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L^p SPACES FROM MATRIX MEASURES

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It is known that a Hilbert space, $L^2(\mu_{ij})$, can be constructed from an $n\times n$ positive matrix measure (μ_{ij}), [5, pp. 1337–1346]. The aim of this note is to show that Banach spaces, corresponding to the usual L^p spaces, can also be constructed and to investigate their properties.

The following notation will be used. If $M=(m_{ij})$ is an $n\times n$ positive semidefinite Hermitian matrix and $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\beta=(\beta_1,\ldots,\beta_n)$ are n-tuples of complex numbers, we shall write the summation $\sum_{i,j=1}^n m_{ij}\alpha_i\bar{\beta}_j$ as $\beta^*M\alpha$.

1. The spaces $L^p(\mu_{ij})$.

DEFINITION. Let (μ_{ij}), $1\le i,j\le n$, be an $n\times n$ positive matrix measure defined on the bounded Borel sets of the real line and let ν be a non-negative regular σ-finite Borel measure with respect to which each μ_{ij} is absolutely continuous. Let the matrix of densities $M=(m_{ij})$ be defined by the equations

$$\mu_{ij}(S)=\int_S m_{ij}(t)\,d\nu(t),\qquad 1\le i,j\le n,$$

where S is any bounded Borel set. For $1\le p<\infty$ the space $L_o^p(\mu_{ij})$ is defined to be the space of all n-tuples of Borel functions $F(t)=(F_1(t),\ldots,F_n(t))$ such that

$$\|F\|=\left[\int_{-\infty}^{\infty}[F^*(t)M(t)F(t)]^{p/2}\,d\nu(t)\right]^{1/p}<\infty.$$

Note that the matrix $M(t)$ is positive semi-definite for ν-almost all t [5, Lemma 7, p. 1338], so that the above integral is non-negative.

It is easily shown that if α is a complex number and $F, G\in L_o^p(\mu_{ij})$, then $\|\alpha F\|=|\alpha|\,\|F\|$ and $\|F+G\|\le\|F\|+\|G\|$. If D denotes the subspace of $L_o^p(\mu_{ij})$ consisting of those F with $\|F\|=0$, we define $L^p(\mu_{ij})$ to be the quotient space $L_o^p(\mu_{ij})/D$.

The space $L_o^\infty(\mu_{ij})$ is defined to be the space of all n-tuples of Borel functions $F(t)=(F_1(t),\ldots,F_n(t))$ such that

$$\|F\|=\nu\text{-ess}\sup[F^*(t)M(t)F(t)]^{1/2}<\infty.$$

Again, it is easily shown that if α is a complex number and $F, G\in L_o^\infty(\mu_{ij})$ then $\|\alpha F\|=|\alpha|\,\|F\|$ and $\|F+G\|\le\|F\|+\|G\|$. If D denotes the subspace of $L_o^\infty(\mu_{ij})$ consisting of those F for which $\|F\|=0$ we define $L^\infty(\mu_{ij})$ to be the quotient space $L_o^\infty(\mu_{ij})/D$.

THEOREM 1. *The spaces $L^p(\mu_{ij})$, $1\le p\le\infty$, are independent of the measure ν used to define them.*

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Proof. We shall deal firstly with the case $1<p<\infty$. Let q be such that $1/p+1/q=1$. For $F\in L^p(\mu_{ij})$, $G\in L^q(\mu_{ij})$ we have by the Schwarz and Hölder inequalities

$$\left|\int_{-\infty}^{\infty} G^*(t)M(t)F(t)\,d\nu(t)\right|$$
$$\le \int_{-\infty}^{\infty} |G^*(t)M(t)F(t)|\,d\nu(t)$$
$$\le \int_{-\infty}^{\infty} [F^*(t)M(t)F(t)]^{1/2}[G^*(t)M(t)G(t)]^{1/2}\,d\nu(t)$$
$$\le \left[\int_{-\infty}^{\infty}[F^*(t)M(t)F(t)]^{p/2}\,d\nu(t)\right]^{1/p}\left[\int_{-\infty}^{\infty}[G^*(t)M(t)G(t)]^{q/2}\,d\nu(t)\right]^{1/q},$$

showing that $\int_{-\infty}^{\infty} G^*(t)M(t)F(t)\,d\nu(t)$ converges absolutely. We now show that this integral is independent of the measure ν.

Let $\tilde{\nu}$ be another σ-finite Borel measure with respect to which each μ_{ij} is absolutely continuous. Let $\tilde{M}=(\tilde{m}_{ij})$ be the corresponding matrix of densities and $N=(n_{ij})$ the matrix of densities of the μ_{ij} with respect to the measure $\nu+\tilde{\nu}$. If m is the density of ν with respect to $\nu+\tilde{\nu}$, then $mM=N$ for $(\nu+\tilde{\nu})$-almost all t. Given Borel functions $F_i(t)$, $G_i(t)$, $1\le i\le n$, we have

$$\int_{-\infty}^{\infty} G^*(t)M(t)F(t)\,d\nu(t) = \int_{-\infty}^{\infty} G^*(t)M(t)F(t)m(t)\,d(\nu+\tilde{\nu})(t)$$
$$= \int_{-\infty}^{\infty} G^*(t)N(t)F(t)\,d(\nu+\tilde{\nu})(t).$$

By a similar argument we obtain an analogous formula in which ν and M are replaced by $\tilde{\nu}$ and $\tilde{M}$ on the left hand side. Thus

$$\int_{-\infty}^{\infty} G^*(t)M(t)F(t)\,d\nu(t) = \int_{-\infty}^{\infty} G^*(t)\tilde{M}(t)F(t)\,d\tilde{\nu}(t).$$

Now given $F\in L^p(\mu_{ij})$, define $G=(G_1,\ldots,G_n)$ by

$$G_i(t) = [F^*(t)M(t)F(t)]^{(p-2)/2}F_i(t), \qquad i=1,2,\ldots,n.$$

(If $p<2$ and $F^*(t)M(t)F(t)=0$, set $G_i(t)=0$.) Then $G\in L^q(\mu_{ij})$ since it is readily shown that

$$\int_{-\infty}^{\infty}[G^*(t)M(t)G(t)]^{q/2}\,d\nu(t) = \int_{-\infty}^{\infty}[F^*(t)M(t)F(t)]^{p/2}\,d\nu(t).$$

Further we see that

$$\int_{-\infty}^{\infty} G^*(t)M(t)F(t)\,d\nu(t) = \int_{-\infty}^{\infty}[F^*(t)M(t)F(t)]^{p/2}\,d\nu(t)$$
$$= \|F\|^p,$$

which by our earlier argument, is independent of ν.

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A similar argument can be constructed for the case $p=1$ with $L^q(\mu_{ij})$ being replaced throughout by $L^\infty(\mu_{ij})$. We shall prove later that $L^\infty(\mu_{ij})$ is the continuous dual of $L^1(\mu_{ij})$. This result will yield the fact that $L^\infty(\mu_{ij})$ is also independent of the measure ν used to define it.

2. **Structure of the spaces** $L^p(\mu_{ij})$. We shall prove that the spaces $L^p(\mu_{ij})$ are Banach spaces. For this purpose we need the following lemmas, the first of which is taken from [5, p. 1341].

LEMMA 1. *Let (μ_{ij}) be an $n\times n$ positive matrix measure whose elements are continuous with respect to a regular σ-finite measure ν. If (m_{ij}) is the matrix of densities of μ_{ij} with respect to ν, then there exist non-negative ν-measurable functions ϕ_i, $1\le i\le n$, ν-integrable over each bounded interval, and ν-measurable functions a_{ij}, $1\le i,j\le n$, such that for ν-almost all t*

(a) $\sum_{j=1}^{n} a_{ij}(t)\overline{a_{kj}(t)} = \delta_{ik}$

and

(b) $\sum_{j=1}^{n} \phi_j(t)a_{ji}(t)\overline{a_{jk}(t)} = m_{ik}(t).$

[Note that we have corrected the misprint in equation (a)].

Let $E=\{t\in R \mid \phi_i(t)=0,\ i=1,2,\dots,n\}$. We note that E is "null" in the sense that each m_{ij} vanishes over E (see Lemma 1(b)). For $t\in R-E$ we define $I(t)=\{i \mid \phi_i(t)\ne 0\}\subset\{1,2,\dots,n\}$ and $l(t)$ as the cardinality of $I(t)$. For $1\le i\le n$ we denote the set $\{1,2,\dots,i\}$ by J_i. A map $\sigma:J_i\to J_j$ is said to be an (i,j) combination if it is one to one and monotonic increasing (necessarily then $i\le j$). There is clearly a unique (i,j) combination corresponding to each i-element subset of J_j. We denote $J_j-\sigma(J_i)$ by $\sim\sigma(J_i)$ and the $(l(t),n)$ combination corresponding to $I(t)\subset J_n$ by $\pi(t)$ so that $\pi(t)(i)\in I(t)$ for $1\le i\le l(t)$.

For $t\notin E$ we now define functions $b_{ij}(t)$ $1\le i,j\le n$ by $b_{ij}(t)=1$ if $j=\pi(t)(i)$, 0 otherwise. Finally we let $S_d=l^{-1}(d)=\{t \mid l(t)=d\}$, $1\le d\le n$.

LEMMA 2. *b_{ij}, $1\le i,j\le n$ are ν-measurable functions and S_d, $1\le d\le n$ are ν-measurable sets.*

Proof. b_{ij} takes only two values, viz. 0 and 1, so to prove measurability of b_{ij} it suffices to prove that $b_{ij}^{-1}(1)$ is a measurable set.

Let σ be an $(i-1,j-1)$ combination and consider the sets $Z_i=\phi_i^{-1}(0)$, $N_i=\{t \mid \phi_i(t)\ne 0\}$. For each i, Z_i and N_i are complementary in R and ν-measurable. We must consider the cases $i=1$ and/or $j=1$ separately. Note that $b_{ij}(t)=1$ implies $\pi(t)(i)=j$ and thus $i\le j$ so that we need only show that each $b_{1j}^{-1}(1)$ is ν-measurable.

First $b_{11}^{-1}(1)=N_1$ which is ν-measurable and for $j>1$

$$\begin{aligned} b_{1j}^{-1}(1) &= \{t \mid \pi(t)(1) = j\} \\ &= \{t \mid \phi_1(t) = \cdots = \phi_{j-1}(t) = 0, \phi_j(t) \neq 0\} \\ &= N_j \cap \bigcap_{k=1}^{j-1} Z_k \end{aligned}$$

which is ν-measurable. Returning to the cases $i>1, j>1$, we see that the set

$$Q_\sigma = \bigcap_{k=1}^{i-1} N_{\sigma(k)} \bigcap_a Z_a$$

where a ranges through $\sim\sigma(J_{i-1})$, is ν-measurable. Further it is easily seen that $t \in Q_\sigma$ if and only if $\phi_r(t)\neq 0$ for $r=\sigma(s)$, $1\leq s\leq i-1$. Let $Q=(\bigcup_\sigma Q_\sigma) \cap N_j$ where σ runs through all $\binom{j-1}{i-1}$ such combinations. Q is a ν-measurable set and $t \in Q$ if and only if the ith index k (in the natural order) for which $\phi_k(t)\neq 0$ is exactly j. On the other hand $Q=\{t \mid \pi(t)(i)=j\}=b_{ij}^{-1}(1)$ and hence b_{ij} is a ν-measurable function.

Finally $S_d=\bigcup_\sigma Q_\sigma$ over all (d, n) combinations σ and so is ν-measurable. This completes the proof.

We are now in a position to define related spaces $L_d^p(\mu_{ij})$ as per $L^p(\mu_{ij})$ on functions F restricted to S_d and with norm given by

$$\|F\|^p = \int_{S_d} [F^*(t)M(t)F(t)]^{p/2}\, d\nu(t). \tag{1}$$

The space $L_d^\infty(\mu_{ij})$ has norm

$$\|F\| = \nu\text{-ess}\sup_{t\in S_d}[F^*(t)M(t)F(t)]^{1/2}. \tag{2}$$

We also define $L^p(C^d)$ as the space of (equivalence classes of) complex-d-vector valued functions G on S_d normed by

$$\|G\| = \left(\int_{S_d}\left[\sum_{i=1}^{d} |G_i(t)|^2\right]^{p/2} d\nu(t)\right)^{1/p} \quad \text{for } 1 \leq p < \infty, \tag{3}$$

$$\|G\| = \nu\text{-ess}\sup_{t\in S_d}\left[\sum_{i=1}^{d} |G_i(t)|^2\right]^{1/2} \quad \text{for } p = \infty. \tag{4}$$

LEMMA 3. *$L_d^p(\mu_{ij})$ is isometrically isomorphic to $L^p(C^d)$ for $1\leq p\leq\infty$.*

Proof. For $F \in L_d^p(\mu_{ij})$ define

$$(TF)_k(t) = \sum_{i,j=1}^{n} b_{ki}(t)a_{ij}(t)F_j(t)\phi_i(t)^{1/2}, \qquad 1 \leq k \leq d.$$

Then TF is clearly a ν-measurable complex-d-vector valued function defined on

S_d. In case $1 \leq p < \infty$ we calculate $\|TF\|^p$ from (3) as

$$\int_{S_d} \left[\sum_{k=1}^{d} \sum_{i,j,r,s=1}^{n} b_{ki}(t) a_{ij}(t) F_j(t) \phi_i(t)^{1/2} b_{kr}(t) \overline{a_{rs}(t)} \overline{F_s(t)} \phi_r(t)^{1/2} \right]^{p/2} d\nu(t)$$
$$= \int_{S_d} \left[\sum_{i,j,s=1}^{n} a_{ij}(t) F_j(t) \phi_i(t)^{1/2} \overline{a_{is}(t)} \overline{F_s(t)} \phi_i(t)^{1/2} \right]^{p/2} d\nu(t)$$
$$= \int_{S_d} \left[\sum_{j,s=1}^{n} m_{js}(t) F_j(t) \overline{F_s(t)} \right]^{p/2} d\nu(t)$$
$$= \|F\|^p.$$

Note that here we have used the identity for $t \in S_d$:

$$\sum_{k=1}^{d} b_{ki}(t) b_{kr}(t) \phi_i(t)^{1/2} \phi_r(t)^{1/2} = \delta_{ir} \phi_i(t)^{1/2} \phi_r(t)^{1/2}$$

which can be readily verified pointwise by considering the cases

(i) $i = r = \pi(t)(k_0)$ for some $1 \leq k_0 \leq d$,
(ii) $i = r \neq \pi(t)(k)$ for any $1 \leq k \leq d$,
(iii) $i \neq r$.

When $p = \infty$, the calculation of $\|TF\|$ uses (4) to give

$$\nu\text{-ess}\sup_{t \in S_d} \left| \sum_{k=1}^{d} |(TF)_k(t)|^2 \right|^{1/2} = \nu\text{-ess}\sup_{t \in S_d} [F^*(t) M(t) F(t)]^{1/2} = \|F\|.$$

The details are as in the previous case.

Thus T is an isometry of $L_d^p(\mu_{ij})$ into $L^p(C^d)$ and is evidently linear. In order to show T is onto, let $G \in L^p(C^d)$ and define, for $t \in S_d$, $1 \leq i \leq n$,

$$F_i(t) = \sum_{u=1}^{d} \sum_{r \in I(t)} a_{ir}(t) b_{ur}(t) G_u(t) \phi_r(t)^{-1/2}.$$

F_i is again ν-measurable by Lemmas 1 and 2, and we claim $F \in L_d^p(\mu_{ij})$, $TF = G$.

With norm (1) for $p < \infty$ we have

$$\|F\|^p = \int_{S_d} [H(t)]^{p/2} \, d\nu(t)$$

where

$$H(t) = \sum_{u,v=1}^{d} \sum_{r,s \in I(t)} \sum_{i,j,k=1}^{n} \overline{a_{ri}(t)} b_{ur}(t) G_u(t) \phi_r(t)^{-1/2} \cdot \phi_k(t) a_{ki}(t) \overline{a_{kj}(t)} a_{sj}(t) b_{vs}(t) \overline{G_v(t)} \phi_s(t)^{-1/2},$$

$m_{ij}(t)$ being rewritten using Lemma 1(b). Applying Lemma 1(a) and simplifying,

$$H(t) = \sum_{u,v=1}^{d} \sum_{r \in I(t)} b_{ur}(t) G_u(t) b_{vr}(t) \overline{G_v(t)}$$
$$= \sum_{u=1}^{d} |G_u(t)|^2.$$

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Here we have used the identity for $t \in S_d$

$$\sum_{r\in I(t)} b_{ur}(t)b_{vr}(t) = \delta_{uv} \quad \text{for} \quad 1 \le u, v \le d,$$

which again is readily checked.

Thus $\|F\|^p = \|G\|^p < \infty$ and so $F \in L^p_d(\mu_{ij})$. Finally we calculate $(TF)_k(t)$ as

$$\sum_{i,j=1}^{n} b_{ki}(t)a_{ij}(t)\phi_i(t)^{1/2} \sum_{u=1}^{d} \sum_{r\in I(t)} \overline{a_{rj}(t)} b_{ur}(t)G_u(t)\phi_r(t)^{-1/2} = \sum_{u=1}^{d} \sum_{r\in I(t)} b_{kr}(t)b_{ur}(t)G_u(t)$$
$$= G_k(t)$$

using the above mentioned identity again.

This completes the proof for the case $1 \le p < \infty$; the case $p = \infty$ is similar.

Given n normed linear spaces $X_1, \ldots, X_n$, we define $l^p(X_i)$, $1 \le p < \infty$, to be the space $\sum_{i=1}^n \oplus X_i$ normed by

$$\|(F_1, \ldots, F_n)\| = \left(\sum_{i=1}^{n} \|F_i\|^p\right)^{1/p}$$

and $l^\infty(X_i)$ to be the space $\sum_{i=1}^n \oplus X_i$ normed by

$$\|(F_1, \ldots, F_n)\| = \sup_{1\le i\le n} \|F_i\|.$$

We are now in a position to state our main result.

THEOREM 2. *$L^p(\mu_{ij})$ is isometrically isomorphic to $l^p(L^p(C^d))$ for $1 \le p \le \infty$.*

Proof. The sets E, S_d, $d = 1, 2, \ldots, n$ form a partition of the real line. Further E being null as mentioned earlier we have $L^p(\mu_{ij}) \cong l^p(L^p_d(\mu_{ij}))$. Lemma 3 completes the proof.

COROLLARY 1. *$L^p(\mu_{ij})^*$, the continuous dual of $L^p(\mu_{ij})$, is isometrically isomorphic to $L^q(\mu_{ij})$ where, for $1 < p < \infty$, $1/p + 1/q = 1$ and for $p = 1$, $q = \infty$.*

Proof. A result of Dinculeanu [4, Corollary 1, p. 282] states that for a Banach space X with separable dual $L^p(X)^* \cong L^q(X^*)$. Since C^d normed by

$$\|C\| = \left|\sum_{i=1}^{d} |C_i|^2\right|^{1/2} \tag{5}$$

is a Hilbert space we have

$$L^p(\mu_{ij})^* \cong l^p(L^p(C^d))^* \cong l^q(L^p(C^d)^*) \cong l^q(L^q(C^d)) \cong L^q(\mu_{ij}).$$

Further applications of our theorem will produce results on uniform convexity and smoothness. Cudia [2] reviews a body of literature involving these and related properties in Banach spaces and their duals, but we give definitions for convenience.

DEFINITION. A normed space X is said to be *uniformly convex with modulus of convexity* $\delta(\varepsilon)$ if given $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|(x-y)/2\| > \varepsilon$ where $0 < \varepsilon < 1$ then $\|(x+y)/2\| \le 1 - \delta(\varepsilon)$.

X is said to be *uniformly smooth with modulus of smoothness* $\eta(\varepsilon)$ if $\|x\|=\|y\|=1$, $\|(x-y)/2\|<\varepsilon$, $0<\varepsilon<1$ imply $\|(x+y)/2\|\geq 1-\eta(\varepsilon)\|(x-y)/2\|$.

COROLLARY 2. *For* $1<p<\infty$, $L^p(\mu_{ij})$ *is uniformly convex with modulus of convexity* $\delta(\varepsilon)=1-(1-\varepsilon^r)^{1/r}$ *where* $r=p$ *if* $2\leq p<\infty$ *and* $r=q=p/(p-1)$ *if* $1<p<2$.

Proof. Clarkson [1, Theorem 2] has given the inequalities

$$\tag{6} \begin{aligned} &|(a+b)/2|^p+|(a-b)/2|^p \leq \tfrac{1}{2}(|a|^p+|b|^p), && 2\leq p<\infty \\ &|(a+b)/2|^p+|(a-b)/2|^p \leq 2^{1-p}(|a|^p+|b|^p), && 1\leq p<2 \end{aligned}$$

valid for $a, b \in C^1$, the complex plane. For $t\in S_d$ and $F, G\in L^p_d(\mu_{ij})$ there is an isometry between C^1 and the smallest two dimensional real subspace of C^d containing $TF(t)$, $TG(t)$, $TF(t)\pm TG(t)$. (The map T used here is the isometry between $L^p_d(\mu_{ij})$ and $L^p(C^d)$ described in Lemma 3.) Thus (6) may be rewritten replacing a and b by $TF(t)$ and $TG(t)$, and modulus by the C^d norm (5). Integrating this new inequality over S_d, and using the fact that T is an isometry we obtain

$$\|(F+G)/2\|^p+\|(F-G)/2\|^p \leq \tfrac{1}{2}(\|F\|^p+\|G\|^p), \qquad 2\leq p<\infty$$

with the corresponding expression for $1\leq p<2$. Such inequalities hold for each $d=1, 2, \ldots, n$. Summing over d gives the corresponding inequalities for F, $G\in L^p(\mu_{ij})$. With $\|F\|=\|G\|=1$ and $\|(F-G)/2\|>\varepsilon$, $(0<\varepsilon<1)$, the result follows by a simple calculation.

COROLLARY 3. *For* $1<p<\infty$, $L^p(\mu_{ij})$ *is uniformly smooth with modulus of smoothness* $\eta(\varepsilon)=[1-(1-\varepsilon)^r]^{1/r}$ *where r is as in Corollary* 2.

Proof. Day [3, Theorem 4.3] has shown (in our notation) that $\eta(\varepsilon)=\delta_*^{-1}(\varepsilon)$ is a modulus of smoothness for X if $\delta_*(\varepsilon)$ is a modulus of convexity for X^*. With $X=L^p(\mu_{ij})$, Corollary 1 gives $X^*\simeq L^q(\mu_{ij})$ and Corollary 2 gives $\delta_*(\varepsilon)=(1-\varepsilon^r)^{1/r}$. Calculating $\delta_*^{-1}(\varepsilon)$ gives the desired result.

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