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ASYMPTOTICS OF TWO INTEGRALS FROM OPTIMIZATION THEORY AND GEOMETRIC PROBABILITY

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Abstract

Asymptotic series are derived for two integrals using a Gaussian identity and Laplace's method, demonstrating an improvement over earlier methods.

LAPLACE'S METHOD; OPTIMIZATION

Anderssen et al. (1976) obtain various bounds and approximations for the expected distance

$$m_k = \int_0^1 \cdots \int_0^1 (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}\, dx_1 \cdots dx_k \tag{1}$$

from the origin of a point uniformly distributed in the cube $[0, 1]^k$. They evaluate m_1, m_2 and m_3 exactly. Otherwise their computationally most efficient formula, by far, is the asymptotic series

$$m_k = (k/3)^{\frac{1}{2}}(1 - 1/10k - 13/280k^2 - 101/2800k^3 - 37533/1232000k^4) + O(k^{-\frac{9}{2}}) \tag{2}$$

as $k \to \infty$. Terms up to k^{-3} give, for example, m_4 accurate to five figures, m_{10} accurate to six figures and m_{20} accurate to seven figures. Their derivation of (2) is, however, cumbersome. We give a simple derivation based on Laplace's method.

The authors also study the expected interpoint distances

$$M_k = \int_0^1 \cdots \int_0^1 \{x_1 - y_1)^2 + \cdots + (x_k - y_k)^2\}^{\frac{1}{2}}\, dx_1\, dy_1 \cdots dx_k\, dy_k, \tag{3}$$

but do not give an asymptotic series like (2), presumably because of the work required using their method. We give a simple derivation of such a series, again using Laplace's method.

Since

$$\lambda^{\frac{1}{2}} = (2/\pi)^{\frac{1}{2}}\lambda \int_0^\infty ds \exp(-\tfrac{1}{2}\lambda s^2), \tag{4}$$

we can write

$$m_k = (2/\pi)^{\frac{1}{2}} k \int_0^\infty f'(-\tfrac{1}{2}s^2) f(-\tfrac{1}{2}s^2)^{k-1}\, ds, \tag{5}$$

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where

$$f(t)=\int_0^1 \exp(tx^2)\,dx. \tag{6}$$

Since $f(t)$ has a maximum at $t=0$, and $f'(0)=\frac{1}{3}$, we write

$$\begin{aligned} f(t)&=\exp(t/3)\int_0^1 \{1+t(x^2-\tfrac{1}{3})+\tfrac{1}{2}t^2(x^2-\tfrac{1}{3})^2+\cdots\}\,dx \\ &=\exp(t/3)(1+2t^2/45+\cdots). \end{aligned} \tag{7}$$

Similarly

$$f'(t)=\exp(t/3)(\tfrac{1}{3}+4t/45+\cdots). \tag{8}$$

Finally

$$\begin{aligned} m_k&=(2/\pi)^{\frac{1}{2}}k\int_0^\infty \exp(-ks^2/6)(\tfrac{1}{3}-\tfrac{2}{45}s^2+\tfrac{1}{90}ks^4+\cdots)\,ds \\ &=\left(\frac{k}{3}\right)^{\frac{1}{2}}(1-1/10k+\cdots). \end{aligned} \tag{9}$$

Turning now to (3) we have, similarly,

$$M_k=(2/\pi)^{\frac{1}{2}}k\int_0^\infty g'(-\tfrac{1}{2}s^2)g(-\tfrac{1}{2}s^2)^{k-1}\,ds, \tag{10}$$

where

$$g(t)=\int_0^1\int_0^1 \exp(t(x-y)^2)\,dx\,dy, \tag{11}$$

which has a maximum at $t=0$, where $g'=\frac{1}{6}$. Thus we write

$$g(t)=\exp(t/6)\sum_{n=0}^\infty t^nI_n. \tag{12}$$

where

$$I_n=\frac{1}{n!}\int_0^1\int_0^1 \{(x-y)^2-\tfrac{1}{6}\}^n\,dx\,dy,$$

and

$$g'(t)=\exp(t/6)\sum_{n=0}^\infty t^nJ_n, \tag{13}$$

where

$$J_n=I_n/6+(n+1)I_{n+1}.$$

Then $I_0=1$, $I_1=0$, $I_2=7/360$, $I_3=11/5670$, $J_0=1/6$, $J_1=7/180$ and $J_2=137/15120$.

Now putting (12) and (13) in (10) and using standard formulae for moments of a normal density gives

$$M_k=(k/6)^{\frac{1}{2}}(1-7/40k-65/896k^2+\cdots). \tag{14}$$

Anderssen et al. (1976) compute M_1, M_2 exactly and $M_3,\cdots,M_{10}$ by a slowly convergent series method. They also obtain an upper bound

(15) $$M_k \leqq (k/6)^{\frac{1}{2}}[\{1+2(1-3/5k)^{\frac{1}{2}}\}/3]^{\frac{1}{2}}.$$

The table lists the $M_1, \cdots, M_{10}$ from Anderssen et al. and their deviations from (14) (as shown) and (15) denoted $(14)-M_k$ and $(15)-M_k$ respectively. This illustrates the accuracy of (14), for k not too small, while its efficiency is obvious.

k	M_k	$(14)-M_k$	$(15)-M_k$
1	0·33333	−0·026	0·021
2	0·52141	−0·005	0·024
3	0·66167	−0·001	0·020
4	0·77766	−0·0006	0·017
5	0·87853	−0·0003	0·015
6	0·96895	−0·0001	0·014
7	1·05159	−0·00007	0·013
8	1·12817	−0·00004	0·012
9	1·19985	−0·00002	0·011
10	1·26748	−0·00001	0·010

References

Anderssen, R. S., Brent, R. P., Daley, D. J. and Moran, P. A. P. (1976) Concerning $\int_0^1 \cdots \int_0^1 (x_1^2+\cdots+x_k^2)^{\frac{1}{2}}\,dx_1 \cdots dx_k$ and a Taylor series method. *SIAM J. Appl. Math.* **30,** 22–30.