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APPROXIMATION OF PIECEWISE CONTINUOUS FUNCTIONS BY QUOTIENTS OF BOUNDED ANALYTIC FUNCTIONS

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1. Introduction. This paper concerns a certain subalgebra of the Banach algebra of complex valued, essentially bounded, Lebesgue measurable functions on the unit circle in the complex plane (denoted here by L^∞). My interest in this subalgebra was prompted by a question of R. G. Douglas. Let H^∞ denote the space of functions in L^∞ whose Fourier coefficients with negative indices vanish (equivalently, the space of boundary functions for bounded analytic functions in the unit disk). Douglas [**5**] has asked whether every closed subalgebra of L^∞ containing H^∞ is determined by the functions in H^∞ that it makes invertible. More precisely, is such an algebra generated by H^∞ and the inverses of the functions in H^∞ that are invertible in the algebra? An affirmative answer is known for L^∞ itself and for certain subalgebras of L^∞ recently studied by Davie, Gamelin, and Garnett [**3**]. At the time of this writing, no algebra is known for which the above question can be answered negatively.

Let C_1 denote the space of complex valued functions on the unit circle that are continuous except possibly at $z = 1$ and have one-sided limits at $z = 1$. Let B_1 denote the closed subalgebra of L^∞ generated by C_1 and H^∞. I began the present investigation in the hope that B_1 would provide a negative answer to Douglas's question. My hope turned out to be unfounded; Douglas's question has an affirmative answer for B_1. I shall prove this here and obtain some additional properties of B_1.

The paper is organized as follows. In § 2 it is shown that on a certain decreasing family of subdomains of the unit disk, the Poisson integral is asymptotically multiplicative on B_1. This yields a necessary condition for a function to be invertible in B_1. In § 3 a Blaschke product is exhibited whose complex conjugate together with H^∞ generates B_1. Besides answering Douglas's question for B_1, this enables one to demonstrate the sufficiency of the invertibility condition of § 2. In § 4 the inner functions that are invertible in B_1 and the inner functions whose inverses generate B_1 are characterized. Section 5 contains some remarks on the Gelfand space of B_1. Finally, in § 6, the results about B_1 are extended to certain larger algebras and some consequent approximation theorems are obtained.

The reader is assumed to be familiar with the basic theory of Hardy spaces in the unit disk (see [**7**; **9**]). The unit disk will be denoted by D. We denote

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by L^1 the Banach space of integrable complex valued functions with respect to normalized Lebesgue measure on ∂D, by H^1 the corresponding Hardy space of functions in L^1 whose Fourier coefficients with negative indices vanish, and by H_0^1 the space of functions in H^1 with mean value 0. We shall identify the functions in H^1 with their natural analytic extensions into D.

I am indebted to R. G. Douglas and H. S. Shapiro for helpful discussions. My treatment of the algebra B_1 borrows ideas used by Douglas [**6**] and M. B. Abrahamse [**1**] to study the algebra $H^\infty + C$.

2. The Poisson integral on B_1. For f in L^∞ and z in D, we let $f(z)$ stand for the value at z of the Poisson integral of f. Thus

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P(r, \theta - t)dt, \quad 0 \leqslant r < 1,$$

where P is the Poisson kernel:

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

For $0 < \epsilon < \pi/2$, we let G_ϵ denote the domain whose boundary is the union of the unit circle, the circular arc $\{e^{i\theta}\cos\epsilon : \epsilon \leqq |\theta| \leqq \pi\}$, and the two segments $[1, e^{i\epsilon}\cos\epsilon]$ and $[1, e^{-i\epsilon}\cos\epsilon]$ (a "pinched annulus"). The two segments are tangent to the circular arc, and each makes an angle with the vertical of absolute value ϵ.

Our aim in this section is to prove the following theorem.

THEOREM 1. *If f and g are in B_1, then*

$$\lim_{\epsilon \to 0} \sup\{|f(z)g(z) - (fg)(z)| : z \in G_\epsilon\} = 0.$$

This has the following immediate consequences.

COROLLARY 1. *If f is an invertible function in B_1, then there is an ϵ such that f is bounded away from* 0 *on G_ϵ.*

In fact, if f is invertible in B_1 then, by the theorem, the product of the Poisson integral of f with the Poisson integral of f^{-1} must be uniformly close to 1 on G_ϵ when ϵ is sufficiently small.

COROLLARY 2. *Let φ be an inner function which is invertible in B_1. Then φ is a Blaschke product whose zeros, if they are infinite in number, tend nontangentially to* 1.

In fact, if φ has infinitely many zeros then these zeros tend nontangentially to 1, because otherwise φ would vanish on every G_ϵ. No point of $\partial D - \{1\}$ can lie in the support of the singular measure associated with φ, because at any point in the support of this measure the cluster set of φ contains 0 [**9**, p. 76]. The point 1 cannot be an atom for the singular measure of φ, for if it were, then φ would be divisible by $\exp[a(z+1)/(z-1)]$ for some $a > 0$, and the latter function, since it tends to 0 as $z \to 1$ nontangentially, is not bounded away from 0 on any G_ϵ. Thus, the singular measure of φ vanishes, that is, φ is a Blaschke product.

Theorem 1 follows without difficulty from the following lemma.

LEMMA 1. *If f is in C_1 and g is in L^∞, then*

$$\lim_{\epsilon\to 0} \sup\{|f(z)g(z) - (fg)(z)| : z \in G_\epsilon\} = 0.$$

To prove the lemma we note that, for $re^{i\theta}$ in D,

$$\begin{aligned} f(re^{i\theta})g(re^{i\theta}) - (fg)(re^{i\theta}) &= [f(re^{i\theta}) - f(e^{i\theta})]g(re^{i\theta}) + [f(e^{i\theta})g(re^{i\theta}) - (fg)(re^{i\theta})] \\ &= A(re^{i\theta}) + B(re^{i\theta}). \end{aligned}$$

We shall show that $\lim_{\epsilon\to 0} \sup\{|B(z)| : z \in G_\epsilon\} = 0$. This result yields the lemma, because the special case when g is the constant function 1 gives $\lim_{\epsilon\to 0} \sup\{|A(z)| : z \in G_\epsilon\} = 0$.

Fix an ϵ and a point $re^{i\theta}$ in G_ϵ. Let η be the angle between 0 and ϵ such that $re^{i\eta}$ is in ∂G_ϵ. Elementary geometric considerations yield the relation $r\cos(\epsilon - \eta) = \cos\epsilon$. We have

$$B(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [f(e^{i\theta}) - f(e^{it})]g(e^{it})P(r, \theta - t)dt.$$

Breaking the integral into two parts, one corresponding to the range of integration $|\theta - t| < \eta$ and the other to the range of integration $\eta < |\theta - t| < \pi$, we find that

$$\begin{aligned} |B(re^{i\theta})| \leqslant{} & ||g||_\infty \sup\{|f(e^{i\theta}) - f(e^{it})| : \theta - \eta < t < \theta + \eta\} \\ & + 2||f||_\infty ||g||_\infty \cdot \frac{1}{\pi}\int_{\eta}^{\pi} P(r, t)dt. \end{aligned}$$

The supremum in the first term on the right side is majorized by

$$\sup\{|f(e^{is}) - f(e^{it})| : \frac{-3\pi}{2} < s, t < \frac{3\pi}{2}, st > 0, |s - t| < \epsilon\},$$

and because f is in C_1, this tends to 0 with ϵ. It remains to show that the integral in the second term on the right side is majorized by a quantity which tends to 0 with ϵ.

From the equality $r\cos(\epsilon - \eta) = \cos\epsilon$ we obtain

$$1 - r = \frac{\cos(\epsilon - \eta) - \cos\epsilon}{\cos(\epsilon - \eta)} = \frac{2\sin\left(\epsilon - \frac{\eta}{2}\right)\sin\frac{\eta}{2}}{\cos(\epsilon - \eta)}.$$

Therefore

$$\eta > 2\sin\frac{\eta}{2} = \frac{(1-r)\cos(\epsilon-\eta)}{\sin\left(\epsilon-\frac{\eta}{2}\right)} > \frac{(1-r)\cos\epsilon}{\sin\epsilon} = (1-r)\cot\epsilon.$$

Because

$$P(r,t) = \frac{1-r^2}{(1-r)^2+4r\sin^2(t/2)} \leqslant \frac{1-r^2}{4rt^2/\pi^2},$$

it follows that

$$\int_\eta^\pi P(r,t)dt \leqslant \frac{\pi^2(1-r^2)}{4r}\int_{(1-r)\cot\epsilon}^\pi t^{-2}dt$$

$$< \frac{\pi^2(1-r^2)}{4r}\cdot\frac{1}{(1-r)\cot\epsilon}$$

$$= \frac{\pi^2(1+r)\tan\epsilon}{4r}.$$

When $\cos\epsilon > 1/2$ the right side is no larger than $\pi^2\tan\epsilon$, and this tends to 0 with ϵ. The proof of Lemma 1 is complete.

To prove Theorem 1 we introduce the function $\sigma(e^{i\theta}) = e^{i\theta/2}$, $0 < \theta < 2\pi$, which will also be useful later. Let B_0 be the set of all functions $\sigma f_1 + f_2$ with f_1 and f_2 in H^∞. It is obvious that B_0 is a subalgebra of B_1 and that it contains H^∞. Also, B_0 contains all the nonnegative powers of σ, and these functions span C_1. Hence B_0 is dense in B_1, and it will suffice to prove Theorem 1 for functions f and g in B_0.

Let $f = \sigma f_1 + f_2$ and $g = \sigma g_1 + g_2$ be two functions in B_0 (where f_1, f_2, g_1, g_2 are in H^∞). For z in D, the difference $f(z)g(z) - (fg)(z)$ can be written as the sum of the following four terms:

$$A_1(z) = (\sigma f_1)(z)(\sigma g_1)(z) - (\sigma^2 f_1 g_1)(z),$$
$$A_2(z) = (\sigma f_1)(z)g_2(z) - (\sigma f_1 g_2)(z),$$
$$A_3(z) = f_2(z)(\sigma g_1)(z) - (\sigma f_2 g_1)(z),$$
$$A_4(z) = f_2(z)g_2(z) - (f_2 g_2)(z).$$

The last term vanishes because the Poisson integral is multiplicative on H^∞. The first term can be broken up as follows:

$$\begin{aligned} A_1(z) = {} & [(\sigma f_1)(z) - \sigma(z)f_1(z)](\sigma g_1)(z) \\ & + \sigma(z)f_1(z)[(\sigma g_1)(z) - \sigma(z)g_1(z)] \\ & + \sigma(z)[\sigma(z)(f_1 g_1)(z) - (\sigma f_1 g_1)(z)] \\ & + \sigma(z)(\sigma f_1 g_1)(z) - (\sigma^2 f_1 g_1)(z). \end{aligned}$$

An application of Lemma 1 to each term on the right side shows that $\lim_{\epsilon\to 0}\sup\{|A_1(z)| : z \in G_\epsilon\} = 0$. As the terms A_2 and A_3 can clearly be handled similarly, the proof of Theorem 1 is complete.

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3. Generation of B_1 by the inverse of a Blaschke product. For f in L^∞ we let $H^\infty[f]$ denote the smallest closed subalgebra of L^∞ containing H^∞ and f. Our aim in this section is to show that there is a Blaschke product φ such that $B_1 = H^\infty[\bar{\varphi}]$. The following lemma is the key.

LEMMA 2. *Let h be a function in H^∞. Then*

$$\operatorname{dist}(h\sigma, H^\infty) \leqq \sup\{|h(x)| : 0 < x < 1\}.$$

Let K denote the above supremum. In estimating the distance of $h\sigma$ from H^∞, we shall use the fact that the quotient space L^∞/H^∞ is the dual of the space H_0^1. The latter implies that $\operatorname{dist}(h\sigma, H^\infty)$ (the norm of the coset of $h\sigma$ in L^∞/H^∞) equals the norm of the functional that $h\sigma$ induces on H_0^1. We can prove the lemma, therefore, by showing that for all g in H^1,

$$(1) \qquad \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} h(e^{i\theta})\sigma(e^{i\theta})g(e^{i\theta})e^{i\theta}d\theta\right| \leqslant K||g||_1.$$

To avoid a minor technical difficulty, we shall prove this directly only for g in H^∞. That will suffice, because H^∞ is L^1-dense in H^1.

Let g be a function in H^∞. The quantity inside the absolute value signs on the left side of (1) can be rewritten as

$$\frac{1}{2\pi i}\int_{\partial D} h(z)\sigma(z)g(z)dz.$$

Let $G = D - [0, 1)$, and let σ be extended in the obvious way to an analytic function in G. (Thus, the extended σ is a branch of $z^{1/2}$. We violate here our convention of extending functions on ∂D by means of the Poisson integral.) The function $h\sigma g$ is then bounded and analytic in G, and it has nontangential boundary values almost everywhere on ∂G, provided we think of each point of $(0, 1)$ as representing, in the obvious manner, two points of ∂G. Applying Cauchy's theorem on a sequence of curves in G converging out towards ∂G and using the bounded convergence theorem, we obtain $\int_{\partial G} h\sigma g dz = 0$. Thus,

$$\begin{aligned}\int_{\partial D} h\sigma g dz &= -\int_{\partial G - \partial D} h\sigma g dz \\ &= -\int_0^1 h(x)\sigma(x + i0)g(x)dx + \int_0^1 h(x)\sigma(x - i0)g(x)dx \\ &= -2\int_0^1 h(x)x^{1/2}g(x)dx.\end{aligned}$$

It follows that the left side of (1) is no larger than

$$\frac{K}{\pi}\int_0^1 |g(x)|dx.$$

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By the Fejér-Riesz inequality [**7**, p. 46],

$$\int_0^1 |g(x)|dx \leqq \pi||g||_1.$$

The proof of Lemma 2 is complete.

COROLLARY 3. *Let φ be an inner function such that*

$$K = \sup\{|\varphi(x)| : 0 < x < 1\} < 1.$$

Then $B_1 \subset H^\infty[\bar{\varphi}]$.

In fact, by Lemma 2 we have, for any nonnegative integer n,

$$\operatorname{dist}(\sigma, \bar{\varphi}^n H^\infty) = \operatorname{dist}(\varphi^n \sigma, H^\infty) \leqq K^n.$$

The left side majorizes $\operatorname{dist}(\sigma, H^\infty[\bar{\varphi}])$, and the right side tends to 0 as n tends to ∞. Thus σ belongs to $H^\infty[\bar{\varphi}]$. Since $B_1 = H^\infty[\sigma]$, the corollary follows.

To answer Douglas's question affirmatively for B_1, therefore, it will suffice to produce an inner function that is invertible in B_1 and satisfies the condition of Corollary 3. We delay this briefly in order to mention another corollary of Lemma 2.

Let C denote the space of continuous complex valued functions on ∂D. We recall that $H^\infty + C$ is a closed subalgebra of L^∞ [**8**, Theorem 2]; it is, in fact, the closed subalgebra of L^∞ generated by H^∞ and $\bar{z}$.

COROLLARY 4. *Let h be in H^∞. Then $h\sigma$ is in $H^\infty + C$ if and only if h has radial limit 0 at the point 1.*

If h has radial limit 0 at 1 then it is immediate from Lemma 2 that $\operatorname{dist}(h\sigma, \bar{z}^n H^\infty) = \operatorname{dist}(z^n h\sigma, H^\infty) \to 0$ as $n \to \infty$, which implies that $h\sigma$ is in $H^\infty + C$. Conversely, if $h\sigma$ is in $H^\infty + C$, then there is a g in H^∞ such that $g - h\sigma$ is in C and takes the value 0 at 1. This means that

$$\lim_{\theta \to 0+} [g(e^{i\theta}) - h(e^{i\theta})] = 0,$$

so it follows from Lindelöf's theorem [**2**, p. 42] that $g - h$ has radial limit 0 at 1. Also

$$\lim_{\theta \to 0-} [g(e^{i\theta}) + h(e^{i\theta})] = 0,$$

so $g + h$ has radial limit 0 at 1. Therefore h has radial limit 0 at 1.

We now produce an inner function which is invertible in B_1 and satisfies the condition of Corollary 3. Let the function f on $\bar{D} - \{1\}$ be defined by $f(z) = \exp[2\pi i \log(1 - z)]$ (the branch of the logarithm is the principal one). The function f is invertible in H^∞, and $|f|$ (restricted to ∂D) belongs to C_1. Hence $\bar{f} = f^{-1}|f|^2$ is in B_1. Therefore $|f - 1|^2 = |f|^2 - f - \bar{f} + 1$ is in B_1. Since $|f - 1|^2$ is bounded away from 0 on ∂D, the function $|f - 1|^{-2}$ can be

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uniformly approximated on ∂D by polynomials in $|f-1|^2$; therefore $|f-1|^{-2}$ is in B_1. Consequently $(f-1)^{-1} = (\bar{f}-1)|f-1|^{-2}$ is in B_1; in other words, the function $f-1$ is invertible in B_1. The outer factor of $f-1$ is invertible in H^∞, and hence the inner factor of $f-1$ is invertible in B_1. We denote this inner factor by φ_0. By Corollary 2, φ_0 is a Blaschke product. Its zeros are at the points $1-e^{-n}$, $n = 0, 1, 2, \ldots$, on the interval $[0, 1)$. Thus

$$|\varphi_0(z)| = \prod_{n=0}^{\infty} \left| \frac{1-e^{-n}-z}{1-(1-e^{-n})z} \right|, \quad |z| < 1.$$

An elementary calculation shows that the nth term in the above product is increasing on the interval $[1-e^{-n}, 1-e^{-n-1}]$ and does not exceed $1-e^{-1}$ at the right endpoint of that interval. Hence $|\varphi_0(x)| \leqq 1-e^{-1}$ for $0 < x < 1$; in other words, φ_0 satisfies the condition of Corollary 3. We have proved

THEOREM 2. $B_1 = H^\infty[\bar{\varphi}_0]$.

The knowledge that $\bar{\varphi}_0$ is in B_1 enables us to prove the following converse of Corollary 3: *If φ is an inner function such that $B_1 \subset H^\infty[\bar{\varphi}]$, then*

$$\sup\{|\varphi(x)| : 0 < x < 1\} < 1.$$

To prove this, let $K = \sup\{|\varphi_0(x)| : 0 < x < 1\}$. If $B_1 \subset H^\infty[\bar{\varphi}]$, then there is a positive integer n and a function h in H^∞ such that $||\bar{\varphi}_0 - \bar{\varphi}^n h||_\infty < (1-K)/(1+K)$. Thus $||\varphi^n - \varphi_0 h||_\infty < (1-K)/(1+K)$ and $||h||_\infty < 1 + (1-K)/(1+K) = 2/(1+K)$. If $0 < x < 1$, then

$$\begin{aligned} |\varphi(x)|^n &\leqq |\varphi_0(x)||h(x)| + |\varphi(x)^n - \varphi_0(x)h(x)| \\ &\leqq K||h||_\infty + ||\varphi^n - \varphi_0 h||_\infty. \end{aligned}$$

The right side is smaller than

$$K \cdot \frac{2}{1+K} + \frac{1-K}{1+K} = 1,$$

and so $\sup\{|\varphi(x)| : 0 < x < 1\} < 1$, as desired.

We shall now use the function φ_0 to show that B_1 contains the boundary functions of the bounded analytic functions in the regions G_ϵ. This will enable us to prove that the invertibility condition of Corollary 1 is sufficient as well as necessary.

THEOREM 3. *Let f be a bounded analytic function in one of the regions G_ϵ $(0 < \epsilon < \pi/2)$. Then the boundary function of f on ∂D belongs to B_1.*

Let $c_n = \operatorname{dist}(f, \bar{\varphi}_0^n H^\infty)$, $n = 1, 2, \ldots$. We wish to show that $\lim_{n\to\infty} c_n = 0$. We have $c_n = \operatorname{dist}(\varphi_0^n f, H^\infty)$, so, as in the proof of Lemma 2, c_n is the norm of the functional that $\varphi_0^n f$ induces on H_0^1. We estimate this norm by the method used in the proof of Lemma 2.

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Let g be any function in H^∞. Then $\varphi_0{}^n fg$ is a bounded analytic function in G_ϵ, so that $\int_{\partial G_\epsilon} \varphi_0{}^n fgdz = 0$. Therefore

$$\text{(2)} \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_0(e^{i\theta})^n f(e^{i\theta}) g(e^{i\theta}) e^{i\theta} d\theta = \frac{1}{2\pi i}\int_{\partial D} \varphi_0{}^n fgdz$$
$$= \frac{-1}{2\pi i}\int_{\partial G_\epsilon - \partial D} \varphi_0{}^n fgdz.$$

Let $K = \sup\{|\varphi_0(z)| : z \in \partial G_\epsilon - \partial D\}$. Because $\sup\{|\varphi_0(x)| : 0 < x < 1\} < 1$, it follows from a theorem of Doob [**4**, Theorem 5.1] that $K < 1$. The right side of (2) is no larger in absolute value than

$$\frac{1}{2\pi} K^n ||f||_\infty \int_{\partial G_\epsilon - \partial D} |g(z)|\ |dz|.$$

By a theorem of Carleson [**7**, p. 157], there is a positive constant M (depending on ϵ) such that

$$\frac{1}{2\pi}\int_{\partial G_\epsilon - \partial D} |g(z)|\ |dz| \leqq M||g||_1.$$

Combining this with the preceding estimate, we may conclude that $c_n \leqq K^n ||f||_\infty M$. Since $K < 1$ we have $c_n \to 0$, as desired. Theorem 3 is proved.

We observed in § 2 that B_1 is the closure in L^∞ of the space of functions $\sigma f_1 + f_2$ with f_1 and f_2 in H^∞. From this and Theorem 3 we see that B_1 can also be described as the closure in L^∞ of the space of boundary functions on ∂D for bounded analytic functions in the slit disk G. Thus, if we transform B_1 by means of a suitable conformal map of G onto D, we find that B_1 is isometrically isomorphic to the closure of H^∞ in L^∞ of Lebesgue measure on the semicircle $\{e^{i\theta} : 0 < \theta < \pi\}$.

The following corollary establishes the sufficiency of the invertibility condition whose necessity was established by Corollary 1.

COROLLARY 5. *Let f be a function in B_1, and assume that there is an ϵ such that f is bounded away from 0 in G_ϵ. Then f^{-1} is in B_1.*

Let $\delta = \inf\{|f(z)| : z \in G_\epsilon\}$. By Theorem 2, there is a non-negative integer n and a function h in H^∞ such that $||\bar{\varphi}_0{}^n h - f||_\infty < \delta/2$. By Theorem 1, if ϵ' is sufficiently small then $|\bar{\varphi}_0{}^n(z) h(z)| > \delta/2$ for z in $G_{\epsilon'}$. Hence h is bounded away from 0 in $G_{\epsilon'}$ for such ϵ', so that h is invertible in B_1 by Theorem 3. Therefore $\bar{\varphi}_0{}^n h$ is invertible in B_1, and the norm of its inverse is obviously no larger than $2/\delta$. Since

$$||1 - (\bar{\varphi}_0{}^n h)^{-1} f||_\infty \leqq ||(\bar{\varphi}_0{}^n h)^{-1}||_\infty ||\bar{\varphi}_0{}^n h - f||_\infty < (2/\delta)(\delta/2) = 1,$$

the function $(\bar{\varphi}_0{}^n h)^{-1} f$ is invertible in B_1. Therefore f is invertible in B_1, as desired.

4. The Blaschke products invertible in B_1. We saw in § 2 that an inner function which is invertible in B_1 must be a Blaschke product whose zeros, if they are infinite in number, tend nontangentially to 1. In the present section we shall obtain a simple criterion for the invertibility of such a Blaschke product in terms of the distribution of its zeros.

Let φ be an infinite Blaschke product whose zeros tend nontangentially to 1. For the sake of simplicity we make a Cayley transformation to the upper half-plane, defining the Blaschke product b in the upper half-plane by

$$b(z) = \varphi\left(\frac{z-i}{z+i}\right).$$

The zero sequence of b will be denoted by $\{z_n\}$. For $r > 0$ let $\nu(r)$ denote the number of indices n such that $\operatorname{Im} z_n < r$.

THEOREM 4. *The following conditions are equivalent.*

(i) *φ is invertible in B_1.*

(ii) *There is a positive constant C such that $\nu(ar) - \nu(r) \leqq aC$ for all $r > 0$ and $a > 1$.*

(iii) *For some $a > 1$, there is a positive constant C such that $\nu(ar) - \nu(r) \leqq C$ for all $r > 0$.*

The equivalence of (ii) and (iii) is elementary. These conditions say, roughly, that the zeros of b tend to ∞ exponentially fast, on the average. They imply, for example, that $\nu(r)/\log r = O(1)$.

We shall prove below that (i) implies (ii) and that (iii) implies (i). Several lemmas of a routine nature are needed.

Since the zeros of φ tend nontangentially to 1, there is a number α in the interval $(0, \pi/2)$ such that $|\pi/2 - \arg z_n| \leqq \alpha$ for all n. By Corollaries 1 and 5, φ is invertible in B_1 if and only if there is a β in $(\alpha, \pi/2)$ such that b is bounded away from 0 in the region $\beta < |\pi/2 - \arg z| < \pi/2$. As we shall see below, if the preceding condition holds for one such β then it holds for every such β. For the remainder of the proof of Theorem 4, we fix a β in $(\alpha, \pi/2)$. We shall write y_n for $\operatorname{Im} z_n$.

LEMMA 3. *If $\beta < |\pi/2 - \arg z| < \pi/2$, then*

$$\left|\frac{z - z_n}{z - \bar{z}_n}\right| \geqq \tfrac{1}{2} \sin(\beta - \alpha)$$

for all n.

In fact, for such a z, the inequalities $|z - z_n| \geqq |z| \sin(\beta - \alpha)$ and $|z - z_n| \geqq |z_n| \sin(\beta - \alpha)$ follow from elementary geometric considerations. Thus $|z - z_n| \geqq \frac{1}{2}(|z| + |z_n|) \sin(\beta - \alpha)$. Since $|z - \bar{z}_n| \leqq |z| + |z_n|$, the desired inequality follows.

LEMMA 4. *The Blaschke product b is bounded away from* 0 *in the region* $\beta < |\pi/2 - \arg z| < \pi/2$ *if and only if the sum*

$$\sum \left(1 - \left|\frac{z - z_n}{z - \bar{z}_n}\right|^2\right)$$

is uniformly bounded in that region.

Let z belong to the region in question. We have

$$|b(z)| = \prod \left|\frac{z - z_n}{z - \bar{z}_n}\right|,$$

and so

$$2 \log |b(z)| = \sum \log \left|\frac{z - z_n}{z - \bar{z}_n}\right|^2.$$

Because the function $-\log t/(1 - t)$ is decreasing on the interval $0 < t \leqq 1$ and takes the value 1 at $t \doteq 1$, it follows from Lemma 3 that

$$1 - \left|\frac{z - z_n}{z - \bar{z}_n}\right|^2 \leqq -\log \left|\frac{z - z_n}{z - \bar{z}_n}\right|^2 \leqq \gamma\left(1 - \left|\frac{z - z_n}{z - \bar{z}_n}\right|^2\right),$$

where

$$\gamma = \frac{-\log[\frac{1}{4}\sin^2(\beta - \alpha)]}{1 - \frac{1}{4}\sin^2(\beta - \alpha)}.$$

Thus

$$-\frac{2}{\gamma}\log |b(z)| \leqq \sum \left(1 - \left|\frac{z - z_n}{z - \bar{z}_n}\right|^2\right) \leqq -2 \log |b(z)|,$$

and the lemma follows.

LEMMA 5. *The Blaschke product b is bounded away from* 0 *in the region* $\beta < |\pi/2 - \arg z| < \pi/2$ *if and only if the sum*

$$\sum \frac{r y_n}{r^2 + y_n^2} \tag{3}$$

remains bounded for $r > 0$.

Let $z = re^{i\theta}$ belong to the upper half-plane. We have

$$1 - \left|\frac{z - z_n}{z - \bar{z}_n}\right|^2 = \frac{4 r y_n \sin \theta}{|z - \bar{z}_n|^2}.$$

Since $|\arg z - \arg \bar{z}_n| > \pi/2 - \alpha$, an application of the law of cosines gives the inequality

$$|z - \bar{z}_n|^2 \geqq (r^2 + y_n^2)(1 - \sin \alpha).$$

On the other hand,

$$\begin{aligned} |z - \bar{z}_n|^2 &\leqq 2(r^2 + |z_n|^2) \\ &\leqq 2(r^2 + y_n^2 \tan^2 \alpha + y_n^2) \\ &\leqq 2 \sec^2 \alpha (r^2 + y_n^2). \end{aligned}$$

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Thus

$$2\cos^2\alpha\sin\theta\sum\frac{ry_n}{r^2+y_n^2}\leqq\sum\left(1-\left|\frac{z-z_n}{z-\bar z_n}\right|^2\right)\leqq\frac{4}{1-\sin\alpha}\sum\frac{ry_n}{r^2+y_n^2}.$$

The desired conclusion is now immediate from Lemma 4.

The proof of Theorem 4 can now be completed in a few lines. Assume first that (iii) holds. For a fixed $r > 0$, the function $ry/(r^2 + y^2)$ is increasing on the interval $0 < y < r$ and decreasing on the interval $r < y < \infty$. Hence

$$\begin{aligned}\sum\frac{ry_n}{r^2+y_n^2} &= \sum_{j=-\infty}^{\infty}\ \sum_{a^jr\leqq y_n<a^{j+1}r}\frac{ry_n}{r^2+y_n^2}\\ &\leqq \sum_{j=-\infty}^{-1}\frac{[\nu(a^{j+1}r)-\nu(a^jr)]r\cdot a^{j+1}r}{r^2+(a^{j+1}r)^2}\\ &\quad+\sum_{j=0}^{\infty}\frac{[\nu(a^{j+1}r)-\nu(a^jr)]r\cdot a^jr}{r^2+(a^jr)^2}\\ &\leqq C\sum_{j=-\infty}^{-1}\frac{a^{j+1}}{1+a^{2j+2}}+C\sum_{j=0}^{\infty}\frac{a^j}{1+a^{2j}}\\ &\leqq 2C\sum_{j=0}^{\infty}a^{-j}=2C(1-a^{-1})^{-1}.\end{aligned}$$

Therefore φ is invertible in B_1 by Lemma 5 and Corollary 5, in other words, (i) holds.

Assume now that (i) holds. Then by Lemma 5 and Corollary 1, there is a positive constant C such that

$$\sum\frac{ry_n}{r^2+y_n^2}\leqq C/2$$

for all $r > 0$. If $y_n \geqq r$ then $y_n/(r^2 + y_n^2) \geqq 1/2y_n$. Therefore, for any $a > 1$,

$$\begin{aligned}\frac{C}{r} &\geqq 2\sum_{y_n\geqq r}\frac{y_n}{r^2+y_n^2}\geqq\sum_{y_n\geqq r}\frac{1}{y_n}\\ &\geqq\sum_{r\leqq y_n<ar}\frac{1}{y_n}\geqq\frac{\nu(ar)-\nu(r)}{ar},\end{aligned}$$

so that (ii) holds. The proof of Theorem 4 is complete.

There exists, also, a simple characterization of the Blaschke products whose inverses generate B_1. We retain the above notations.

THEOREM 5. *If the Blaschke product φ is invertible in B_1, then the following conditions are equivalent.*

(i′) $B_1 = H^\infty[\bar\varphi]$.

(ii′) *There is an $a > 1$ and an $r_0 > 0$ such that $\nu(ar) - \nu(r) > 0$ for all $r > r_0$.*

By Corollary 3 and the remark following Theorem 2, the condition $B_1 \subset H^\infty[\bar{\varphi}]$ is equivalent to the condition $\limsup_{r\to\infty}|b(ir)| < 1$. We shall prove the theorem by showing that if $\bar{\varphi}$ is in B_1, then the last condition is equivalent to (ii′). A lemma is needed.

LEMMA 6. *The conditions* $\limsup_{r\to\infty}|b(ir)| < 1$ *and*

$$\liminf_{r\to\infty} \sum \frac{ry_n}{r^2 + y_n^2} > 0$$

are equivalent.

We have

$$\begin{aligned} 2 \log |b(ir)| &= \sum \log \left| \frac{ir - z_n}{ir - \bar{z}_n} \right|^2 \\ &= \sum \log \left(1 - \frac{4ry_n}{|ir - \bar{z}_n|^2} \right) \\ &\leqq - \sum \frac{4ry_n}{|ir - \bar{z}_n|^2} . \end{aligned}$$

Since $|ir - \bar{z}_n|^2 \leqq 2 \sec^2 \alpha (r^2 + y_n^2)$ (see the proof of Lemma 5), we obtain

$$\log |b(ir)| \leqq - \cos^2 \alpha \sum \frac{4ry_n}{r^2 + y_n^2} .$$

This inequality shows that the second condition of Lemma 6 implies the first condition.

For the other direction, assume that there is a sequence $\{r_j\}$ tending to ∞ such that

$$\lim_{j\to\infty} \sum_n \frac{r_j y_n}{r_j^2 + y_n^2} = 0.$$

We may assume, without loss of generality, that $4r_j y_n/(r_j^2 + y_n^2) < 1/2$ for all n and j. This implies that

$$\log \left(1 - \frac{4r_j y_n}{r_j^2 + y_n^2} \right) \geqq -2 \log 2 \cdot \frac{4r_j y_n}{r_j^2 + v_n^2} .$$

Hence

$$\begin{aligned} 2 \log |b(ir_j)| &= \sum_n \log \left(1 - \frac{4r_j y_n}{|ir_j - \bar{z}_n|^2} \right) \\ &\geqq \sum_n \log \left(1 - \frac{4r_j y_n}{r_j^2 + y_n^2} \right) \\ &\geqq -8 \log 2 \sum_n \frac{r_j y_n}{r_j^2 + y_n^2} , \end{aligned}$$

and it follows that $|b(ir_j)| \to 1$. Lemma 6 is proved.

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To prove Theorem 5 it is convenient to note that

$$\tfrac{1}{2}\min\left(\frac{y}{r},\frac{r}{y}\right) \leqq \frac{ry}{r^2+y^2} \leqq \min\left(\frac{y}{r},\frac{r}{y}\right).$$

Consequently,

$$\sum \frac{ry_n}{r^2+{y_n}^2} \leqq \frac{1}{r}\sum_{y_n\leqq r} y_n + r\sum_{y_n>r}\frac{1}{y_n} \leqq 2\sum \frac{ry_n}{r^2+{y_n}^2}.$$

Let the quantity in the middle of this pair of inequalities be denoted by $S(r)$. In view of Lemma 6, it will suffice, for the proof of Theorem 5, to show that, if φ is invertible in B_1, condition (ii′) is equivalent to the condition $\liminf_{r\to\infty} S(r) > 0$. One direction is completely trivial, for if (ii′) holds then, obviously, $S(r) \geqq 1/a$ for $r > r_0$. For the other direction, assume that (ii′) fails. Fix $a > 1$ and $r_0 > 0$. Then there is an $R > r_0$ such that $\nu(a^2R) - \nu(a^{-2}R) = 0$. Using condition (ii) of Theorem 4, we obtain

$$\begin{aligned} S(R) &\leqq \frac{1}{R}\sum_{j=-\infty}^{-3} [\nu(a^{j+1}R) - \nu(a^jR)]a^{j+1}R \\ &\quad + R\sum_{j=2}^{\infty} [\nu(a^{j+1}R) - \nu(a^jR)](a^jR)^{-1} \\ &\leqq aC\sum_{j=-\infty}^{-3} a^{j+1} + aC\sum_{j=2}^{\infty} a^{-j} \\ &= \frac{2C}{a-1}. \end{aligned}$$

We may conclude that $\liminf_{r\to\infty} S(r) \leqq 2C/(a-1)$. Since a can be chosen arbitrarily large, it follows that $\liminf_{r\to\infty} S(r) = 0$. The proof of Theorem 5 is complete.

Condition (ii′) obviously implies that $\liminf_{r\to\infty} \nu(r)/\log r > 0$. The last condition does not imply (ii′), however, as one can show by simple examples.

5. The Gelfand space of B_1. By the Gelfand space of a commutative Banach algebra, we shall mean here the set of (nontrivial) multiplicative linear functionals on the algebra, endowed with the Gelfand topology. We denote the Gelfand spaces of L^∞, H^∞, and B_1 by X, Y, and Z, respectively. We shall not distinguish notationally between the functions in L^∞, H^∞, and B_1 and their Gelfand transforms. The spaces X and Y are discussed in detail in [**9**, Chapter 10], and we shall use without further reference the basic facts presented there. For λ in ∂D we let X_λ and Y_λ denote the fibers of X and Y above λ.

Each functional in Z is the extension to B_1 of a functional in Y. Because functionals in Y have unique representing measures on X, no functional in Y

has more than one extension to a functional in Z. We may thus identify Z with the set of functionals in Y that extend multiplicatively to B_1, or, equivalently, with the set of functionals in Y whose representing measures are multiplicative on B_1.

If λ is in ∂D and $\lambda \neq 1$, then $B_1|X_\lambda = H^\infty|X_\lambda$, so every functional in Y_λ extends multiplicatively to B_1. It is clear that none of the evaluation functionals in Y at points of D extend multiplicatively to B_1. Thus, to describe Z, it only remains to describe the functionals in Y_1 that extend multiplicatively to B_1.

The fiber X_1 splits naturally into two parts, the set X_1^+ of functionals that assign to σ the value 1 and the set X_1^- of functionals that assign to σ the value -1. If f is any function in C_1, then f assumes on X_1^+ and X_1^- the constant values

$$\lim_{\theta \to 0+} f(e^{i\theta}) \quad \text{and} \quad \lim_{\theta \to 0-} f(e^{i\theta}),$$

respectively. Thus $B_1|X_1^+$ is the uniform closure of $H^\infty|X_1^+$, and $B_1|X_1^-$ is the uniform closure of $H^\infty|X_1^-$. It follows that any functional in Y_1 whose representing measure is supported either entirely by X_1^+ or entirely by X_1^- extends multiplicatively to B_1. On the other hand, if the support of a representing measure for a functional in Y_1 meets both X_1^+ and X_1^-, the measure is easily seen not to be multiplicative on C_1.

Let Y_1^+ (Y_1^-) be the set of functionals in Y_1 whose representing measures are supported by X_1^+ (X_1^-). The above remarks show that

$$Z = (\bigcup_{\lambda \neq 1} Y_\lambda) \cup Y_1^+ \cup Y_1^-.$$

Using the function $\exp[i \log(1 - z)]$, it is simple to prove from this that in order for a net in D to have all of its cluster points in Z, it is necessary and sufficient that the net be eventually in every region G_ϵ. Theorem 1 is an easy consequence of this observation. (Nevertheless, the more "down to earth" proof of Theorem 1 given in § 2, although more computational, seemed preferable to the author.) Corollary 5 can also be obtained via the preceding observation. One seems to need the corona theorem for that, however.

Corollary 4 yields the following curious fact about H^∞: *If h is in H^∞ and h has radial limit* 0 *at the point* 1, *then there is a g in H^∞ such that $g|X_1^+ = h|X_1^+$ and $g|X_1^- = -h|X_1^-$.*

6. The algebras B_E. For E a subset of ∂D, let C_E denote the space of functions in L^∞ that are continuous at each point of $\partial D - E$ and have one-sided limits at each point of E. Let B_E denote the closed subalgebra of L^∞ generated by C_E and H^∞. If E is a singleton $\{\lambda\}$, we shall write B_λ in place of B_E.

The algebra B_E can be described alternatively as the closed subalgebra of L^∞ generated by the algebras B_λ with λ in E. Because the question of Douglas has an affirmative answer for each B_λ, it also has an affirmative answer for B_E:

the algebra B_E is the closed subalgebra of L^∞ generated by H^∞ and the inverses of the Blaschke products that are invertible in B_E.

The problem now arises of characterizing the inner functions that are invertible in B_E. To answer this, it is sufficient to consider the case where E is finite.

LEMMA 7. *Let h be a function in H^∞ which is invertible in B_E. Then there is a finite subset F of E such that h is invertible in B_F.*

In fact, because B_E is generated by $\bigcup_{\lambda \in E} B_\lambda$, there is a finite subset F of E and a function g in B_F such that $\|g - h^{-1}\|_\infty < 1/\|h\|_\infty$. Then $\|1 - gh\|_\infty < 1$, so gh is invertible in B_F. Therefore h is invertible in B_F, as desired.

Now fix a finite subset F of ∂D. For $|\lambda| = 1$ and $0 < \epsilon < \pi/2$, let $G_\epsilon(\lambda)$ be the domain obtained by rotating G_ϵ about the origin through an angle arg λ. Thus, $G_\epsilon(\lambda)$ plays the same role relative to B_λ as G_ϵ does relative to B_1. Let $G_\epsilon(F) = \bigcap_{\lambda \in F} G_\epsilon(\lambda)$. We note that, if F is not a singleton, $G_\epsilon(F)$ is not connected when ϵ is small. The following theorem, analogous to Theorem 1, is an easy consequence of Lemma 1.

THEOREM 6. *If f and g are in B_F, then*

$$\lim_{\epsilon \to 0} \sup\{|f(z)g(z) - (fg)(z)| : z \in G_\epsilon(F)\} = 0.$$

As an immediate consequence we obtain a necessary condition for invertibility in B_F.

COROLLARY 6. *If f is an invertible function in B_F, then there is an ϵ such that f is bounded away from 0 on $G_\epsilon(F)$.*

On the other hand, a trivial modification of the proof of Theorem 3 yields its analogue in the present situation.

THEOREM 7. *If f is a bounded analytic function in one of the regions $G_\epsilon(F)$, then the boundary function of f on ∂D belongs to B_F.*

This yields, in the same way as before, the sufficiency of the invertibility criterion.

COROLLARY 7. *If f is a function in B_F and f is bounded away from 0 in one of the regions $G_\epsilon(F)$, then f is invertible in B_F.*

Suppose now that φ is an inner function which is invertible in B_F. From Corollary 6 we see that φ must be a Blaschke product whose zeros, if they are infinite in number, tend nontangentially to F. Assuming φ does have infinitely many zeros, we can partition the zero sequence of φ into finitely many subsequences, each of which tends to a single point of F. In the corresponding factorization of φ, each factor is invertible in B_λ for λ the limit of its zero sequence. (The latter follows from Corollary 5.)

Combining the preceding observations with Lemma 7, we obtain the following conclusion: *If φ is an inner function which is invertible in B_E for some subset E of ∂D, then there is a finite subset $\{\lambda_1, \ldots, \lambda_p\}$ of E and a factorization*

$\varphi = \varphi_1 \ldots \varphi_p$ of φ such that φ_j is a Blaschke product invertible in B_{λ_j}. This in conjunction with Theorem 4 describes precisely how the zeros of a Blaschke product must be distributed for the product to be invertible in B_E.

The above results can be used to establish some approximation theorems, of which the following is a sample. We let D_+ denote the intersection of D with the upper half-plane and $(\partial D)_+$ the intersection of ∂D with the upper half-plane.

THEOREM 8. *Let f be a bounded analytic function in D_+. Then there is a sequence in H^∞ converging uniformly on $(\partial D)_+$ to the boundary function of f.*

Let g be the function on ∂D that equals the boundary function of f on $(\partial D)_+$ and equals 0 on the rest of ∂D. By Theorem 7, g is in $B_{\{1,-1\}}$. Hence we can uniformly approximate g by a finite sum of the form $u_1h_1 + \ldots + u_ph_p$, where $h_1, \ldots, h_p$ are in H^∞ and $u_1, \ldots, u_p$ are in $C_{\{1,-1\}}$. The functions $u_j|(\partial D)_+$ extend continuously to the closure of $(\partial D)_+$, so each u_j can be uniformly approximated on $(\partial D)_+$ by polynomials. Replacing each u_j by a suitable approximating polynomial, we obtain a function in H^∞ that uniformly approximates g on $(\partial D)_+$.

The same reasoning gives a stronger result.

THEOREM 9. *Let $I_1, \ldots, I_p$ be disjoint closed subarcs of the unit circle. For $k = 1, \ldots, p$ let J_k be the Jordan curve formed from I_k and the line segment having the same end-points as I_k, and let f_k be a bounded analytic function in the interior of J_k. Then there is a sequence in H^∞ which, for each k, converges uniformly on I_k to the boundary function of f_k.*

REFERENCES

1. M. B. Abrahamse, *Toeplitz operators in multiply connected regions*, Doctoral Dissertation, University of Michigan, 1971.
2. C. Carathéodory, *Theory of functions of a complex variable*, Vol. II (Chelsea, New York, 1954).
3. A. M. Davie, T. W. Gamelin, and J. Garnett, *Distance estimates and pointwise bounded density* (to appear).
4. J. L. Doob, *The boundary values of analytic functions*. II, Trans. Amer. Math. Soc. *35* (1933), 418–451.
5. R. G. Douglas, *On the spectrum of Toeplitz and Wiener-Hopf operators*, Proc. Conference on Abstract Spaces and Approximation (Oberwohlfach, 1968), I.S.N.M., vol. 10 (Birkhäuser Verlag, Basel, 1969), pp. 53–66.
6. ——— *Banach algebra techniques in operator theory* (Academic Press, New York and London, 1972).
7. P. L. Duren, *Theory of H^p spaces* (Academic Press, New York and London, 1970).
8. H. Helson and D. Sarason, *Past and future*, Math. Scand. *21* (1967), 5–16.
9. K. Hoffman, *Banach spaces of analytic functions* (Prentice-Hall, Englewood Cliffs, N.J., 1962).

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