AN UNPLEASANT SET IN A NON-LOCALLY-CONVEX VECTOR LATTICE

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1. In a linear topological space E one often carries out various " smoothing " operations on a subset A, such as taking the *convex hull* co A and the *closure* A^-. If E is also a (real) vector lattice, the *solid hull*

$$\text{sol } A = \{y \in E: |y| \leqq |x| \text{ for some } x \in A\}$$

is also a natural " smoothing out " of A. If sol $A = A$ then A is called *solid,* and if E has a base of solid neighbourhoods of 0 as do all the common topological vector lattices such as $C(X)$, L_p, Köthe spaces and so on—then E is called a *locally solid space.*

The example here constructed shows that these smoothing operations act much more pathologically on bounded, and indeed on compact, sets in non-locally-convex spaces than they do in locally-convex ones. Several examples of this kind are known: let us describe briefly some of the well-known results to put our example into perspective.

In any linear topological space, closure preserves convexity (that is, if A is convex so is A^-) and it is this that makes the *closed convex hull* $\overline{\text{co}}\, A$ of a set A, defined as the smallest closed convex set containing A, coincide with the closure of the convex hull of A. In a locally solid space it follows from the basic decomposition lemma for vector lattices that solidity is preserved both by convex hulls and by closure: hence, similarly, the *closed convex solid hull* of a set A, that is the smallest closed convex solid set containing A, coincides with (co (sol A))$^-$ and is therefore denoted $\overline{\text{co}}$ sol A. Similarly with the *closed solid hull* $\overline{\text{sol}}\, A$ of A. Note that $\overline{\text{sol}}$ (co A) is not generally the same as $\overline{\text{co}}$ sol A, the latter being the larger set.

It is almost trivial that in a locally convex space both closure and convex hull preserve boundedness. In a locally solid space, solid hull also preserves boundedness, and so in a locally convex *and* locally solid topological vector lattice, if A is bounded so also is $\overline{\text{co}}$ sol A.

On the other hand, if E is not locally convex then boundedness need not be preserved by convex hulls—there are well-known examples of a compact set C such that co C is unbounded and indeed $\overline{\text{sol}}$ (co C) is the whole space.

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To summarise the situation in a general locally solid space:

Operation	Property preserved			
	closed	convex	solid	bounded
closure	✓	✓	✓	✓
convex hull	×	✓	✓	× (✓ if loc. convex)
solid hull	×	×	✓	✓

Below, we produce a set A in a locally solid space E such that $\overline{\text{co}}\, A$ is bounded —indeed compact—but $\overline{\text{co}}$ sol A is all of E. In other words, A behaves itself with regard to convex hulls whereas the (in view of local solidity seemingly not much larger) set sol A misbehaves badly. Further, $\overline{\text{co}}\, A$ provides an example of a bounded and even compact convex set C such that $\overline{\text{co}}$ sol C is all of E, and thus answers the following question raised by D. H. Fremlin:

It has been known for some time (**1**) that if E and F are L_1 spaces, any continuous linear map $T: E \to F$ can be decomposed into continuous, linear, positive and negative parts. This may fairly easily be generalised (**2**; p. 21, p. 86) to the case where F is one of a large class of order-complete vector lattices with locally convex, locally solid topologies, which includes all L_p spaces ($1 \leqq p \leqq \infty$), all Köthe spaces, etc.

Fremlin's question is: Is the local convexity of F necessary? The answer is Yes, for the map $T: l_1 \to \mathcal{M}$ constructed in the proof of Lemma 2.4 satisfies the required conditions except for local convexity of the range space, but is easily seen not to be decomposable in the desired way. By composing T with the operator S on l_1 defined by

$$Sx = y \text{ where } y_n = \beta_n^{-1} x_n$$

(in the notation of Theorem 2.3) one can even construct a compact operator which misbehaves in this way.

2. Construction of example

The example is set in the space $\mathcal{M} = \mathcal{M}(S) = \mathcal{M}(S, \underset{\sim}{S}, \mu)$ of measurable real functions (= *random variables*) on a measure space $S = (S, \underset{\sim}{S}, \mu)$ with the topology of convergence in measure; as usual, functions which are equal almost everywhere are counted identical. It is well known (**6**, p. 54) that $\mathcal{M}$ is a complete metrizable linear topological space, non-locally-convex if μ is non-atomic. The natural ideas here are those of probability theory, and we refer the reader to the excellent books of Breiman, or Feller, or Kingman and Taylor (**3**, **4**, **5**) for the definitions and properties of the terms used—chiefly of the *distribution* and *density function* of an $x \in \mathcal{M}$, of its *variance*

$$V(x) = \int (x - \int x)^2 = \int x^2 - (\int x)^2,$$

and of *independence* of random variables.

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Note 2.1. To start, we need to produce a sequence of independent random variables ϕ_k each having the uniform distribution over $0 \leqq x \leqq 1$. A simple way of doing this is to let $S = [0, 1]^{\mathbf{N}}$ with the product of Lebesgue measure on each factor, and to let ϕ_k be the kth coordinate projection. But since it is pleasant to work with the simpler measure space $S = (0, 1)$ with Lebesgue measure, we note that standard results [e.g. (**3**; p. 16, p. 37)] show that the functions

$$\phi_1(t) = 0 \cdot t_1 t_3\ t_5\ \ldots$$
$$\phi_2(t) = 0 \cdot t_2 t_6\ t_{10} \ldots$$
$$\phi_3(t) = 0 \cdot t_4 t_{12} t_{20} \ldots$$
$$\ldots$$

have the required property, where $0 \cdot t_1 t_2 t_3 \ldots$ is the binary expansion of $t \in (0, 1)$. (In ambiguous cases choose the expansion that ends in zeros.)

Recall that a function x has the *Cauchy distribution* (**4**, p. 50; **5**, pp. 310, 349) if its distribution has the density function $(\pi(1+x^2))^{-1}$. Let us say x *has the* $C(\lambda)$ *distribution*, where $\lambda > 0$, if $\lambda^{-1}x$ has the Cauchy distribution.

A simple computation verifies:

Lemma 2.2. *With the ϕ_k as above, the functions x_k defined by*

$$x_k(t) = -\cot \phi_k(t), \quad (t \in S)$$

are independent random variables with the $C(1)$ (*i.e. Cauchy*) *distribution.*

The main result of the paper is:

Theorem 2.3. *Let* $\beta_n = \log \log (n+2)$, $n = 1, 2 \ldots$ (*the* 2 *just ensures that* $\beta_n > 0$ *for all* n) *and define a subset of* $\mathcal{M}$ *by*

$$A = \{\beta_n^{-1} x_n : n = 1, 2, \ldots\}.$$

Then $\overline{\text{co}}\ A$ *is compact but* $\overline{\text{co}}$ sol A *is all of* $\mathcal{M}$.

The proof is contained in the following lemmas.

Lemma 2.4. *For any real sequence* $\alpha_n \to 0$, $\overline{\text{co}}\{\alpha_n x_n : n \in \mathbf{N}\}$ *is compact in* $\mathcal{M}$.

Proof. We shall construct a compact convex set of which the set in question is the continuous linear image. Let l_0 denote the space of real sequences with only finitely many non-zero terms, with the l_1 norm. Denote the nth coordinate of $u \in l_0$ by $u(n)$; let e_k denote the kth basis vector, with 1 in the kth place, 0 elsewhere. By a standard result (**4**, p. 50), if functions x_n are independent and distributed $C(1)$ then any finite linear combination $\sum_1^k \lambda_n x_n$ is distributed $C\left(\sum_1^k |\lambda_n|\right)$. Thus if we define the linear map $T: l_0 \to \mathcal{M}$ by

$$Tu = \sum_n u(n) x_n$$

then Tu is distributed $C(\| u \|)$ whenever $u \neq 0$. It follows trivially that for any sequence $\{u_n\}$ in l_0 with $\| u_n \| \to 0$ one has $Tu_n \to 0$ in measure; hence T is continuous. Since $\mathcal{M}$ is complete we can thus extend T uniquely by continuity to a continuous map from l_1 to $\mathcal{M}$. Now $\{\alpha_n x_n : n \in \mathbf{N}\}$ is the image of the set $S = \{\alpha_n e_n : n \in \mathbf{N}\}$; the latter forms a sequence converging to 0, so that $\overline{\text{co}}\, S$ is compact in l_1. Since $\overline{\text{co}}\, \{\alpha_n x_n : n \in \mathbf{N}\}$ is clearly contained in $T(\overline{\text{co}}\, S)$ the result follows.

Lemma 2.5. *Let A be as in the theorem, and let $m > 0$. Then there is a sequence of convex combinations of elements of* sol A *converging in measure to the constant function $m1$.*

Proof. Define $F_k : \mathbf{R} \to \mathbf{R}$ for each $k > 0$ by

$$F_k(s) = \begin{cases} | s | & (| s | \leqq k) \\ 0 & \text{otherwise.} \end{cases}$$

With β_n as in the theorem, define $k_n > 0$ such that $\log(1 + k_n^2) = \pi m \beta_n$, that is

$$k_n = (e^{\pi m \beta_n} - 1)^{\frac{1}{2}};$$

and define the function $y_n \in \mathcal{M}$ by

$$y_n = \beta_n^{-1} F_{k_n} \circ x_n.$$

Clearly $0 \leqq y_n \leqq \beta_n^{-1} | x_n |$ so that $y_n \in \text{sol}\, A$; and since the x_n are independent, so are the y_n. Each y_n is bounded, and since x_n has the density function $1/(\pi(1+s^2))$ one has

$$\int y_n = \int_{-\infty}^{\infty} \beta_n^{-1} F_{k_n}(s) ds/(\pi(1+s^2))$$

$$= 2 \int_0^{k_n} \beta_n^{-1} s ds/(\pi(1+s^2)) = \log(1 + k_n^2)/(\pi \beta_n)$$

$$= m, \text{ independent of } n.$$

Also, in the same way,

$$V(y_n) \leqq \int y_n^2 = 2 \int_0^{k_n} (\beta_n^{-1} s)^2 ds/(\pi(1+s^2))$$

$$= \frac{2}{\pi \beta_n^2} \int_0^{k_n} s^2 ds/(1+s^2) \leqq 2k_n/(\pi \beta_n^2).$$

Now define convex combinations z_n of the y_n by

$$z_n = \left(\sum_1^n \frac{1}{r} y_r \right) \Big/ \left(\sum_1^n \frac{1}{r} \right).$$

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Then since $\int y_r = m$ for each r we have $\int z_n = m$, and the standard computation rules for variances give

$$\int (z_n - m)^2 = V(z_n)$$
$$= \left(\sum_1^n \frac{1}{r^2} V(y_r)\right) \Big/ \left(\sum_1^n \frac{1}{r}\right)^2$$
$$\leq \left(\sum_{r=1}^n \frac{2k_r}{\pi \beta_r^2 r^2}\right) \Big/ \left(\sum_1^n \frac{1}{r}\right)^2. \qquad (*)$$

But

$$\sum_{r=1}^\infty \frac{k_r}{\beta_r^2 r^2} \leq \sum_{r=1}^\infty \frac{e^{\pi m \beta_r/2}}{\beta_r^2 r^2}$$
$$= \sum_{r=1}^\infty \frac{(\log (r+2))^{\pi m/2}}{(\log \log (r+2))^2 r^2}$$

which is easily seen to converge. Hence the numerator of (*) is bounded, while the denominator tends to ∞, and therefore $\int (z_n - m)^2 \to 0$ as $n \to \infty$. Thus $z_n \to m1$, in the L_2 norm and therefore in measure, as $n \to \infty$, and the result is proved.

Theorem 2.3 follows at once from this. For $\overline{\text{co}}\, A$ is compact by Lemma 2.4, while Lemma 2.5 shows that $\overline{\text{co}}$ sol A contains the positive constant functions. Since $\overline{\text{co}}$ sol A is solid it therefore contains all bounded functions in $\mathcal{M}$; these functions are clearly dense in $\mathcal{M}$, and therefore $\overline{\text{co}}$ sol $A = \mathcal{M}$.

The reason for the original choice of the sequence $\{\beta_n\}$ was in order to ensure that $\beta_n \to \infty$ but $\sum_r \exp(M\beta_r)/r^2$ converges for each $M > 0$. If one is satisfied with a set A such that $\overline{\text{co}}\, A$ is bounded (rather than compact) while $\overline{\text{co}}$ sol A is all of $\mathcal{M}$, one can take $\beta_n = 1$ and avoid the rather delicate estimates involved in the above proof: the details are left to the reader.

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