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RESEARCH ARTICLE

Quaternionic hyperbolic lattices of minimal covolume

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Abstract
For any $n > 1$ we determine the uniform and nonuniform lattices of the smallest covolume in the Lie group $\mathrm{Sp}(n, 1)$. We explicitly describe them in terms of the ring of Hurwitz integers in the nonuniform case with n even, respectively, of the icosian ring in the uniform case for all $n > 1$.

1. Introduction

1.1. The problem

The purpose of this article is to determine the lattices in the Lie group $G = \mathrm{PSp}(n, 1)$ of minimal covolume for any integer $n > 1$. For other rank one real simple Lie groups (namely $G = \mathrm{PO}(n, 1)$, and $G = \mathrm{PU}(n, 1)$), this problem has been addressed in several different papers (see, for instance, [12, 14, 3, 11]). The result in the case $G = \mathrm{PGL}_2(\mathbb{R})$ is a classical theorem of Siegel [34]. Many of the results mentioned above are restricted to the class of arithmetic lattices: This allows the use of Prasad's volume formula [28] along with techniques from Borel–Prasad [4] as the main ingredient in the proof, and we shall adopt the same strategy in this paper. A significant advantage when treating the case $G = \mathrm{PSp}(n, 1)$ is that all lattices are arithmetic (since superrigidity holds) so that the results obtained below solve the problem in this Lie group.

It will be more convenient to work with lattices in the group $\mathrm{Sp}(n, 1)$, which is a double cover of $\mathrm{PSp}(n, 1) = \mathrm{Sp}(n, 1)/\{\pm I\}$. Let $\mathbb{H}$ denote the Hamiltonian quaternions. By definition, $\mathrm{Sp}(n, 1)$ is the unitary group $\mathbf{U}(V_{\mathbb{R}}, h)$ of $\mathbb{H}$-linear automorphisms of $V_{\mathbb{R}} = \mathbb{H}^{n+1}$ preserving the Hermitian form

$$h(x, y) = -\overline{x_0}y_0 + \overline{x_1}y_1 + \cdots + \overline{x_n}y_n. \tag{1.1}$$

In Section 1.5, we explain how our results translate back into the original problem in $\mathrm{PSp}(n, 1)$, and we discuss their geometric meaning in terms of quaternionic hyperbolic orbifolds.

We will use the Euler–Poincaré characteristic χ (defined in the sense of C.T.C. Wall) as a measure of the covolume: There exists a normalization μ^{EP} of the Haar measure on $\mathrm{Sp}(n, 1)$ such that $\mu^{\mathrm{EP}}(\Gamma\backslash \mathrm{Sp}(n, 1)) = \chi(\Gamma)$ for any lattice $\Gamma \subset \mathrm{Sp}(n, 1)$; see Section 4.2. The problem is then to find the lattices $\Gamma \subset \mathrm{Sp}(n, 1)$ with minimal value for $\chi(\Gamma)$. It is usual (and natural) to separate the problem into the subcases of Γ uniform (i.e., the quotient $\Gamma\backslash \mathrm{Sp}(n, 1)$ being compact), respectively, Γ nonuniform.

1.2. The nonuniform case

Denote by $\mathscr{H} \subset \mathbb{H}$ the ring of Hurwitz integers, which consists of elements of the form $\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \in \mathbb{H}$ with either all $\alpha_i \in \mathbb{Z}$ or all $\alpha_i \in \mathbb{Z} + \frac{1}{2}$. Let $\mathrm{Sp}(n, 1, \mathscr{H})$ be the subgroup $\mathbf{U}(L, h) \subset \mathbf{U}(V_{\mathbb{R}}, h)$ stabilizing the lattice $L = \mathscr{H}^{n+1} \subset V_{\mathbb{R}}$. In matrix notation, it corresponds to the set of elements of $\mathrm{Sp}(n, 1)$ with coefficients in $\mathscr{H}$, whence the notation. The group $\mathrm{Sp}(n, 1, \mathscr{H})$ is a nonuniform lattice of $\mathrm{Sp}(n, 1)$ (see Section 2.3). It is easily checked that it is normalized by the scalar matrix $g = I(1+i)/\sqrt{2}$, for which $g^2 \in \mathrm{Sp}(n, 1, \mathscr{H})$ holds. We denote by Γ_n^0 the subgroup extension of $\mathrm{Sp}(n, 1, \mathscr{H})$ by g. Thus, Γ_n^0 contains $\mathrm{Sp}(n, 1, \mathscr{H})$ as a subgroup of index 2. For $n = 2$, this lattice has been considered in [15, Prop. 5.8]. We will compute the following (see Corollary 5.8):

$$\chi(\Gamma_n^0) = \frac{(n+1)}{2} \prod_{j=1}^{n+1} \frac{2^j + (-1)^j}{4j} |B_{2j}|, \tag{1.2}$$

where B_m is the m-th Bernoulli number. For the reader's convenience, we list the first few values for $\chi(\Gamma_n^0)$ in Table 1. Note the few distinct prime factors (namely, $p = 2, 3$) appearing for $n = 2$; this compares with $\chi(\mathrm{SL}_2(\mathbb{Z})) = -1/12$.

Theorem 1. *For any n even, the lattice Γ_n^0 realizes the smallest covolume among nonuniform lattices in* $\mathrm{Sp}(n, 1)$. *Up to conjugacy, it is the unique lattice with this property.*

At this point, we would like to stress the relative simplicity of the description of the lattice Γ_n^0. In comparison, the results of [2, 3, 11] concerning $\mathrm{PO}(n, 1)$ and $\mathrm{PU}(n, 1)$ describe the minimal covolume lattices as normalizers of *principal* arithmetic subgroups, i.e., by using a local-to-global (adelic) description that heavily depends on Bruhat–Tits theory (see Section 4). A more concrete description in those cases is only available in low dimensions (in the form of Coxeter groups) or in a few special cases (see, for instance, [8, 9]). Another situation where satisfactory descriptions are available is the case of a *split* Lie group G (see, for instance, [35] and [18, 31] in the positive characteristic case).

The adelic description of arithmetic subgroups is the setting needed to apply Prasad's formula, and in this respect, the proof of Theorem 1 (and Theorem 3 below) follows the same strategy as in those previous articles. The improvement in the present case is stated in Theorem 5.7, where we have been able to express a large class of stabilizers of Hermitian lattices—including $\mathrm{Sp}(n, 1, \mathscr{H})$—as principal arithmetic subgroups, in particular permitting the computation of their covolumes. This makes these subgroups more tractable to geometric or algebraic investigation; for instance, the reflectivity of $\mathrm{Sp}(n, 1, \mathscr{H})$ has already been studied by Allcock in [1].

For n odd, there is a nonuniform lattice of covolume smaller than Γ_n^0:

Theorem 2. *Let $n > 1$ be odd. There exists a unique (up to conjugacy) nonuniform lattice $\Gamma_n^1 \subset \mathrm{Sp}(n, 1)$ of minimal covolume. It is commensurable with Γ_n^0, and*

$$\chi(\Gamma_n^1) = \frac{(n+1)}{2} \prod_{j=1}^{n+1} \frac{2^{2j} - 1}{4j} |B_{2j}| \prod_{j=1}^{\frac{n+1}{2}} \frac{1}{2^{4j} - 1}. \tag{1.3}$$

Table 1. *Some explicit values for $n \leq 5$.*

$n =$	2	3	4	5
$\chi(\Gamma_n^0)$	$\frac{1}{2^{11} \cdot 3^3}$	$\frac{17}{2^{14} \cdot 3^5 \cdot 5}$	$\frac{17 \cdot 31}{2^{19} \cdot 3^6 \cdot 11}$	$\frac{17 \cdot 31 \cdot 691}{2^{22} \cdot 3^7 \cdot 5 \cdot 7 \cdot 11}$
$\chi(\Gamma_n^1)$		$\frac{1}{2^{14} \cdot 3^2 \cdot 5^2}$		$\frac{31 \cdot 691}{2^{22} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13}$
$\chi(\Delta_n)$	$\frac{67}{2^{10} \cdot 3^3 \cdot 5^3 \cdot 7}$	$\frac{19^2 \cdot 67}{2^{13} \cdot 3^5 \cdot 5^4 \cdot 7}$	$\frac{19^2 \cdot 67 \cdot 191 \cdot 2161}{2^{18} \cdot 3^6 \cdot 5^5 \cdot 7 \cdot 11}$	

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Table 2. *Some approximate values.*

n	$\chi(\Gamma_n^s)$	$\chi(\Delta_n)$
2	1.808×10^{-5}	2.769×10^{-6}
3	2.712×10^{-7}	2.777×10^{-6}
4	1.253×10^{-7}	2.171×10^{-4}
5	1.662×10^{-8}	3.162
10	1.736×10^{8}	5.771×10^{64}
15	8.624×10^{55}	3.510×10^{218}
20	1.654×10^{151}	1.833×10^{478}

For the lattice Γ_n^1, we did not manage to find an alternative to the construction relying on principal arithmetic subgroups. Thus, a possible improvement of our result would be to obtain a more concrete description for it.

Remark 1.1. The notation has been chosen so that Γ_n^s denotes the nonuniform lattice of the smallest covolume for any n when setting $s = (n \bmod 2) \in \{0, 1\}$.

1.3. The uniform case

Let $k = \mathbb{Q}(\sqrt{5})$, and let $\mathscr{I}$ denote the *icosian ring*, i.e., $\mathscr{I}$ is the unique (up to conjugacy) maximal order in the quaternion k-algebra $\left(\frac{-1,-1}{k}\right)$ (see [5, Sect. 8.2] or [38, p.141]). We have an inclusion $\mathscr{I} \subset \mathbb{H}$. The following Hermitian form restricts to the standard $\mathscr{I}$-lattice $\mathscr{I}^{n+1}$ in $V_{\mathbb{R}} = \mathbb{H}^{n+1}$:

$$h(x, y) = \tfrac{1-\sqrt{5}}{2}\overline{x_0}y_0 + \overline{x_1}y_1 + \cdots + \overline{x_n}y_n. \tag{1.4}$$

The stabilizer $\mathbf{U}(\mathscr{I}^{n+1}, h)$ is a uniform lattice in $\mathrm{Sp}(n, 1)$ (see Section 2.3), which we will denote by the symbol Δ_n in the following.

Theorem 3. *For any $n > 1$, the lattice Δ_n realizes the smallest covolume among uniform lattices in* $\mathrm{Sp}(n, 1)$. *Up to conjugacy, it is the unique lattice with this property. Its Euler characteristic is given by*

$$\chi(\Delta_n) = (n + 1) \prod_{j=1}^{n+1} \frac{\zeta_k(1 - 2j)}{4}, \tag{1.5}$$

where ζ_k denotes the Dedekind zeta function of $k = \mathbb{Q}(\sqrt{5})$.

Remark 1.2. The special values $\zeta_k(1 - 2j)$ appearing in equation (1.5) are known to be rational by the Klingen–Siegel theorem (more generally, for any totally real k), and they can be precisely evaluated (see [33, Sect. 3.7]). A list for $j = 1, \ldots, 5$ is given, for instance, in [10, Table 2], from which we obtain the explicit values for $\chi(\Delta_n)$ listed in Table 1. We omit $n = 5$ for reason of space.

1.4. Numerical values, growth

We can now compare the nonuniform and uniform lattices and study the asymptotic of their covolumes with respect to the dimension. We give a few numerical values in Table 2. One sees that $\chi(\Gamma_n^s)$ and $\chi(\Delta_n)$ are very close for $n = 2$, but then $\chi(\Delta_n)$ starts growing much faster than $\chi(\Gamma_n^s)$ (which also grows with $n > 5$). More precisely, we can state the following result, which essentially follows from Theorems 1–3; see Section 6.5 for the discussion of the proof.

Corollary 1. *Each of the sequences $\chi(\Gamma_n^s)$, $\chi(\Delta_n)$, and $\chi(\Delta_n)/\chi(\Gamma_n^s)$ grows superexponentially as $n \to \infty$.*

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Geometric lower bounds—by means of embedded balls—for the volume of quaternionic hyperbolic manifolds have been obtained by Philippe in her thesis (see [26, Cor. 5.2]), and in [15, Cor. 5.3] for noncompact manifolds. In contrast to Corollary 1, these bounds decrease fast with the dimension.

It is clear from Corollary 1 that the proof of the next result now follows by inspecting a finite number of values.

Corollary 2. *For $n = 2$, the lattice of the smallest covolume in* $\mathrm{Sp}(n, 1)$ *is uniform. For any $n > 2$, this lattice is nonuniform. The smallest Euler characteristic of a lattice in* $\mathrm{Sp}(n, 1)$ *(with $n > 1$ arbitrary) is given by $\chi(\Gamma_5^1)$.*

1.5. Quaternionic hyperbolic orbifolds

Let $\pi : \mathrm{Sp}(n, 1) \to \mathrm{PSp}(n, 1)$ denote the projection. Any lattice $\Gamma' \subset \mathrm{PSp}(n, 1)$ is the image of a lattice $\Gamma = \pi^{-1}(\Gamma')$ that contains the center $\{\pm I\}$. Then we have $\chi(\Gamma') = 2\chi(\Gamma)$. It follows that Γ' is of minimal covolume in $\mathrm{PSp}(n, 1)$ if and only if so is Γ in $\mathrm{Sp}(n, 1)$ (note that a lattice of minimal covolume in $\mathrm{Sp}(n, 1)$ necessarily contains the center $\{\pm I\}$).

The group $\mathrm{PSp}(n, 1)$ identifies with the isometries of the quaternionic hyperbolic n-space $\mathbf{H}^n_{\mathbb{H}}$. For any lattice $\Gamma' \subset \mathrm{PSp}(n, 1)$, we consider the finite-volume quaternionic hyperbolic orbifold $M = \Gamma' \backslash \mathbf{H}^n_{\mathbb{H}}$. Alternatively, we may write M as the quotient $M = \Gamma \backslash \mathbf{H}^n_{\mathbb{H}}$, where $\Gamma = \pi^{-1}(\Gamma')$. Then the *orbifold* Euler–Poincaré characteristic of M is given by $\chi(M) = \chi(\Gamma') = 2\chi(\Gamma)$. In case Γ' is torsion-free, M is a quaternionic hyperbolic *manifold*, and $\chi(M)$ corresponds to the usual (i.e., topological) Euler characteristic. The volume of the orbifold M is proportional to $\chi(M)$ (see below). Thus, Theorems 1–3 determine the quaternionic hyperbolic orbifolds (compact or noncompact) of the smallest volume.

The choice of a normalization of the volume form on $\mathbf{H}^n_{\mathbb{H}}$ induces a volume form on its compact dual, i.e., on the quaternionic projective space $\mathbb{H}P^n$. For the induced volume on a quotient $M = \Gamma' \backslash \mathbf{H}^n_{\mathbb{H}}$, we have

$$\mathrm{vol}(M) = \frac{\mathrm{vol}(\mathbb{H}P^n)}{n+1} \chi(M), \tag{1.6}$$

where $n + 1$ appears as the Euler characteristic of $\mathbb{H}P^n$.

1.6. Outline

The classification of arithmetic subgroups in $\mathrm{Sp}(n, 1)$ is discussed in Section 2. In Section 3, we recall some materials from Bruhat–Tits theory, in particular to prepare the discussion of Prasad's volume formula in Section 4. Section 5 deals with lattices that are defined as stabilizers of Hermitian modules. The proofs of the results stated in the introduction are contained in Section 6, with the exception of the uniqueness, which is proved in Section 7.

2. Arithmetic subgroups in $\mathrm{Sp}(n, 1)$

2.1. Admissible groups

Let k be a number field, and consider an absolutely simple algebraic k-group $\mathbf{G}$ such that

$$\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R}) \cong \mathrm{Sp}(n, 1) \times K \tag{2.1}$$

for some compact group K. Then $\mathbf{G}$ is simply connected of type C_{n+1}, and k is totally real. Moreover, we can fix an embedding $k \subset \mathbb{R}$ so that $\mathbf{G}(\mathbb{R})$ identifies with $\mathrm{Sp}(n, 1)$. It follows from the classification of simple algebraic group (see [36] and Remark 2.2 below) that $\mathbf{G}$ is isomorphic over k to some unitary group $\mathbf{U}(V, h)$, where

- ○ D is a quaternion algebra over k, with the standard involution $x \mapsto \overline{x}$.
- ○ V is a right D-vector space.
- ○ h is a nondegenerate hermitian form on V (sesquilinear with respect to the standard involution).

Such a k-group $\mathbf{G} = \mathbf{U}(V, h)$ satisfying equation (2.1) will be called *admissible*, and in this case, we shall use the same terminology for the Hermitian space (V, h). We call D the *defining algebra* of $\mathbf{G}$. Facts concerning quaternion algebras will be recalled along the lines; we refer to [38] or [19, Ch. 2 and 6].

Remark 2.1. For any field extension K/k, we have by definition $\mathbf{G}(K) = \mathbf{U}(V_K, h)$, where $V_K = V \otimes_k K$. The latter is seen as a D_K-module for $D_K = D \otimes_k K$. In particular, we can use the notation $\mathbf{U}(V_k, h)$ to denote the k-points $\mathbf{G}(k)$.

Remark 2.2. In [36, p. 56] Tits describes the classification in terms of the *special* unitary group $\mathbf{SU}$; however, in our case $\mathbf{SU} = \mathbf{U}$ since symplectic transformations have determinant 1.

2.2. Admissible defining algebras

We denote by $\mathcal{V}_k = \mathcal{V}_k^\infty \cup \mathcal{V}_k^{\mathrm{f}}$ the set of (infinite or finite) places of k, and, for any $v \in \mathcal{V}_k$, by D_v the quaternion algebra $D_{k_v} = D \otimes_k k_v$. The algebra D is completely determined by the set of places $v \in \mathcal{V}_k$ where it ramifies (i.e., for which D_v is a division algebra), and the set of such places is of even (finite) cardinality. There is no other obstruction to the existence of a quaternion algebra D with prescribed localizations $\{D_v \mid v \in V\}$; see [19, Sect. 7.3].

Let $\mathbf{G} = \mathbf{U}(V, h)$ as above, with V over D. The following (well-known) result appears as a special case of Lemma 5.2 below. Recall that D_v is said *split* if it is isomorphic to $M_2(k_v)$, and this happens exactly when D_v is not ramified.

Lemma 2.3. *Let $v \in \mathcal{V}_k$. The algebraic group $\mathbf{G}_{k_v}$ (obtained by scalars extension) splits if and only if D_v splits (i.e., is not ramified).*

Proof. It follows from the classification in [36, p. 56] that $\mathbf{G}_{k_v}$ has relative rank less than $n+1$ if D_v is a division algebra; thus, in this case, $\mathbf{G}_{k_v}$ is not split. If D_v splits the fact that $\mathbf{G}_{k_v}$ splits will follow from Lemma 5.2 below with $R = k_v$. □

Corollary 2.4. *The k-isomorphism class of $\mathbf{G}$ determines its defining algebra D uniquely up to k-isomorphism.*

Proof. This is now clear since D is determined by the set of places where it splits. □

Corollary 2.5. *In order for the Hermitian space (V, h) over D to be admissible, it is necessary that D ramifies at any $v \in \mathcal{V}_k^\infty$, i.e., $D_v \cong \mathbb{H}$ for any $v \in \mathcal{V}_k^\infty$.*

Proof. By the admissibility condition, for any $v \in \mathcal{V}_k^\infty$ the group $\mathbf{G}(k_v)$ is either $\mathrm{Sp}(n,1)$ or $\mathrm{Sp}(n+1) = \mathrm{Sp}(n+1, 0)$. For $n > 1$, these groups are not split. It follows from Lemma 2.3 that D ramifies at each $v \in \mathcal{V}_k^\infty$ so that $D_v \cong \mathbb{H}$ (see [19, Sect. 2.5]). □

A pair (k, D) with $k \subset \mathbb{R}$ a totally real number field and D a quaternion algebra over k will be called *admissible* if D satisfies the necessary condition of Corollary 2.5. More simply, we say that 'D is admissible'.

Proposition 2.6. *Let (V, h) and (V', h') be two admissible Hermitian spaces of the same dimension over the same quaternion k-algebra D. Then $\mathbf{U}(V, h)$ is k-isomorphic to $\mathbf{U}(V', h')$.*

Proof. Being admissible, the two Hermitian spaces (V, h) and (V', h') have the same signature over k_v for any $v \in \mathcal{V}_k^\infty$, and it follows from [32, 10.1.8 (iii)] that $(V, h) \cong (V', h')$. □

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Remark 2.7. There is actually a bijection between the set of admissible pairs (k, D) for $k \subset \mathbb{R}$ totally real and the set of algebraic groups that are admissible for $\mathrm{Sp}(n, 1)$; see [21, Sect. 4]. We will not need this fact in its full generality.

2.3. The classification of lattices

Let $\mathbf{G}$ be an admissible k-group, and let $\mathcal{O}_k$ denote the ring of integers in k. By the Theorem of Borel and Harish-Chandra, any subgroup $\Gamma \subset \mathbf{G}(\mathbb{R})$ that is commensurable with $\mathbf{G}(\mathcal{O}_k)$ (for some embedding $\mathbf{G} \subset \mathrm{GL}_N$) is a lattice in $\mathrm{Sp}(n, 1)$. Such a subgroup is called *arithmetic*. The work of Margulis [20] has shown that superrigidity for $\mathrm{Sp}(n, 1)$ (later proved by Gromov and Schoen [13] in the non-Archimedean case and Corlette [6] in the Archimedean) implies the arithmeticity of any lattice in $\mathrm{Sp}(n, 1)$, i.e., any lattice in $\mathrm{Sp}(n, 1)$ can be constructed as an arithmetic subgroup, as above. A pair $(k, \mathbf{G})$ with $\mathbf{G}$ admissible determines exactly one commensurability class of lattices in $\mathrm{Sp}(n, 1)$ (up to conjugacy); see [29, Prop. 2.5]. We will say that the lattices Γ in such a commensurability class are *defined over* k. Moreover, with Corollary 2.4, we see that the defining k-algebra D of $\mathbf{G}$ is an invariant of the commensurability class. We take over the terminology and say that that D is the *defining algebra* of Γ. Conversely, by Proposition 2.6, the pair (k, D) uniquely determines the commensurability class.

Proposition 2.8 (Compactness criterion). *A lattice* $\Gamma \subset \mathrm{Sp}(n, 1)$ *is nonuniform if and only if it is defined over* $\mathbb{Q}$.

Proof. This is specialization of Godement's compactness criterion, which asserts that an arithmetic subgroup of $\mathbf{G}$ semisimple is nonuniform in $\mathbf{G}(k \otimes_{\mathbb{Q}} \mathbb{R})$ if and only if $\mathbf{G}$ is k-isotropic. If $k \neq \mathbb{Q}$, an admissible k-group $\mathbf{G}$ has a compact factor $\mathbf{G}(k_v)$ for some $v \in \mathcal{V}_k^\infty$ so that $\mathbf{G}$ cannot be isotropic. Let $k = \mathbb{Q}$, and let $\mathbf{G} = \mathbf{U}(V, h)$ admissible defined over $\mathbb{Q}$. Then $\mathbf{G}$ is isotropic when (V, h) is, and by [32, Theorem 10.1.1], this happens exactly when its trace form q_h (which is a quadratic form over $\mathbb{Q}$ in $4(n+1)$ variables) is isotropic. Then $\mathbf{G}$ is isotropic by [32, Cor. 5.7.3 (iii)]. □

We will describe in Section 5 a concrete way to construct arithmetic subgroups in $\mathrm{Sp}(n, 1)$.

3. Parahoric subgroups and Galois cohomology

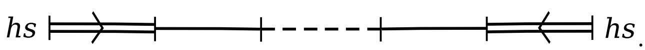
We collect in this section some notions of the Bruhat–Tits theory; we refer to [37]. Section 3.1 and 3.2 are needed for the volume computation in Section 4. The content of Section 3.3 will appear later, in Section 6.3, and the reader might want to skip it until they reach this point.

In this section, $\mathbf{G}$ denotes an admissible k-group. For any finite place $v \in \mathcal{V}_k^{\mathrm{f}}$, we denote by $\mathfrak{o}_v$ the valuation ring in k_v.

3.1. Parahoric subgroups

For any $v \in \mathcal{V}_k^{\mathrm{f}}$, the group $\mathbf{G}(k_v)$ acts on its associated Bruhat–Tits building X_v. A *parahoric subgroup* $P_v \subset \mathbf{G}(k_v)$ is by definition a stabilizer of a facet of X_v (note that we are working with $\mathbf{G}$ simply connected). Maximal parahoric subgroups correspond to stabilizers of vertices on X_v. If Δ_v denotes the affine root system of $\mathbf{G}(k_v)$, then the conjugacy classes of parahoric subgroups $P_v \subset \mathbf{G}(k_v)$ are in correspondence with the subsets $\Theta_v \subsetneq \Delta_v$; then Θ_v is called the *type* of P_v. The correspondence preserves the inclusion, and thus, maximal subgroups have types that omit exactly one element in Δ_v.

Assume first that $\mathbf{G}_{k_v}$ is split. Then its affine root system is given by the following local Dynkin diagram:



(3.1)

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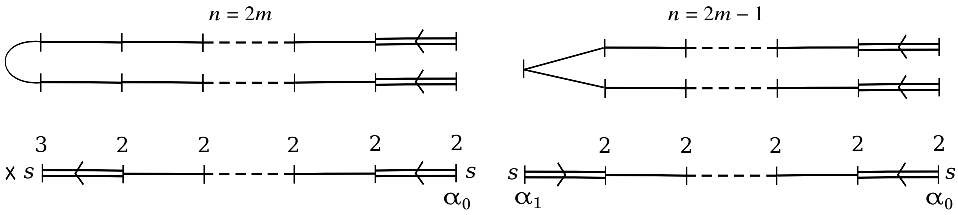
Table 3. *Local indices for the type* ${}^2\mathrm{C}_{n+1}$.



The parahoric subgroups of maximal volume in $\mathbf{G}(k_v)$ are those that are *hyperspecial*, i.e., of type $\Delta_v \setminus \{\alpha\}$, where α is any of the two hyperspecial vertices (labelled 'hs' in equation (3.1)).

There is exactly one nonsplit form of $\mathbf{G}_{k_v}$ of type C_{n+1}, and it splits over the maximal unramified extension $\widehat{k}_v/k_v$. If $\mathbf{G}_{k_v}$ is not split, it corresponds to the local type named ${}^2\mathrm{C}_{n+1}$ in [37, Sect. 4.3]; we have reproduced in Table 3 the corresponding local indices (their description depends on the parity of n). It is a general fact, proved in [4, App. A], that parahoric subgroups of maximal volume in $\mathbf{G}(k_v)$ are those of type $\Delta_v \setminus \{\alpha\}$, where α is *very special* (see *loc. cit.* for the definition). In our case, the very special vertex in Δ_v is α_0 for n even, respectively, α_1 for n odd (as shown in Table 3; note that α_1 is defined only for n odd). Thus, for $s = n \bmod 2$, a parahoric subgroup of $\mathbf{G}(k_v)$ is of maximal volume exactly when it is of type $\Theta_v = \Delta_v \setminus \{\alpha_s\}$.

3.2. The group scheme structure

Let $\mathbf{G}(k_v)$ as above, split or not. Each parahoric subgroup $P_v \subset \mathbf{G}(k_v)$ can be written as $P_v = \mathscr{G}(\mathfrak{o}_v)$ for some canonical smooth group scheme $\mathscr{G}$ over $\mathfrak{o}_v$. Let $\mathfrak{f}_v$ be the residue field of k_v. For a fixed P_v, following the notation of [28], we denote by $\overline{\mathrm{M}}_v$ the maximal reductive quotient of the $\mathfrak{f}_v$-group $\mathscr{G}_{\mathfrak{f}_v}$ obtained from $\mathscr{G}$ by base change; $\overline{\mathrm{M}}_v$ is connected. The structure of $\overline{\mathrm{M}}_v$ can be obtained from the type of P_v by using the procedure explained in [37, Sect. 3.5.2]. If $\mathbf{G}_{k_v}$ is split, then P_v is hyperspecial if and only if $\overline{\mathrm{M}}_v$ is simple of type C_{n+1} (i.e., of the same type as $\mathbf{G}_{k_v}$). In this case, we will write $\overline{\mathrm{M}}_v = \overline{\mathscr{M}}_v$.

3.3. The Galois cohomology action

For $\mathbf{G}$ admissible over k, we let $\mathbf{C}$ denote its center and $\overline{\mathbf{G}} = \mathbf{G}/\mathbf{C}$ its adjoint quotient. For any field extension K/k, the group $\overline{\mathbf{G}}(K)$ identifies with the group of inner K-automorphisms of $\mathbf{G}$. We denote by δ the connecting map in the Galois cohomology exact sequence:

$$1 \to \mathbf{C}(K) \to \mathbf{G}(K) \xrightarrow{\pi} \overline{\mathbf{G}}(K) \xrightarrow{\delta} H^1(K, \mathbf{C}) \to H^1(K, \mathbf{G}). \tag{3.2}$$

For $K = k_v$ non-Archimedean, this provides an action of $H^1(k_v, \mathbf{C})$ on the local Dynkin diagram Δ_v (see [4, Sect. 2.8]); the action respects the symmetries of Δ_v so that $H^1(k_v, \mathbf{C}) \to \mathrm{Aut}(\Delta_v)$. Note in particular that, for $\mathbf{G}_{k_v}$ nonsplit of type C_{n+1}, we have $\mathrm{Aut}(\Delta_v) = 1$. In the split case, there is exactly one nontrivial symmetry. We denote by ξ the induced 'global' map $H^1(k, \mathbf{C}) \to \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} \mathrm{Aut}(\Delta_v)$ (the image actually lies in the direct product).

Of particular interest to us is the subgroup $\delta(\overline{\mathbf{G}}(k))' = \delta(\overline{\mathbf{G}}(k) \cap \pi(\mathbf{G}(\mathbb{R})))$ (the notation follows [4, Sect. 2.8]; recall that we have fixed an inclusion $k \subset \mathbb{R}$). We will make use of the following alternative description.

Lemma 3.1. *The group $\delta(\overline{\mathbf{G}}(k))'$ coincides with the kernel of the diagonal map*

$$H^1(k, \mathbf{C}) \to \prod_{v \in \mathcal{V}_k^\infty} H^1(k_v, \mathbf{C}).$$

Proof. For $x \in H^1(k, \mathbf{C})$, we denote by $(x_v)_{v \in \mathcal{V}_k^\infty}$ its image in $\prod_{v \in \mathcal{V}_k^\infty} H^1(k_v, \mathbf{C})$. Let us first assume $x \in \delta(\overline{\mathbf{G}}(k))'$. Then for the place $v = \mathrm{id}$ corresponding to the inclusion $k \subset \mathbb{R}$, we have $x_v \in (\delta \circ \pi)(\mathbf{G}(\mathbb{R}))$, which by exactness of equation (3.2) (with $K = \mathbb{R}$) is equivalent to $x_v = 1$. For $v \neq \mathrm{id}$, the group $\mathbf{G}(k_v)$ is compact, and in this case, it is known that $\pi : \mathbf{G}(k_v) \to \overline{\mathbf{G}}(k_v)$ is surjective (see [27, Sect. 3.2: Cor. 3]). Since $x_v \in \delta(\overline{\mathbf{G}}(k_v))$, we thus have $x_v = 1$ again by the exactness of equation (3.2). Conversely, suppose that x has trivial image in $\prod_{v \in \mathcal{V}_k^\infty} H^1(k_v, \mathbf{C})$. Then (x_v) has certainly trivial image in $\prod_{v \in \mathcal{V}_k^\infty} H^1(k_v, \mathbf{G})$. But for $\mathbf{G}$ simply connected the latter identifies with $H^1(k, \mathbf{G})$ by the Hasse principle (see [27, Theorem 6.6]). By exactness of equation (3.2) with $K = k$, it follows that $x \in \delta(\overline{\mathbf{G}}(k))'$. □

Remark 3.2. We will see in Lemma 7.1 that actually $\mathbf{G}(k_v) \to \overline{\mathbf{G}}(k_v)$ is surjective for $v = \mathrm{id}$ as well, which slightly simplifies the proof. We have preferred giving the above proof, which works quite generally when $\mathbf{G}$ is simply connected; it appears in [7, Sect. 12.2].

4. Principal arithmetic subgroups and volumes

4.1. Principal arithmetic subgroups

For $\mathbf{G}$ an admissible k-group, we will denote by $P = (P_v)_{v \in \mathcal{V}_k^{\mathrm{f}}}$ a collection of parahoric subgroups $P_v \subset \mathbf{G}(k_v)$ $(v \in \mathcal{V}_k^{\mathrm{f}})$. Such a collection is called *coherent* if the product $\prod_{v \in \mathcal{V}_k^{\mathrm{f}}} P_v$ is open in the group $\mathbf{G}(\mathbb{A}_{\mathrm{f}})$, where $\mathbb{A}_{\mathrm{f}}$ denotes the finite adèles of k. This condition implies that P_v is hyperspecial for all but finitely many $v \in \mathcal{V}_k^{\mathrm{f}}$. Moreover, one has that the subgroup $\Lambda_P = \mathbf{G}(k) \cap \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} P_v$ is an arithmetic subgroup of $\mathbf{G}(k)$, called *principal*.

4.2. The normalized Haar measure

The covolume of the principal arithmetic subgroup $\Lambda_P \subset \mathbf{G}(k)$ can be computed with Prasad's volume formula [28] in terms of the volumes of the parahoric subgroups P_v $(v \in \mathcal{V}_k^{\mathrm{f}})$. In the notation of *loc. cit.*, our situation corresponds to the case $\mathbf{G}_S = \mathbf{G}(\mathbb{R})$ (i.e., S contains a single infinite place corresponding to the inclusion $k \subset \mathbb{R}$). We write $\mu = \mu_S$ for the normalization of the Haar measure on $\mathbf{G}(\mathbb{R})$ used in [28, Sect. 3.6]. Then for the Euler–Poincaré characteristic (in the sense of C.T.C. Wall) of $\Gamma \subset \mathbf{G}(\mathbb{R})$, one has $|\chi(\Gamma)| = |\chi(X_u)|\, \mu(\Gamma \backslash \mathbf{G}(\mathbb{R}))$, where X_u is the compact dual symmetric space associated with $\mathbf{H}^n_{\mathbb{H}}$ (see [4, §4]). Explicitly, $X_u = \mathrm{Sp}(n+1)/(\mathrm{Sp}(n) \times \mathrm{Sp}(1))$ is the quaternionic projective space $\mathbb{H}P^n$, for which $\chi(X_u) = n+1$. Moreover, since the symmetric space of $\mathrm{Sp}(n,1)$ has dimension $4n$, it follows from [33, Prop. 23] that $\chi(\Gamma)$ is positive. Thus,

$$\chi(\Gamma) = (n+1) \cdot \mu(\Gamma \backslash \mathbf{G}(\mathbb{R})). \tag{4.1}$$

4.3. Prasad's volume formula

To state the volume formula for $\Lambda_P \subset \mathbf{G}(k)$ in an explicit way, we need to introduce some notation; we mostly follow [28]. The symbol $\mathscr{D}_k$ denotes the absolute value of the discriminant of k, and we write $d = [k : \mathbb{Q}]$ for the degree. For each $v \in \mathcal{V}_k^{\mathrm{f}}$, let $\mathfrak{f}_v$ be the residue field of k_v, and let q_v be the cardinality of $\mathfrak{f}_v$. For each parahoric subgroup P_v, the reductive $\mathfrak{f}_v$-group $\overline{\mathrm{M}}_v$ is defined in Section 3.2. The reductive $\mathfrak{f}_v$-group corresponding to a hyperspecial parahoric subgroup in the split form of $\mathbf{G}$ is denoted by $\overline{\mathscr{M}}_v$. For all but finitely many $v \in \mathcal{V}_k^{\mathrm{f}}$, we have that P_v is hyperspecial and thus $\overline{\mathrm{M}}_v \cong \overline{\mathscr{M}}_v$.

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In our situation, Prasad's volume formula [28, Theorem 3.7] takes the following form (note that in our case $\ell = k$ since $\mathbf{G}$ is of type C and thus has no outer symmetries):

$$\mu(\Lambda_P \backslash \mathbf{G}(\mathbb{R})) = \mathscr{D}_k^{\dim \mathbf{G}/2} \left(\prod_{j=1}^{n+1} \frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \mathscr{E}(P), \tag{4.2}$$

where the 'Euler product' $\mathscr{E}(P)$ is given by

$$\mathscr{E}(P) = \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} \frac{q_v^{(\dim \overline{\mathrm{M}}_v + \dim \overline{\mathscr{M}}_v)/2}}{|\overline{\mathrm{M}}_v(\mathfrak{f}_v)|}. \tag{4.3}$$

4.4. The Euler product and zeta functions

Let T be the finite set of places $v \in \mathcal{V}_k^{\mathrm{f}}$ such that P_v is not hyperspecial. For $v \notin T$, we have that $\overline{\mathrm{M}}_v \cong \overline{\mathscr{M}}_v$, which is an $\mathfrak{f}_v$-split simple group of type C_{n+1}, for which $|\overline{\mathscr{M}}_v(\mathfrak{f}_v)| = q_v^{(n+1)^2} \prod_{j=1}^{n+1} (q_v^{2j} - 1)$ (see [24, Tab. 1]), and $\dim \overline{\mathscr{M}}_v = \dim \mathbf{G} = (n+1)(2n+3)$. By a direct computation, we may rewrite $\mathscr{E}(P)$ as the following:

$$\begin{aligned} \mathscr{E}(P) &= \prod_{v \in T} e'(P_v) \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} \frac{q_v^{\dim \overline{\mathscr{M}}_v}}{|\overline{\mathscr{M}}_v(\mathfrak{f}_v)|} \\ &= \prod_{v \in T} e'(P_v) \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} \prod_{j=1}^{n+1} \frac{1}{1 - q_v^{-2j}} \\ &= \prod_{v \in T} e'(P_v) \prod_{j=1}^{n+1} \zeta_k(2j), \end{aligned} \tag{4.4}$$

where ζ_k is the Dedekind zeta function of k, and the correcting factors $e'(P_v)$ (so called 'lambda factors' in [3]) are given by

$$e'(P_v) = q_v^{(\dim \overline{\mathrm{M}}_v - \dim \overline{\mathscr{M}}_v)/2} \frac{|\overline{\mathscr{M}}_v(\mathfrak{f}_v)|}{|\overline{\mathrm{M}}_v(\mathfrak{f}_v)|}. \tag{4.5}$$

Putting together equations (4.1), (4.2) and (4.4), we can finally write (where the second line is obtained from the functional equation of ζ_k; see [22, Ch.VII (5.11)]):

$$\chi(\Lambda_P) = (n+1) \mathscr{D}_k^{\dim \mathbf{G}/2} \prod_{v \in T} e'(P_v) \prod_{j=1}^{n+1} \left(\frac{(2j-1)!}{(2\pi)^{2j}} \right)^d \zeta_k(2j) \tag{4.6}$$

$$= (n+1) \prod_{v \in T} e'(P_v) \prod_{j=1}^{n+1} 2^{-d} |\zeta_k(1-2j)|. \tag{4.7}$$

4.5. The nonsplit local factors

We compute in the following lemma the local factors $e'(P_v)$ of interest to us.

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Lemma 4.1. *Suppose that $\mathbf{G}_{k_v}$ is nonsplit, and let $P_v^t \subset \mathbf{G}(k_v)$ be a special parahoric subgroup of type $\Delta_v \setminus \{\alpha_t\}$ (assuming n odd if $t = 1$). Then*

$$e'(P_v^0) = \prod_{j=1}^{n+1} (q_v^j + (-1)^j); \tag{4.8}$$

$$e'(P_v^1) = \frac{\prod_{j=1}^{2m} (q_v^{2j} - 1)}{\prod_{j=1}^{m} (q_v^{4j} - 1)}, \tag{4.9}$$

where $n + 1 = 2m$ in the latter.

Remark 4.2. Note that $e'(P_v^1)$ is clearly an integer. The fact that P_v^1 has larger volume than P_v^0 (see Section 3.1) is reflected by the fact that $e'(P_v^1)$ is smaller than $e'(P_v^0)$; this inequality can be checked empirically (and probably rigorously with some effort) from the formulas in Lemma 4.1. One may notice that, as polynomials in q_v, these two local factors have quite similar order of magnitude though.

Proof. The definition of $e'(P_v^t)$ is given in equation (4.5). The dimension and order for $\overline{\mathscr{M}}_v$ are given in Section 4.4. We obtain the description of $\overline{\mathrm{M}}_v$ (see Section 3.2) in each case by [37, Sec. 3.5.2]. We refer to [24, Tab. 1] for the order of the classical finite simple groups. For P_v^0, we have that $\overline{\mathrm{M}}_v$ is given as an almost direct product $\overline{\mathrm{M}}_v = T \cdot H$, where T is a nonsplit one-dimensional torus, and H is a simple of type ${}^2\mathrm{A}_n$. In particular, $\dim \overline{\mathrm{M}}_v = (n+1)^2$, and $|T(\mathfrak{f}_v)| = q_v + 1$. By Lang's isogeny theorem, we have $|\overline{\mathrm{M}}_v(\mathfrak{f}_v)| = (q_v + 1)|H(\mathfrak{f}_v)|$, and the formula for $e'(P_v^0)$ now follows from a straightforward computation. Note that, for $n = 2$, the local index ${}^2\mathrm{C}_3$ needs to be listed separately (see [37, p.63]); however, the description for $\overline{\mathrm{M}}_v$ is similar, and the formula remains the same.

For P_v^1 with $n + 1 = 2m$, we have that $\overline{\mathrm{M}}_v$ is obtained by Weil restriction of scalars as $\overline{\mathrm{M}}_v = \mathrm{Res}_{\mathfrak{k}_v/\mathfrak{f}_v}(H)$, where H is simple of type C_m and $\mathfrak{k}_v/\mathfrak{f}_v$ is quadratic (i.e., $\mathfrak{k}_v$ has cardinality q_v^2). Thus, $\overline{\mathrm{M}}_v$ has twice the dimension of H, which is $m(2m+1)$, and $|\overline{\mathrm{M}}_v(\mathfrak{f}_v)| = |H(\mathfrak{k}_v)|$. The result for $e'(P_v^1)$ follows directly. □

5. Stabilizers of Hermitian lattices

In this section, we obtain the covolume of the lattices Γ_n^0 and Δ_n in $\mathrm{Sp}(n, 1)$ as a consequence of Theorem 5.4. To prove the latter, we first need to study the structure of the stabilizers of Hermitian lattices; this is done in Section 5.1 and 5.2. In those sections, R will denote an integral domain containing the ring of integers $\mathcal{O}_k$, and K will be the field of fractions of R.

5.1. Hermitian lattices over orders

Let us fix an order $\mathcal{O}_D$ in an admissible quaternion k-algebra D, and consider the right $\mathcal{O}_D$-module $L = \mathcal{O}_D^{n+1}$. We set $\mathcal{O}_{D,R} = \mathcal{O}_D \otimes_{\mathcal{O}_k} R$. Then $L_R = L \otimes_{\mathcal{O}_k} R$ is a right $\mathcal{O}_{D,R}$-module. Consider a Hermitian form h on L, described as follows:

$$h(x, y) = \sum_{i=0}^{n} a_i \bar{x}_i y_i, \tag{5.1}$$

where the coefficients a_i are taken in $\mathcal{O}_k$. Note that the standard involution on D restricts to $\mathcal{O}_D$ (use the trace) so that (L, h) is indeed a Hermitian module in the sense of [32, Ch. 7] (and so is (L_R, h) for any ring extension R of $\mathcal{O}_k$). We write $V = L_k = D^{n+1}$, and we will assume that (V, h) is admissible. We say in this case that the module (L, h) itself is *admissible*.

Recall that by definition the Hermitian module (L_R, h) is *regular* (or *nonsingular*) if the map $\phi_h : x \mapsto h(x, \cdot)$ induces an isomorphism of $\mathcal{O}_{D,R}$-modules from L_R onto its dual module $(L_R)^*$, seen

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as a right module via $f\alpha = \overline{\alpha} f$ (see [32, Sect. 7.1]). When R is a field this is equivalent to (L_R, h) being nondegenerate, i.e., $(L_R)^\perp = 0$. For more general R, we will need the following result.

Lemma 5.1. *If the coefficients of h are invertible in R (i.e., $a_i \in R^\times$ for $i = 0, \ldots, n$), then the Hermitian module (L_R, h) is regular.*

Proof. Let $\{\mathbf{e}_i\}$ be the standard basis of $L_R = \mathcal{O}_{D,R}^{n+1}$, and let $\{\mathbf{e}_i^*\} \subset (L_R)^*$ be the associated dual basis. We have $\phi_h(\mathbf{e}_i) = a_i \mathbf{e}_i^* = \mathbf{e}_i^* a_i$ (note that $a_i = \overline{a}_i$ since $a_i \in \mathcal{O}_k$). The map $\mathbf{e}_i^* \mapsto \mathbf{e}_i a_i^{-1}$ from $(L_R)^*$ to L_R is then inverse to ϕ_h. □

5.2. *A key lemma*

For (L, h) and (V, h) as above, consider the stabilizer of L in $\mathbf{G}(k) = \mathbf{U}(V, h)$, i.e., the subgroup

$$\mathbf{U}(L, h) = \{g \in \mathbf{U}(V, h) \mid gL = L\}. \tag{5.2}$$

This is an arithmetic subgroup of $\mathbf{G}(k)$. More generally, we will denote by $\mathbf{U}(L_R, h) \subset \mathbf{G}(K)$ the stabilizer of $L_R \subset V_K$. The following lemma is the key result that will be used in Section 5.3.

Lemma 5.2. *Assume that the following conditions hold:*

1. *R is a principal ideal domain;*
2. *$\mathcal{O}_{D,R}$ splits, i.e., $\mathcal{O}_{D,R} \cong M_2(R)$;*
3. *the hermitian module (L_R, h) is regular.*

Then there is an isomorphism $\phi : \mathbf{U}(V_K, h) \longrightarrow \mathrm{Sp}_{2n+2}(K)$ such that $\phi(\mathbf{U}(L_R, h)) = \mathrm{Sp}_{2n+2}(R)$.

Proof. We adapt the discussion from [32, pp. 361-362] (which considers *skew*-Hermitian spaces, only over fields) to our setting. First, we may fix an identification $\mathcal{O}_{D,R} = M_2(R)$; the standard involution is then given by

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{5.3}$$

Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ so that V_K has the following splitting: $V_K = V_K e_1 \oplus V_K e_2$. We set $V_1 = V_K e_1$. As for *loc. cit.*, we obtain from h a bilinear form b_h on V_1 determined by

$$h(xe_1, ye_1) = \begin{pmatrix} 0 & 0 \\ b_h(xe_1, ye_1) & 0 \end{pmatrix}, \tag{5.4}$$

and in our case, b_h is easily seen to be antisymmetric. Since (L_R, h) is a Hermitian module, the form b_h actually restricts to a bilinear (antisymmetric) form on the R-lattice $L_1 = L_R e_1$ of V_1. Note that L_1 is free over R of rank $2n + 2$. If $f \in L_1^*$, then we can extend f to L_R by setting for any $x \in L_R$:

$$\tilde{f}(x) = f(xe_1) + f(xee_1)e, \tag{5.5}$$

where $e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. One computes that this is indeed an extension, which is actually $\mathcal{O}_{D,R}$-linear, i.e., $\tilde{f} \in (L_R)^*$. In particular, we have that the symplectic module (L_1, b_h) is regular, as (L_R, h) itself is assumed to be regular. Since by assumption R is a principal ideal domain (PID), we can now deduce that (L_1, h) is a orthogonal sum of hyperbolic modules (see, for instance, [16, Prop. 2.1]), and thus, its isometry group is isomorphic to $\mathrm{Sp}_{2n+2}(R)$.

An analogous formula to equation (5.5) can be used to extend any isometry σ of (L_1, h) to an isometry $\tilde{\sigma} \in \mathbf{U}(L_R, h)$ (see [32, p.362]). This shows that $g \mapsto g|_{L_1}$ yields an isomorphism from $\mathbf{U}(L_R, h)$ to $\mathrm{Sp}_{2n+2}(R)$. The same construction with $R = K$ thus provides the isomorphism ϕ in the statement. □

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5.3. The local structure of lattice stabilizers

Let again (L, h) denote an admissible lattice over $\mathcal{O}_D$, with D defined over the number field k and $\mathcal{O}_D \subset D$ an order. The following (nonstandard) terminology will be convenient for us.

Definition 5.3. We say that (L, h) is *of maximal type* if $\mathcal{O}_D$ is maximal and (L, h) is regular.

Remark 5.4. Given D, the existence of an admissible (L, h) of maximal type does not seem to be obvious (and we believe that it is wrong in general).

For each finite place $v \in \mathcal{V}_k^{\mathrm{f}}$, we shall abbreviate the notation from Section 5.1 (with $R = \mathfrak{o}_v$) as follows: $L_v = L_{\mathfrak{o}_v}$. As above, $\mathcal{R}$ denotes the set of finite places $v \in \mathcal{V}_k^{\mathrm{f}}$, where D_v ramifies, and $\mathbf{G} = \mathbf{U}(V, h)$.

Lemma 5.5. *Assume that (L, h) is of maximal type, and let $v \in \mathcal{V}_k^{\mathrm{f}}$ be a finite place with $v \notin \mathcal{R}$. Then $\mathbf{U}(L_v, h)$ is a hyperspecial parahoric subgroup in $\mathbf{G}(k_v) \cong \mathrm{Sp}_{2n+2}(k_v)$.*

Proof. The order $\mathcal{O}_{D,\mathfrak{o}_v}$, being maximal in $D_v \cong M_2(k_v)$, must be conjugate to $M_2(\mathfrak{o}_v)$ (see [19, Ch. 6]). Thus, we can apply Lemma 5.2: It implies that $\mathbf{U}(L_v, h)$ identifies with $\mathrm{Sp}_{2n+2}(\mathfrak{o}_v)$, which is hyperspecial parahoric by [37, Sect. 3.4.2]. □

We now turn our attention to the case of places where D ramifies.

Lemma 5.6. *For (L, h) of maximal type and $v \in \mathcal{R}$, the subgroup $\mathbf{U}(L_v, h)$ is a special parahoric subgroup in $\mathbf{G}(k_v)$ of type $\Delta_v \setminus \{\alpha_0\}$.*

Proof. Let $\widehat{k}_v$ be the maximal unramified extension of k_v, with ring of integers $\widehat{\mathfrak{o}}_v$. Let $\mathcal{O}_{D,v} = \mathcal{O}_{D,\mathfrak{o}_v}$ and $\widehat{\mathcal{O}} = \mathcal{O}_{D,v} \otimes_{\mathfrak{o}_v} \widehat{\mathfrak{o}}_v$. The latter in an order in $\widehat{D}_v = D_v \otimes \widehat{k}_v$, and we consider a maximal order $\mathcal{O}' \subset \widehat{D}_v$ containing $\widehat{\mathcal{O}}$. That is,

$$\mathcal{O}_{D,v} \subset \widehat{\mathcal{O}} \subset \mathcal{O}' . \tag{5.6}$$

Thus $\mathcal{O}' \cap D_v$ is an order in D_v, which equals $\mathcal{O}_{D,v}$ since the latter is maximal. Note that $\widehat{D}_v$ is split (see [19, Theorem 2.6.5]).

From the inclusions (5.6), we may interpret the subgroup $P_v = \mathbf{U}(L_v, h) \cong \mathbf{U}(\mathcal{O}_{D,v}^{n+1}, h)$ of $\mathbf{G}(k_v)$ as a subgroup of the matrix group $P'_v = \mathbf{U}((\mathcal{O}')^{n+1}, h)$. The latter is a hyperspecial parahoric of $\mathbf{G}(\widehat{k}_v)$ by Lemma 5.5. From the equality $\mathcal{O}' \cap D_v = \mathcal{O}_{D,v}$, we deduce $P'_v \cap \mathbf{G}(k_v) = P_v$. But in view of the local indices in Table 3, this means that P_v is a special parahoric subgroup of type $\Delta_v \setminus \{\alpha_0\}$ (since α_0 is the unique affine root in Δ_v that appears as the restriction of hyperspecial roots of $\mathbf{G}(\widehat{k}_v)$). □

5.4. The volume formula for the maximal type

Lattices of maximal type are particularly interesting because of the following result. Recall that q_v denotes the cardinality of the residue field of k_v (for $v \in \mathcal{V}_k^{\mathrm{f}}$). See Definition 5.3 for ‘maximal type’.

Theorem 5.7. *Let (L, h) be an admissible Hermitian $\mathcal{O}_D$-lattice of maximal type. Then $\mathbf{U}(L, h)$ is a principal arithmetic subgroup of $\mathbf{G}(k) = \mathbf{U}(V, h)$, and*

$$\chi(\mathbf{U}(L, h)) = (n+1) \prod_{j=1}^{n+1} \left(\frac{\zeta_k(1-2j)}{2^{[k:\mathbb{Q}]}} \prod_{v \in \mathcal{R}} q_v^j + (-1)^j \right), \tag{5.7}$$

where $\mathcal{R}$ is the set of finite places where D ramifies.

Proof. We can write $L = V_k \cap \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} L_v$, from which we obtain (for $\mathbf{G}(k)$ diagonally embedded in $\prod_v \mathbf{G}(k_v)$):

$$\mathbf{U}(L, h) = \mathbf{G}(k) \cap \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} \mathbf{U}(L_v, h).$$

With Lemmas 5.5 and 5.6, this shows that $\mathbf{U}(L, h)$ is principal, and the formula for $\chi(\mathbf{U}(L, h))$ is deduced from equation (4.7) and Lemma 4.1. □

We emphasize the special case of the standard Hermitian form over the Hurwitz integers in the next corollary. It implies the formula in equation (1.2) since we have $\chi(\Gamma_n^0) = \chi(\mathrm{Sp}(n, 1, \mathscr{H}))/2$ by construction.

Corollary 5.8. *The arithmetic subgroup* $\mathrm{Sp}(n, 1, \mathscr{H})$ *is principal, and*

$$\chi(\mathrm{Sp}(n, 1, \mathscr{H})) = (n+1) \prod_{j=1}^{n+1} \frac{2^j + (-1)^j}{4j} |B_{2j}|, \tag{5.8}$$

where B_m *is the m-th Bernoulli number.*

Proof. We have that $\mathscr{H}$ is a maximal order in $D = \mathscr{H} \otimes \mathbb{Q} = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, and the latter is the quaternion $\mathbb{Q}$-algebra that ramifies exactly at $p = 2$ and $p = \infty$ (see [38, p.79]). By Lemma 5.1, it is clear that $(\mathscr{H}^{n+1}, h)$, with h given in equation (1.1), is of maximal type. Thus, we can apply the theorem, and the formula in equation (5.8) follows immediately from the known expression:

$$\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1}.$$

□

5.5. *The covolume of* Δ_n

Let $L = \mathscr{I}^{n+1}$, where $\mathscr{I}$ is the icosian ring. The Hermitian form equation (1.4) has been chosen so that (L, h) is of maximal type. By definition, $\Delta_n = \mathbf{U}(L, h)$. The formula in equation (1.5) for $\chi(\Delta_n)$ is thus an immediate consequence of Theorem 5.7 since in this case $\mathcal{R} = \emptyset$ (see [38, p.150]).

6. The minimality of $\chi(\Gamma_n^s)$ and $\chi(\Delta_n)$

6.1. *Normalizers of minimal covolume*

Let $\Gamma \subset \mathrm{Sp}(n, 1)$ be a *maximal lattice*, i.e., maximal with respect to inclusion as in Section 2.3. We have $\Gamma \subset \mathbf{G}(\overline{k} \cap \mathbb{R})$ for some admissible k-group $\mathbf{G}$, but the stricter inclusion $\Gamma \subset \mathbf{G}(k)$ does not hold in general. By [4, Prop. 1.4], we have that Γ is a normalizer $N_{\mathbf{G}(\mathbb{R})}(\Lambda_P)$, where $\Lambda_P \subset \mathbf{G}(k)$ is a principal arithmetic subgroup.

Let $P = (P_v)$ be a coherent collection such that any parahoric subgroup P_v is of maximal volume. Then Λ_P is of minimal covolume among arithmetic lattices contained in $\mathbf{G}(k)$. It is a priori not clear—but turns out to be true—that the normalizer $N_{\mathbf{G}(\mathbb{R})}(\Lambda_P)$ for such a choice of P is of minimal covolume in its commensurability class in $\mathbf{G}(\mathbb{R})$. This can be proved in the same way as in [3, Sect. 4.3] (see also [7, Sect. 12.3] for a more detailed exposition). We state the result in the following lemma. In the rest of this section, $\mathcal{R}$ denotes the set of finite places where $\mathbf{G}$ does not split (equivalently, where its defining algebra ramifies).

Lemma 6.1. *The lattice* $\Gamma = N_{\mathbf{G}(\mathbb{R})}(\Lambda_P)$ *is of minimal covolume in its commensurability class if and only if the coherent collection* $P = (P_v)$ *satisfies*

1. *P_v is hyperspecial for each $v \notin \mathcal{R}$; and*
2. *P_v is special of maximal volume for $v \in \mathcal{R}$.*

Recall that for $v \in \mathcal{R}$ a parahoric subgroup $P_v \subset \mathbf{G}(k_v)$ is special of maximal volume exactly when it has type $\Delta_v \setminus \{\alpha_s\}$, where $s = n \bmod 2$. Using Lemmas 5.5 and 5.6, we thus have:

Corollary 6.2. *Let (L, h) be an admissible $\mathcal{O}_D$-lattice of maximal type. If n is even or $\mathcal{R} = \emptyset$, then the lattice $N_{\mathrm{Sp}(n,1)}(\mathbf{U}(L, h))$ is of minimal covolume in its commensurability class.*

6.2. *The index computation*

We will need to estimate the index $[\Gamma : \Lambda_P]$ for $\Gamma = N_{\mathbf{G}(\mathbb{R})}(\Lambda_P)$ of minimal covolume in its commensurability class. For this, we state the following lemma, which considers a slightly more general situation. The symbol h_k denotes the class number of k, and U_k (respectively, U_k^+) are the units (respectively, totally positive units) in $\mathcal{O}_k$.

Lemma 6.3. *Let $P = (P_v)$ such that P_v is hyperspecial for any $v \in \mathcal{V}_k^{\mathrm{f}} \setminus \mathcal{R}$. Then*

$$[\Gamma : \Lambda_P] \leq 2^{\#\mathcal{R}} \cdot h_k \cdot |U_k^+/U_k^2|.$$

Proof. We assume the notation of Section 3.3; in particular, $\mathbf{C}$ is the center of $\mathbf{G}$. We set $A = \delta(\overline{\mathbf{G}}(k))'$. Let $\Theta = (\Theta_v)_{v \in \mathcal{V}_k^{\mathrm{f}}}$ be the type of the coherent collection P. From the assumption, it follows that none of the types Θ_v has symmetries, and thus, the stabilizer of Θ in A equals the kernel A_ξ of ξ. By [4, Prop. 2.9], we thus have the exact sequence

$$1 \to \mathbf{C}(\mathbb{R})/(\mathbf{C}(k) \cap \Lambda_P) \to \Gamma/\Lambda_P \to A_\xi \to 1. \tag{6.1}$$

In our case, $\mathbf{C} = \mu_2$, and it follows that the left part vanishes. Hence, $[\Gamma : \Lambda_P] = |A_\xi|$. We now use the identification $H^1(k, \mu_2) = k^\times/(k^\times)^2$. For $S \subset \mathcal{V}_k^{\mathrm{f}}$ any finite set of places, we define

$$k_{2,S} = \left\{ x \in k^\times \;\middle|\; v(x) \in 2\mathbb{Z} \;\; \forall v \in \mathcal{V}_k^{\mathrm{f}} \setminus S \right\},$$

and $k_2 = k_{2,\emptyset}$. Since $\mathrm{Aut}(\Delta_v)$ is trivial for any $v \in \mathcal{R}$, it follows from [4, Prop. 2.7] that $H^1(k, \mathbf{C})_\xi = k_{2,\mathcal{R}}/(k^\times)^2$. See [4] for the definitions. For $k_{2,S}^+ \subset k_{2,S}$ denoting the subgroup consisting of totally positive elements, we conclude from Lemma 3.1 that $A_\xi = k_{2,\mathcal{R}}^+/(k^\times)^2$. This group contains $k_2^+/(k^\times)^2$ with index at most $2^{\#\mathcal{R}}$, and the order of the latter can be bounded by $h_k \cdot |U_k^+/U_k^2|$ by using the same argument as in the proof of [4, Prop. 0.12]. □

The following is obtained as a corollary of the proof.

Corollary 6.4. *Let $k = \mathbb{Q}$ and $\#\mathcal{R} = 1$, and assume P as above. Then $[\Gamma : \Lambda_P] = 2$.*

Proof. Let $\mathcal{R} = \{p\}$, with $p > 0$. Lemma 6.3 shows $[\Gamma : \Lambda_P] \leq 2$, but on the other hand, p provides a nontrivial element in $\mathbb{Q}_{2,\mathcal{R}}^+/(\mathbb{Q}^\times)^2$. □

6.3. *The nonuniform lattices Γ_n^s*

By definition, $\Gamma_n^0 = \langle g, \mathrm{Sp}(n, 1, \mathscr{H}) \rangle$ with g of order 2 that normalizes $\mathrm{Sp}(n, 1, \mathscr{H})$. Let D be the defining algebra of Γ_n^0 (i.e., $D = \mathscr{H} \otimes \mathbb{Q}$). It ramifies precisely at $p = 2$ and $p = \infty$. Thus, we can apply Corollary 6.4, and it follows that Γ_n^0 coincides with the normalizer of $\mathrm{Sp}(n, 1, \mathscr{H})$ in $\mathrm{Sp}(n, 1)$. For n even, Corollary 6.2 then shows that Γ_n^0 is of minimal covolume in its commensurability class. On the other hand, Rohlfs' criterion [30, Satz 3.5] shows that Γ_n^0 is maximal (w.r.t. inclusion) for any $n > 1$.

For $n > 1$ odd, we construct Γ_n^1 as follows. Let $\mathbf{G}$ be the $\mathbb{Q}$-group that contains $\mathrm{Sp}(n, 1, \mathscr{H})$. We choose a coherent collection $P = (P_v)$ with P_v hyperspecial for $v \neq 2$, and $P_v = P_v^1$ of type $\Delta_v \setminus \{\alpha_1\}$

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for $v = 2$. Let $\Gamma_n^1 = N_{\mathbf{G}(\mathbb{R})}(\Lambda_P)$. By Lemma 6.1, it is of minimal covolume in its commensurability class. Moreover, by Corollary 6.4, we have $[\Gamma_n^1 : \Lambda_P] = 2$. In particular,

$$\chi(\Gamma_n^1) = \frac{e'(P_2^1)}{e'(P_2^0)}\chi(\Gamma_n^0), \tag{6.2}$$

from which we obtain the formula in equation (1.3) with Lemma 4.1.

The following proposition now implies—up to the uniqueness—Theorems 1 and 2.

Proposition 6.5. *Let* $\Gamma \subset \mathrm{Sp}(n,1)$ *be a nonuniform lattice of minimal covolume, and let* $s = (n \bmod 2)$. *Then* Γ *is commensurable to* Γ_n^s, *and they have the same covolume.*

Proof. We have seen that Γ_n^s is of minimal covolume in its commensurability class, so it suffices to prove the commensurability. By Section 2, Γ nonuniform is constructed as an arithmetic subgroup of $\mathbf{G} = \mathbf{U}(V,h)$ for (V,h) admissible, and V a vector space over a quaternion $\mathbb{Q}$-algebra D. Let $\mathcal{R}$ be the set of places v such that D_v ramifies. Being of minimal covolume, we may write $\Gamma = N_{\mathbf{G}(\mathbb{R})}(\Lambda_P)$, with P_v hyperspecial unless $v \in \mathcal{R}$ (by Lemma 6.1). Moreover, for $v \in \mathcal{R}$ the subgroup P_v is of maximal volume and thus of type $\Delta \setminus \{\alpha_s\}$ by Section 3.1. By Lemma 6.3 (with $k = \mathbb{Q}$), we have $[\Gamma : \Lambda_P] \leq 2^{\#\mathcal{R}}$. Together with equation (4.7), this gives

$$\begin{aligned} \chi(\Gamma) &\geq \frac{\chi(\Lambda_P)}{[\Gamma : \Lambda_P]} \\ &\geq (n+1) \prod_{v \in \mathcal{R}} \frac{e'(P_v)}{2} \prod_{j=1}^{n+1} \frac{\zeta(1-2j)}{2}. \end{aligned} \tag{6.3}$$

Only the middle factor in equation (6.3) depends on the choice of Λ_P, and it takes the smallest possible value for $\mathcal{R} = \{2\}$ (note that $\mathcal{R} = \emptyset$ cannot appear here; see Section 2.2). But in that case, this lower bound is precisely $\chi(\Gamma_n^s)$, whence $\chi(\Gamma) = \chi(\Gamma_n^s)$ (by minimality of $\chi(\Gamma)$). Since D is now ramified exactly at $p = 2$ and $p = \infty$, it has same defining algebra as Γ_n^0, and the commensurability follows from Section 2.3. □

6.4. The minimality of $\chi(\Delta_n)$

We now discuss the uniform case. Recall that the covolume of Δ_n has been discussed in Section 5.5. Lemma 6.3 (with $k = \mathbb{Q}(\sqrt{5})$ and $\mathcal{R} = \emptyset$) implies that Δ_n coincides with its own normalizer in $\mathrm{Sp}(n,1)$. Corollary 6.2 thus implies that Δ_n is of minimal covolume in its commensurability class. The following proposition proves the first statement in Theorem 3. In the proof, we omit details that should be clear from the proof of Proposition 6.5.

Proposition 6.6. *Let* $\Gamma \subset \mathrm{Sp}(n,1)$ *be a uniform lattice of minimal covolume. Then* Γ *is commensurable to* Δ_n, *and they have the same covolume.*

Proof. Let Γ be a uniform lattice of minimal covolume, and let k be its field of definition. Then k is totally real, of degree $d \geq 2$. Assume first that $k = \mathbb{Q}(\sqrt{5})$, and let D be the defining algebra of Γ. It is clear that any nontrivial local factor $e'(P_v)$ would contribute to increase the volume formula, and this shows that D does not ramify at any finite place; this implies that Γ is commensurable to Δ_n. Thus, it suffices to prove that $k = \mathbb{Q}(\sqrt{5})$, i.e., that $d = 2$ and $\mathscr{D}_k = 5$ (recall that $\mathscr{D}_k$ denotes the discriminant in absolute value).

Let $\mathbf{G}$ be the algebraic k-group used to construct the arithmetic subgroup Γ, and let us write $\Gamma = N_{\mathbf{G}(\mathbb{R})}(\Lambda_P)$ with $P = (P_v)$ a coherent collection of parahoric subgroups $P_v \subset \mathbf{G}(k_v)$ (of maximal

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volume). Combining equation (4.6) and Lemma 6.3, we find

$$\chi(\Gamma) \geq \frac{(n+1)\mathscr{D}_k^{\dim \mathbf{G}/2}}{2^{\#\mathcal{R}}\ h_k\ |U_k^+/U_k^2|}\, C(n)^d \prod_{v\in\mathcal{R}} e'(P_v) \prod_{j=1}^{n+1} \zeta_k(2j),$$

with

$$C(n) = \prod_{j=1}^{n+1} \frac{(2j-1)!}{(2\pi)^{2j}}. \tag{6.4}$$

We clearly have $\zeta_k(2j) > 1$, and $|U_k^+/U_k^2| \leq 2^{d-1}$ (by Dirichlet's unit theorem). Moreover, $e'(P_v) > 2$ for any $v \in \mathcal{R}$, so that the factor $2^{-\#\mathcal{R}}$ is compensated by the product of those local factors. We can use the bound $h_k \leq 16(\pi/12)^d \mathscr{D}_k$ (see [3, Sect. 7.2]: the argument given there for a non-totally real field ℓ provides the same bound for k; see [7, Sect. 15.2] for details). This gives

$$\chi(\Gamma) \geq \frac{(n+1)}{16\ \mathscr{D}_k\ 2^{d-1}} \left(\frac{12}{\pi}\right)^d \mathscr{D}_k^{\dim \mathbf{G}/2}\, C(n)^d.$$

On the other hand, we have

$$\begin{aligned}\chi(\Delta_n) &= (n+1)\, 5^{\dim \mathbf{G}/2}\, C(n)^2 \prod_{j=1}^{n+1} \zeta_{\mathbb{Q}(\sqrt{5})}(2j)\\ &< 1.2 \cdot (n+1)\, 5^{\dim \mathbf{G}/2}\, C(n)^2.\end{aligned}$$

Here we bound the product of zeta functions by the value 1.2 by adapting the proof in [2, p.760: proof of $(*)$] as follows (see *loc. cit.* for details):

$$\begin{aligned}\prod_{j=1}^{n+1} \zeta_{\mathbb{Q}(\sqrt{5})}(2j) &< \zeta_{\mathbb{Q}(\sqrt{5})}(2)\, \zeta_{\mathbb{Q}(\sqrt{5})}(4)\, \zeta_{\mathbb{Q}(\sqrt{5})}(6) \prod_{j=4}^{\infty} \left(1 + \frac{2}{2^{2j}}\right)^2\\ &< \zeta_{\mathbb{Q}(\sqrt{5})}(2)\, \zeta_{\mathbb{Q}(\sqrt{5})}(4)\, \zeta_{\mathbb{Q}(\sqrt{5})}(6)\, e^{1/48},\end{aligned}$$

and we evaluate this last bound with Pari/GP [25].

For the quotient, this gives

$$\frac{\chi(\Gamma)}{\chi(\Delta_n)} > \frac{1}{39\,\mathscr{D}_k} \left(\frac{12}{\pi}\right)^d \left(\frac{\mathscr{D}_k}{5}\right)^{\frac{(n+1)(2n+3)}{2}} \left(\frac{C(n)}{2}\right)^{d-2}. \tag{6.5}$$

Let us write $f(n, d, \mathscr{D}_k)$ for the bound on the right-hand side. We have to show that $f(n, d, \mathscr{D}_k) \geq 1$ unless $d = 2$ and $\mathscr{D}_k = 5$.

The constant $C(n)/2$ is larger than 1 for $n \geq 13$, and grows monotone from that point. Thus, $f(n, d, \mathscr{D}_k) \geq f(13, d, \mathscr{D}_k)$, and it suffices to consider the range $n \in \{2, \ldots, 13\}$. For $d = 2$, the smallest discriminant after 5 is $\mathscr{D}_k = 8$, and numerical evaluation shows that $f(n, 2, 8) > 1$ for all $n \leq 13$. This shows $d > 2$. For k totally real of degree $d = 3$, the lowest discriminant is $\mathscr{D}_k = 49$. Again, we check that $f(n, 3, 49) > 1$. And similarly with $d = 4$ and $\mathscr{D}_k \geq 725$.

It remains to exclude $d \geq 5$. In that case, we use the following bound due to Odlyzko (see [23, Tab. 4]): $\mathscr{D}_k > (6.5)^d$. Then equation (6.5) transforms into

$$\frac{\chi(\Gamma)}{\chi(\Delta_n)} > \frac{1}{39 \cdot 5} \left(\frac{12}{\pi}\right)^2 \left(\frac{(6.5)^2}{5}\right)^{\delta(n)} a(n)^{d-2}, \tag{6.6}$$

where $a(n) = (6/\pi)C(n)(6.5)^{\delta(n)}$ and $\delta(n) = \dim \mathbf{G}/2 - 1$. We check that $a(n) > 1$ for all $n \in \{2, \dots, 13\}$. The product preceding $a(n)$ in equation (6.6) is also easily seen to be (much) larger than 1. This finishes the proof. □

6.5. *The proof of Corollary 1*

We have that each of $\chi(\Delta_n)$, $\chi(\Gamma_n^s)$ and their quotients contains a factor $C(n)$ (which is given in (6.4)); see equation (4.6). For n large enough, it is easily seen that this factor grows faster than (say) $(2n+1)!$, i.e., it grows superexponentially. This implies immediately that $\chi(\Delta_n)$ and $\chi(\Gamma_n^s)$ grow superexponentially, as their remaining factors also increase with n. Moreover, the other factors appearing in $\chi(\Gamma_n^s)$ grow at most exponentially so that the $\chi(\Delta_n)/\chi(\Gamma_n^s)$ has a superexponentially growth as well.

7. Proof of the uniqueness

In this section, we complete the proof of Theorems 1–3 by showing the uniqueness statements.

7.1. *The surjectivity of the adjoint map*

We start by proving the following auxiliary result.

Lemma 7.1. *For $n > 1$ and $\mathbf{G}$ admissible for* $\mathrm{Sp}(n, 1)$*, the map $\pi : \mathbf{G}(\mathbb{R}) \to \overline{\mathbf{G}}(\mathbb{R})$ is surjective.*

Proof. We have to show that $\delta : \overline{\mathbf{G}}(\mathbb{R}) \to H^1(\mathbb{R}, \mathbf{C})$ has trivial image; see (3.2). Recall that $H^1(\mathbb{R}, \mathbf{C}) = \mathbb{R}^\times/(\mathbb{R}^\times)^2$. Let $\mathbf{G} = \mathbf{U}(V, h)$. By [17, Prop. 12.20 and Sect. 31.A], the image of δ corresponds (modulo squares) to elements $\alpha \in \mathbb{R}^\times$ such that $(V, \alpha h)$ is isomorphic to (V, h). For h of signature $(n, 1)$ with $n > 1$, this requires $\alpha > 0$, whence the result. □

7.2. *Counting the conjugacy classes*

Let $\Gamma \subset \mathrm{Sp}(n, 1)$ be a nonuniform (respectively, uniform) lattice that realizes the smallest covolume. Then by Section 6, we have $\Gamma = N_{\mathrm{Sp}(n,1)}(\Lambda_P)$, where $P = (P_v)$ is a coherent collection of parahoric subgroups $P_v \subset \mathbf{G}(k_v)$ of maximal volume for each $v \in \mathcal{V}_k^{\mathrm{f}}$, and $\mathbf{G}$ is precisely the admissible k-group that determines Γ_n^s (resp. Δ_n). For $v \in \mathcal{R}$, this determines the type Θ_v of P_v uniquely (see Lemma 6.1), and for $v \notin \mathcal{R}$ we have that Θ_v is one of the two conjugate hyperspecial types. These two hyperspecial types are conjugate by $\overline{\mathbf{G}}(k_v)$. Up to $\overline{\mathbf{G}}(k)$-conjugacy, the number of principal arithmetic subgroups Λ_P with such a type $\Theta = (\Theta_v)$ is given by the order of following *class group* (see, for instance, [3, Sect. 6.2]):

$$\mathfrak{C}_P = \frac{\prod'_{v \in \mathcal{V}_k^{\mathrm{f}}} H^1(k_v, \mathbf{C})}{\delta(\overline{\mathbf{G}}(k)) \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} \delta(\overline{P}_v)}, \tag{7.1}$$

where $\overline{P}_v \subset \overline{\mathbf{G}}(k_v)$ is the stabilizer of P_v, and in the numerator, $\prod'$ denotes the restricted product with respect to the collection of subgroups $\delta(\overline{P}_v)$.

By Lemma 7.1, this order also gives an upper bound on the number of $\mathbf{G}(\mathbb{R})$-conjugacy classes of Λ_P (note that $k \subset \mathbb{R}$). Thus, the uniqueness in Theorems 1–3 follows immediately from the following.

Proposition 7.2. *For $k = \mathbb{Q}(\sqrt{5})$ (respectively, $k = \mathbb{Q}$), we have $\mathfrak{C}_P = 1$.*

Proof. We have $H^1(k_v, \mathbf{C}) = k_v^\times/(k_v^\times)^2$. The subgroup $\delta(\overline{P}_v) \subset H^1(k_v, \mathbf{C})$ equals the type stabilizer $H^1(k_v, \mathbf{C})_{\Theta_v}$. If $v \in \mathcal{R}$, then $\mathrm{Aut}(\Delta_v) = 1$ (see Table 3) so that this stabilizer is trivially the whole

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$H^1(k_v, \mathbf{C})$. For $v \notin \mathcal{R}$, we have $\delta(\overline{P}_v) = \mathfrak{o}_v^\times (k_v^\times)^2/(k_v^\times)^2$ by [4, Prop. 2.7]. Thus, $\mathfrak{C}_P$ is a quotient of

$$\mathfrak{C}'_P = \frac{\prod'_{v \in \mathcal{V}_k^{\mathrm{f}}} k_v^\times/(k_v^\times)^2}{\delta(\overline{\mathbf{G}}(k)) \prod_{v \in \mathcal{V}_k^{\mathrm{f}}} \mathfrak{o}_v^\times (k_v^\times)^2/(k_v^\times)^2}. \tag{7.2}$$

Now by Lemma 7.1 and the proof of Lemma 3.1, we have that $\delta(\overline{\mathbf{G}}(k)) = \delta(\overline{\mathbf{G}}(k))' = k^{(+)}/(k^\times)^2$, where

$$k^{(+)} = \left\{ x \in k^\times \mid x_v > 0 \quad \forall v \in \mathcal{V}_k^\infty \right\}. \tag{7.3}$$

From equation (7.2), we obtain an isomorphism

$$\mathfrak{C}'_P \cong \mathcal{J}_k / \left(k^{(+)} \mathcal{J}_k^\infty \mathcal{J}_k^2 \right),$$

where $\mathcal{J}_k$ is the group of finite idèles of k, and $\mathcal{J}_k^\infty \subset \mathcal{J}_k$ its subgroup consisting of integral idèles. For both $k = \mathbb{Q}$ and $k = \mathbb{Q}(\sqrt{5})$, the unit group U_k contains a representative of each class of $k^\times/k^{(+)}$. Thus,

$$\begin{aligned} k^{(+)} \mathcal{J}_k^\infty &= k^{(+)} U_k \mathcal{J}_k^\infty \\ &= k^\times \mathcal{J}_k^\infty \end{aligned}$$

so that

$$\mathfrak{C}'_P \cong \mathcal{J}_k / \left(k^\times \mathcal{J}_k^\infty \mathcal{J}_k^2 \right).$$

But the latter is a quotient of the class group of k (see [27, Sect. 1.2.1]), which is trivial here. □

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Conflicts of Interest. None

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