'$ whose values are unitary operators on $H^{\cdot}$. For the moment, we see little use for such a result and merely wish to point out the analogy with finite-dimensional representations: The operator $d\pi(e_\phi)$ shifts the weight spaces by an amount ϕ and then operates on the shifted space.

In much the same spirit, the operators $\pi(g)$ may be partially described, where g is a coset representation of an element of the Weyl group, i.e. $\mathrm{Ad}\, g(\mathfrak{H}) \subseteq \mathfrak{H}$. Let $\omega(g)$ be defined on H by

$$\omega(g)f(\psi) = f(\psi \circ \mathrm{Ad}(g)).$$

Since $\mathrm{Ad}(g)$ is of determinant 1 on $\mathfrak{H}$, $\omega(g)$ is unitary. Also for any $x \in \mathfrak{H}$,

$$\begin{aligned}\omega(g)\pi(\exp x)f(\psi) &= \exp(i\psi(\mathrm{Ad}\, g(x)))f(\psi \circ \mathrm{Ad}\, g)\\ &= \pi(\exp(\mathrm{Ad}\, g(x)))\omega(g)f(\psi) = \pi(g \exp(x) g^{-1}\omega(g)f(\psi)\\ &\qquad = \pi(g)\pi(\exp x)\pi(g^{-1})\omega(g)f(\psi).\end{aligned}$$

So $\pi(g)\omega(g^{-1})$ commutes with $\pi(\exp x)$ for all x in $\mathfrak{H}$. Thus $\pi(g) = U_g\omega(g)$, where U_g commutes with $\pi(\exp \mathfrak{H})$ and may thus be considered a function on $\mathfrak{H}'$ whose values are unitary operators on $H^{\cdot}$.

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