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q-ANALOGS OF SOME BIORTHOGONAL FUNCTIONS

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ABSTRACT. In this note we obtain a q-analog of a pair of biorthogonal sets of rational functions which have been obtained recently by M. Rahman in connection with the addition theorem for the Hahn polynomials.

1. **Introduction.** Recently Rahman, in trying to find product formulae and addition theorem for the Hahn polynomials, discovered a new family of biorthogonal rational functions [3]. Since the q-Hahn polynomials [2] have been of interest recently it would be of equal interest to find q-analogs of Rahman's biorthogonal system, which, when $q \to 1$, reduce to those of Rahman. In §2 we present q-analogs of Rahman's $R_n^{(1)}(x)$ and $S_n^{(1)}(x)$ biorthogonal functions.

For notation we shall adopt the following

$$(a;q)_0 = 1, \qquad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) \quad \text{for} \quad n \geq 1.$$

However we shall use $[a]_n$ to mean $(a;q)_n$ and use $(a;q)_n$ only when we wish to indicate the base q explicitly. $[a]_\infty = (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$.

Basic hypergeometric series are defined by

$${}_{p+1}\phi_p\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_{p+1}; q, z\\ \beta_1, \beta_2, \ldots, \beta_p\end{matrix}\right] = \sum_{k=0}^{\infty} \frac{[\alpha_1]_k[\alpha_2]_k\cdots[\alpha_{p+1}]_k}{[q]_k[\beta_1]_k\cdots[\beta_p]_k} z^k.$$

We shall also use the bilateral q-integral

$$\int_{-\infty}^{\infty} f(x)\, d_qx = (1-q) \sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)]q^n.$$

The q-derivative $D_qf(x)$ is defined by

$$D_qf(x) = \frac{f(x) - f(qx)}{x}.$$

2. In this note we prove that the functions

$$R_n(x;q) = {}_3\phi_2\left[\begin{matrix}b, ac/d, q^{-n}; q, q\\ bc/q, aqx\end{matrix}\right] \tag{2.1}$$

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and

$$S_n(x;q) = {}_3\phi_2\left[\begin{matrix} c, bd/a, q^{-n}; q, q \\ bc/q, dqx \end{matrix}\right] \tag{2.2}$$

are biorthogonal. In fact we prove that

$$I_{n,m} = \int_{-\infty}^{\infty} w(x) R_n(x;q) S_m(x;q) d_q x = A_n \delta_{nm} \tag{2.3}$$

where

$$w(x) = \frac{[axq]_\infty [dxq]_\infty}{[abxq]_\infty [cdxq]_\infty}, \qquad A_n = K \frac{[q]_n}{[bc/q]_n} (bc/q)^n,$$

$$K = \frac{2[-adq^2/bc]_\infty [bc/adq]_\infty \{(q^2;q^2)_\infty\}^2 [b]_\infty [c]_\infty [ac/d]_\infty [bd/a]_\infty}{[q^2]_\infty (a^2q^2/b^2;q^2)_\infty (b^2/a^2;q^2)_\infty (d^2q^2/c^2;q^2)_\infty (c^2/d^2;q^2)_\infty [bc/q]_\infty}.$$

Proof of (2.3). Substituting for $R_n(x;q)$ and $S_m(x;q)$ from (2.1) and (2.2), changing the order of summation, we have

$$I_{n,m} = \sum_{j=0}^{n} \sum_{k=0}^{m} \frac{[b]_j [ac/d]_j [q^{-n}]_j [c]_j [bd/a]_k [q^{-m}]_k}{[q]_j [bc/q]_j [q]_k [bc/q]_k} q^{j+k} \times \int_{-\infty}^{\infty} \frac{[axq^{1+j}]_\infty [dxq^{1+k}]_\infty}{[axq/b]_\infty [dxq/c]_\infty} d_q x.$$

Evaluating the inner q-integral using the following result of Askey [1; 3.12] (which also follows from [5; (5)] on setting $c = -(\beta/\alpha)q$, $e = -\beta q$, $a = -(\beta/\gamma)q$, $b = (\beta/\gamma)q$, $f = \beta q$):

$$\int_{-\infty}^{\infty} \frac{[atq^x]_\infty [-btq^y]_\infty}{[at]_\infty [-bt]_\infty} d_q t = \frac{2\Gamma q^{(x+y-1)}}{\Gamma_q(x)\Gamma_q(y)} \times \frac{\left[-\frac{a}{b} q^x\right]_\infty \left[-\frac{b}{a} q^y\right]_\infty [ab]_\infty \left[\frac{q}{ab}\right]_\infty \{(q^2;q^2)_\infty\}^2}{(a^2;q^2)_\infty (q^2/a^2;q^2)_\infty (b^2;q^2)_\infty (q^2/b^2;q^2)_\infty}, \tag{2.4}$$

we get

$$I_{nm,} = K \sum_{j=0}^{n} \frac{[q^{-n}]_j}{[q]_j} q^j {}_2\phi_1\left[\begin{matrix} bcq^{-1+j}, q^{-m}; q, q \\ bc/q \end{matrix}\right].$$

Summing the inner ${}_2\phi_1[q]$ by the q-Vandermonde theorem, we get (2.3). Formula (2.3) is a q-analog of a result of Rahman [3]. Following Rahman [3] one can prove the following Rodrigue's type representations for $R_n(x;q)$ and $S_n(x;q)$

$$D_q^n \left\{ \frac{[aqx]_\infty [dx]_\infty}{[axq/b]_\infty [dxq/c]_\infty} \right\} = (dq/bc)^n (1-dx) \frac{[bc/q]_n w(x) R_n(x;q)}{[dx]_n}, \tag{2.5}$$

$$D_q^n \left\{ \frac{[axq]_\infty [dqx]_\infty}{[aqx/b]_\infty [dxq/c]_\infty} \right\} = (aq/bc)^n (1-ax) \frac{[bc/q]_n w(x) S_n(x;q)}{[ax]_n}. \tag{2.6}$$

Indeed if we calculate the left hand side of these formulae and apply the transformation

$$ {}_3\phi_2\left[\begin{matrix} a, b, q^{-n}; q, q \\ e, g \end{matrix}\right] = \frac{[eg/ab]_n}{[g]_n}(ab/e)^n \, {}_3\phi_2\left[\begin{matrix} e/a, e/b, q^{-n}; q, q \\ e, eg/ab \end{matrix}\right], $$

(which is obtained from [4, (8.3)] by letting $c \to 0$), we get the right hand of the formulae (2.5) and (2.6). Then the biorthogonality relation (2.3) can be proved by successive summation by parts.

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