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SOME CONNECTIONS BETWEEN AN OPERATOR AND ITS ALUTHGE TRANSFORM

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Abstract. Associated with $T = U|T|$ (polar decomposition) in $\mathcal{L}(\mathbf{H})$ is a related operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, called the Aluthge transform of T. In this paper we study some connections between T and $\tilde{T}$, including the following relations; the single valued extension property, an analogue of the single valued extension property on $W^m(D, \mathbf{H})$, Dunford's property (C) and the property (β).

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Let $\mathbf{H}$ be a complex Hilbert space, and denote by $\mathcal{L}(\mathbf{H})$ the algebra of all bounded linear operators on $\mathbf{H}$. If $T \in \mathcal{L}(\mathbf{H})$, we write $\sigma(T)$, $\sigma_{ap}(T)$, and $\sigma_p(T)$ for the spectrum, the approximate point spectrum, and the point spectrum of T, respectively.

An arbitrary operator $T \in \mathcal{L}(\mathbf{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $ker U = ker|T| = ker T$ and $ker U^* = ker T^*$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, called the *Aluthge transform of* T, and denoted throughout this paper by $\tilde{T}$.

An operator $T \in \mathcal{L}(\mathbf{H})$ is said to be *p-hyponormal*, where $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$, where T^* is the adjoint of T. In particular, if $p = 1$, T is called *hyponormal*. There is a vast literature concerning p-hyponormal operators.

An operator $T \in \mathcal{L}(\mathbf{H})$ is said to satisfy the *single-valued extension property* if for any open subset V in $\mathbf{C}$, the function

$$T - \lambda : \mathcal{O}(V, \mathbf{H}) \longrightarrow \mathcal{O}(V, \mathbf{H})$$

defined by the obvious pointwise multiplication, is one-to-one. Here $\mathcal{O}(V, \mathbf{H})$ denotes the Fréchet space of $\mathbf{H}$-valued analytic functions on V with respect to uniform topology. If T has the single valued extension property, then for any $x \in \mathbf{H}$ there exists a unique maximal open set $\rho_T(x)$ ($\supset \rho(T)$, the resolvent set) and a unique $\mathbf{H}$-valued analytic function f defined in $\rho_T(x)$ such that

$$(T - \lambda)f(\lambda) = x \quad (\lambda \in \rho_T(x)).$$

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In the following theorem we show that Aluthge transforms preserve the single valued extension property.

THEOREM 1.1. *An operator T with polar decomposition $U|T|$ has the single valued extension property if and only if $\tilde{T}$ has.*

Proof. Assume that T has the single valued extension property. Suppose that W is an open subset of $\mathbf{C}$ and $f : W \to \mathbf{H}$ is an analytic function satisfying $(\tilde{T} - \lambda)f(\lambda) = 0$, for each $\lambda \in W$. Since $T(U|T|^{\frac{1}{2}}) = (U|T|^{\frac{1}{2}})\tilde{T}$,

$$(T - \lambda)U|T|^{\frac{1}{2}}f(\lambda) = U|T|^{\frac{1}{2}}(\tilde{T} - \lambda)f(\lambda) = 0,$$

for each $\lambda \in W$. Since T has the single valued extension property, $U|T|^{\frac{1}{2}}f(\lambda) = 0$ for each $\lambda \in W$. Since $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, $\tilde{T}f(\lambda) = 0$ for each $\lambda \in W$. Since $(\tilde{T} - \lambda)f(\lambda) = 0$ for each $\lambda \in W$, $\lambda f(\lambda) = 0$ for each $\lambda \in W$. Since $f(\lambda) = 0$ on $W\backslash\{0\}$ and is analytic on W, f is identically 0 on W. Therefore, $\tilde{T}$ has the single valued extension property.

The proof of the converse implication is similar. □

The following corollary shows the relationships between the local spectra of T and $\tilde{T}$.

COROLLARY 1.2. *If an operator T with polar decomposition $U|T|$ has the single valued extension property, then*

$$\sigma_{\tilde{T}}\big(|T|^{\frac{1}{2}}x\big) \subset \sigma_T(x) \quad \text{and} \quad \sigma_T\big(U|T|^{\frac{1}{2}}x\big) \subset \sigma_{\tilde{T}}(x).$$

Proof. For $\lambda \in \rho_T(x)$, we have $(T - \lambda)x(\lambda) \equiv x$, where $\lambda \to x(\lambda)$ is the analytic function defined on $\rho_T(x)$. Since $|T|^{\frac{1}{2}}T = \tilde{T}|T|^{\frac{1}{2}}$,

$$(\tilde{T} - \lambda)|T|^{\frac{1}{2}}x(\lambda) = |T|^{\frac{1}{2}}(T - \lambda)x(\lambda) \equiv |T|^{\frac{1}{2}}x.$$

Hence $\rho_T(x) \subset \rho_{\tilde{T}}(|T|^{\frac{1}{2}}x)$, so that $\sigma_{\tilde{T}}(|T|^{\frac{1}{2}}x) \subset \sigma_T(x)$.

Similarly, we can prove the second inclusion. □

COROLLARY 1.3. *If an operator T with polar decomposition $U|T|$ has the single valued extension property, then*

$$|T|^{\frac{1}{2}}H_T(F) \subseteq H_{\tilde{T}}(F) \quad \text{and} \quad U|T|^{\frac{1}{2}}H_{\tilde{T}}(F) \subseteq H_T(F),$$

where $H_T(F) = \{x \in \mathbf{H} : \sigma_T(x) \subseteq F\}$ for $F \subset \mathbf{C}$.

Proof. If $x \in H_T(F)$, then $\sigma_T(x) \subseteq F$. By Corollary 1.2, we get $\sigma_{\tilde{T}}(|T|^{\frac{1}{2}}x) \subseteq F$. Hence $|T|^{\frac{1}{2}}x \in H_{\tilde{T}}(F)$. Thus $|T|^{\frac{1}{2}}H_T(F) \subseteq H_{\tilde{T}}(F)$.

Similarly, we get $U|T|^{\frac{1}{2}}H_{\tilde{T}}(F) \subseteq H_T(F)$. □

Our next result shows that the Aluthge transform preserves an analogue of the single valued extension property for $W^m(D, \mathbf{H})$ and an operator T on $\mathbf{H}$; that is, $T - \lambda : W^m(D, \mathbf{H}) \to W^m(D, \mathbf{H})$ is one-to-one if and only if $\tilde{T} - \lambda$ is. First of all, let us define a special Sobolev type space. Let D be a bounded open subset of $\mathbf{C}$ and m a fixed non-negative integer. The vector valued Sobolev space $W^m(D, \mathbf{H})$ with respect to $\bar{\partial}$ and order m will be the space of those functions $f \in L^2(D, \mathbf{H})$ whose derivatives $\bar{\partial}f, \cdots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(D, \mathbf{H})$.

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Endowed with the norm

$$\|f\|_{W^m}^2 = \sum_{i=0}^{m} \|\bar{\partial}^m f\|_{2,D}^2,$$

$W^m(D, \mathbf{H})$ becomes a Hilbert space contained continuously in $L^2(D, \mathbf{H})$.

THEOREM 1.4. *Let $T = U|T|$ be the polar decomposition of T in $\mathcal{L}(\mathbf{H})$ and let D be an arbitrary bounded disk containing $\sigma(T) \cup \{0\}$ in $\mathbf{C}$. Then $T - \lambda : W^2(D, \mathbf{H}) \to W^2(D, \mathbf{H})$ is one-to-one if and only if $\tilde{T} - \lambda : W^2(D, \mathbf{H}) \to W^2(D, \mathbf{H})$ is one-to-one.*

Proof. Assume $T - \lambda$ is one-to-one. If $f \in W^2(D, \mathbf{H})$ is such that $(\tilde{T} - \lambda)f = 0$, then $(T - \lambda)U|T|^{\frac{1}{2}} f = 0$. By the hypothesis, $U|T|^{\frac{1}{2}} = 0$. Hence $\tilde{T}f = 0$. Thus $\lambda f = 0$; i.e., $\lambda \bar{\partial}^i f = 0$ for $i = 0, 1, 2$. By applications of [**9**, Proposition 3.2] with $T = 0$, we get

$$\|(I - P)f\|_{2,D} \leq C_D(\| - \lambda\bar{\partial} f\|_{2,D} + \| - \lambda\bar{\partial}^2 f\|_{2,D}), \tag{1}$$

where P denotes the orthogonal projection of $L^2(D, \mathbf{H})$ onto the Bergman space $A^2(D, \mathbf{H})$. From (1) we have $f = Pf$. Hence $\lambda f = \lambda Pf = 0$. From [**3**, Corollary 10.7], there exists a constant $c > 0$ such that

$$c\|Pf\|_{2,D} \leq \|\lambda Pf\|_{2,D}.$$

Hence $f = Pf = 0$.

Conversely, if $\tilde{T} - \lambda$ is one-to-one, we can prove the required result by the same argument. □

The following corollary shows that, for every p-hyponormal operator T, the equality *supp*$((T - \lambda)f) =$ *supp*(f) holds for every $f \in W^2(D, \mathbf{H})$.

COROLLARY 1.5. *If T is p-hyponormal, then the operator $T - \lambda : W^2(D, \mathbf{H}) \to W^2(D, \mathbf{H})$ is one-to-one.*

Proof. Since $\tilde{\tilde{T}}$ is hyponormal by [**1**], it is known from [**9**] that $\tilde{\tilde{T}} - \lambda$ is one-to-one. By two applications of Theorem 1.4 we conclude that $T - \lambda$ is one-to-one. □

COROLLARY 1.6. *If an operator $T \in \mathcal{L}(\mathbf{H})$ satisfies $T = S + N$, where S is p-hyponormal, S and N commute, and $N^m = 0$, then $T - \lambda$ is one-to-one on $W^2(D, \mathbf{H})$.*

Proof. Let $f \in W^2(D, \mathbf{H})$ be such that $(T - \lambda)f = 0$. Then

$$(S - \lambda)f = -Nf. \tag{2}$$

Hence $(S - \lambda)N^{j-1}f = -N^j f$ for $j = 1, 2, \ldots, m$. We prove that $N^j f = 0$ for $j = 0, 1, \ldots, m - 1$ by induction. Since $N^m = 0$,

$$(S - \lambda)N^{m-1}f = -N^m f = 0.$$

Since $S - \lambda$ is one-to-one from Corollary 1.5, $N^{m-1}f = 0$. Assume it is true when $j = k$, i.e., $N^k f = 0$. From (2), we get

$$(S - \lambda)N^{k-1}f = -N^k f = 0.$$

Since $S - \lambda$ is one-to-one from Corollary 1.5, $N^{k-1}f = 0$. By induction, we have $f = 0$. Hence $T - \lambda$ is one-to-one. □

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The following theorem shows that if $\lim_{n\to\infty}\|(T-\lambda)f_n\|_{W^m}=0$, then we cannot obtain by the same method more than $\lim_{n\to\infty}\|f_n\|_{W^{m-2}}=0$ for $m\ge 2$.

THEOREM 1.7. *Let $T=U|T|$ be the polar decomposition of T in $\mathcal{L}(\mathbf{H})$ and let D be an arbitrary bounded disk containing $\sigma(T)\cup\{0\}$ in $\mathbf{C}$. Assume that $\tilde{T}-\lambda : W^m(D,\mathbf{H})\to W^m(D,\mathbf{H})$ is bounded below. If f_n is a sequence in $W^m(D,\mathbf{H})$ such that we have $\lim_{n\to\infty}\|(T-\lambda)f_n\|_{W^m}=0$, then $\lim_{n\to\infty}\|f_n\|_{W^{m-2}}=0$ for $m\ge 2$.*

Proof. If f_n is a sequence in $W^m(D,\mathbf{H})$ such that $\lim_{n\to\infty}\|(T-\lambda)f_n\|_{W^m}=0$, then by the definition of the norm in $W^m(D,\mathbf{H})$ we have

$$\lim_{n\to\infty}\|(T-\lambda)\bar{\partial}^i f_n\|_{2,D}=0 \tag{3}$$

for $i=0,1,\dots,m$. Since $|T|^{1/2}T=\tilde{T}|T|^{1/2}$, we get

$$\lim_{n\to\infty}\left\|(\tilde{T}-\lambda)|T|^{\frac{1}{2}}\bar{\partial}^i f_n\right\|_{2,D}=0$$

for $i=0,1,\dots,m$. Since $\tilde{T}-\lambda$ is bounded below, we have

$$\lim_{n\to\infty}\left\||T|^{\frac{1}{2}}\bar{\partial}^i f_n\right\|_{2,D}=0$$

for $i=0,1,\dots,m$. Since $T=U|T|$, we get

$$\lim_{n\to\infty}\|T\bar{\partial}^i f_n\|_{2,D}=0 \tag{4}$$

for $i=0,1,\dots,m$. Hence by (3) and (4) we obtain

$$\lim_{n\to\infty}\|\lambda\bar{\partial}^i f_n\|_{2,D}=0 \tag{5}$$

for $i=0,1,\dots,m$. By an application of [**7**, Proposition 2.2] with $T=0$,

$$\lim_{n\to\infty}\|(I-P)\bar{\partial}^i f_n\|_{2,D}=0 \tag{6}$$

for $i=0,1,\dots,m-2$, where P denotes the orthogonal projection of $L^2(D,\mathbf{H})$ onto the Bergman space $A^2(D,\mathbf{H})=L^2(D,\mathbf{H})\cap\mathcal{O}(U,\mathbf{H})$. Then (5) and (6) imply that

$$\lim_{n\to\infty}\|\lambda P\bar{\partial}^i f_n\|_{2,D}=0$$

for $i=0,1,\dots,m-2$. Since $\lambda P\bar{\partial}^i f_n$ is bounded below, by [**3**, Corollary 10.7], we get

$$\lim_{n\to\infty}\|P\bar{\partial}^i f_n\|_{2,D}=0 \tag{7}$$

for $i=0,1,\dots,m-2$. By (6) and (7) we conclude that $\lim_{n\to\infty}\|f_n\|_{W^{m-2}}=0$. □

Next we show that Aluthge transforms preserve the finite ascent except for $\lambda=0$.

THEOREM 1.8. *For arbitrary $\lambda\in\mathbf{C}\backslash\{0\}$, $ker(T-\lambda)^n=ker(T-\lambda)^{n+1}$ if and only if $ker(\tilde{T}-\lambda)^n=ker(\tilde{T}-\lambda)^{n+1}$, for some $n\in\mathbf{N}$.*

Proof. Assume that for all $\lambda\in\mathbf{C}\backslash\{0\}$, there is an $n\in\mathbf{N}$ such that $ker(T-\lambda)^n=ker(T-\lambda)^{n+1}$. Since $ker(\tilde{T}-\lambda)^n\subset ker(\tilde{T}-\lambda)^{n+1}$, it suffices to show that

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$ker\,(\tilde{T}-\lambda)^n \supset ker(\tilde{T}-\lambda)^{n+1}$. Let $x \in ker\,(\tilde{T}-\lambda)^{n+1}$. Since $T(U|T|^{\frac{1}{2}}) = (U|T|^{\frac{1}{2}})\tilde{T}$,

$$(T-\lambda)^{n+1}U|T|^{\frac{1}{2}}x = U|T|^{\frac{1}{2}}(\tilde{T}-\lambda)^{n+1}x = 0.$$

Therefore, $U|T|^{\frac{1}{2}}x \in ker(T-\lambda)^{n+1} = ker(T-\lambda)^n$. Since

$$U|T|^{\frac{1}{2}}(\tilde{T}-\lambda)^n x = (T-\lambda)^n U|T|^{\frac{1}{2}}x = 0,$$

$\tilde{T}(\tilde{T}-\lambda)^n x = 0$. We obtain $\lambda(\tilde{T}-\lambda)^n x = 0$. Since $\lambda \neq 0$, $(\tilde{T}-\lambda)^n x = 0$.

The proof of the converse implication is similar. □

THEOREM 1.9. *Let $T \in \mathcal{L}(\mathbf{H})$ have polar decomposition $U|T|$. Then for all nonzero $\lambda \in \mathbf{C}$, $ran(T-\lambda)$ is closed if and only if $ran(\tilde{T}-\lambda)$ is closed.*

Proof. Assume that $ran(\tilde{T}-\lambda)$ is closed, for all nonzero $\lambda \in \mathbf{C}$. If $y \in \overline{ran(T-\lambda)}$, for all nonzero $\lambda \in \mathbf{C}$, then there exists a sequence $\{x_n\}$ in $\mathbf{H}$ such that

$$\lim_{n\to\infty}(T-\lambda)x_n = y.$$

Since $|T|^{\frac{1}{2}}T = \tilde{T}|T|^{\frac{1}{2}}$, we have

$$\lim_{n\to\infty}(\tilde{T}-\lambda)|T|^{\frac{1}{2}}x_n = \lim_{n\to\infty}|T|^{\frac{1}{2}}(T-\lambda)x_n = |T|^{\frac{1}{2}}y.$$

Since $ran(\tilde{T}-\lambda)$ is closed, for all nonzero $\lambda \in \mathbf{C}$, there exists a $z \in \mathbf{H}$ such that

$$\lim_{n\to\infty}(\tilde{T}-\lambda)|T|^{\frac{1}{2}}x_n = (\tilde{T}-\lambda)z.$$

Since the limit is unique, $(\tilde{T}-\lambda)z = |T|^{\frac{1}{2}}y$. Thus $\tilde{T}z = |T|^{\frac{1}{2}}y + \lambda z$. Set $w = U|T|^{\frac{1}{2}}z - y$. Then

$$|T|^{\frac{1}{2}}w = \tilde{T}z - |T|^{\frac{1}{2}}y = \lambda z.$$

Hence we get

$$\begin{aligned}(T-\lambda)w &= U|T|^{\frac{1}{2}}\big(|T|^{\frac{1}{2}}w\big) - \lambda w \\ &= U|T|^{\frac{1}{2}}(\lambda z) - \lambda\big(U|T|^{\frac{1}{2}}z - y\big) \\ &= \lambda y.\end{aligned}$$

Since λ is nonzero,

$$(T-\lambda)\left(\frac{w}{\lambda}\right) = y.$$

Hence $y \in ran(T-\lambda)$. Thus $ran(T-\lambda)$ is closed, for all nonzero $\lambda \in \mathbf{C}$.

The proof of the converse is similar. □

COROLLARY 1.10. *For all nonzero $\lambda \in \mathbf{C}$, $T-\lambda$ is bounded below if and only if $\tilde{T}-\lambda$ is.*

Proof. Let $T = U|T|$ be the polar decomposition of T. If $T-\lambda$ is bounded below for all nonzero $\lambda \in \mathbf{C}$, then it is one-to-one and has closed range. From Theorem 1.9,

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it suffices to show that $\tilde{T} - \lambda$ is one-to-one. If $(\tilde{T} - \lambda)x = 0$, then $(T - \lambda)U|T|^{\frac{1}{2}}x = 0$. Hence $U|T|^{\frac{1}{2}}x = 0$, i.e., $\tilde{T}x = 0$. Since $\lambda \neq 0$, $x = 0$.

The proof of the converse is similar. □

The following theorem shows that the Aluthge transform preserves the finite descent except for $\lambda = 0$.

THEOREM 1.11. *For all nonzero $\lambda \in \mathbf{C}$, $ran(T - \lambda)^n = ran(T - \lambda)^{n+1}$ if and only if $ran(\tilde{T} - \lambda)^n = ran(\tilde{T} - \lambda)^{n+1}$ for some $n \in \mathbf{N}$.*

Proof. Assume that $ran(T - \lambda)^n = ran(T - \lambda)^{n+1}$ for some $n \in \mathbf{N}$ and for all nonzero $\lambda \in \mathbf{C}$. Since $ran(\tilde{T} - \lambda)^n \supset ran(\tilde{T} - \lambda)^{n+1}$, it suffices to show that $ran(\tilde{T} - \lambda)^n \subset ran(\tilde{T} - \lambda)^{n+1}$. If $y \in ran(\tilde{T} - \lambda)^n$, there exists an $x \in \mathbf{H}$ such that $y = ran(\tilde{T} - \lambda)^n x$. Since $U|T|^{\frac{1}{2}}\tilde{T} = TU|T|^{\frac{1}{2}}$,

$$U|T|^{\frac{1}{2}}y = (T - \lambda)^n U|T|^{\frac{1}{2}}x.$$

Since $U|T|^{\frac{1}{2}}y \in ran(T - \lambda)^n = ran(T - \lambda)^{n+1}$, there exists a $z \in \mathbf{H}$ such that $\tilde{T}y = |T|^{\frac{1}{2}}(T - \lambda)^{n+1}z = (\tilde{T} - \lambda)^{n+1}|T|^{\frac{1}{2}}z$. Hence $\tilde{T}y \in ran(\tilde{T} - \lambda)^{n+1}$ and so there exists an $s \in \mathbf{H}$ such that $\tilde{T}y = (\tilde{T} - \lambda)^{n+1}s$. Set $w = (\tilde{T} - 2\lambda)s - (\tilde{T} - \lambda)^2 s$. Then

$$(\tilde{T} - \lambda)^{n+1}w = -\lambda^2 y.$$

Since $\lambda \neq 0$,

$$(\tilde{T} - \lambda)^{n+1}\left(-\frac{w}{\lambda^2}\right) = y.$$

Hence $y \in ran(\tilde{T} - \lambda)^{n+1}$.

The proof of the converse is similar. □

Suppose that $T \in \mathcal{L}(\mathbf{H})$ has the single valued extension property. The operator T is said to satisfy *Dunford's property* (C) if the linear submanifold

$$H_T(F) := \{x \in \mathbf{H} : \sigma_T(x) \subseteq F\}$$

is closed, for each closed subset F of $\mathbf{C}$, where $\sigma_T(x) := \mathbf{C} \backslash \rho_T(x)$.

The following theorem shows that Aluthge transforms preserve Dunford's property (C) in some cases.

Recall that an operator $X \in \mathcal{L}(\mathbf{H}, \mathbf{K})$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator $A \in \mathcal{L}(\mathbf{H})$ is said to be a *quasiaffine transform* of an operator $T \in \mathcal{L}(\mathbf{K})$ if there is a quasiaffinity $X \in \mathcal{L}(\mathbf{H}, \mathbf{K})$ such that $XA = TX$. Furthermore, operators A and T are said to be *quasisimilar* if there are quasiaffinities X and Y such that $XA = TX$ and $AY = YT$.

THEOREM 1.12. *If T, with polar decomposition $U|T|$ is a quasiaffinity in $\mathcal{L}(\mathbf{H})$, then T satisfies Dunford's property (C) if and only if $\tilde{T}$ does.*

Proof. Assume that T satisfies Dunford's property (C). Consider

$$H_{\tilde{T}}(F) := \{x \in \mathbf{H} : \sigma_{\tilde{T}}(x) \subseteq F\},$$

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for every closed subset F of $\mathbf{C}$. Since $\tilde{T}$ has the single valued extension property from Theorem 1.1, it suffices to show that $H_{\tilde{T}}(F)$ is closed. If $x \in \overline{H_{\tilde{T}}(F)}$, then there exist a sequence $\{x_n\}$ in $H_{\tilde{T}}(F)$ such that $x_n \to x$. Since $x_n \in H_{\tilde{T}}(F)$, $\sigma_{\tilde{T}}(x_n) \subseteq F$. For any $\lambda \in F^c$ we have $\lambda \in \rho_{\tilde{T}}(x_n)$. Hence $(\tilde{T} - \lambda)x_n(\lambda) \equiv x_n$, where $\lambda \to x_n(\lambda)$ is the analytic function defined on $\rho_{\tilde{T}}(x_n)$. Since $U|T|^{\frac{1}{2}}\tilde{T} = TU|T|^{\frac{1}{2}}$,

$$(T - \lambda)U|T|^{\frac{1}{2}}x_n(\lambda) \equiv U|T|^{\frac{1}{2}}x_n.$$

Hence $\lambda \in \rho_T(U|T|^{\frac{1}{2}}x_n)$. Thus $\sigma_T(U|T|^{\frac{1}{2}}x_n) \subseteq F$. Therefore,

$$U|T|^{\frac{1}{2}}x_n \in H_T(F).$$

Since $H_T(F)$ is closed by hypothesis, $U|T|^{\frac{1}{2}}x \in H_T(F)$. For any $\lambda \in F^c$, we have

$$(T - \lambda)U|T|^{\frac{1}{2}}x(\lambda) \equiv U|T|^{\frac{1}{2}}x.$$

Since $U|T|^{\frac{1}{2}}\tilde{T} = TU|T|^{\frac{1}{2}}$, we have

$$U|T|^{\frac{1}{2}}(\tilde{T} - \lambda)x(\lambda) \equiv U|T|^{\frac{1}{2}}x.$$

Since T is a quasiaffinity, we get

$$(\tilde{T} - \lambda)x(\lambda) \equiv x.$$

Thus $\lambda \in \rho_{\tilde{T}}(x)$. Hence $\sigma_{\tilde{T}}(x) \subseteq F$.

The proof of the converse implication is similar. □

An operator $T \in \mathcal{L}(\mathbf{H})$ is called *decomposable* if for every finite open covering $\{G_1, \ldots, G_n\}$ of $\mathbf{C}$ there exists a system $\{Y_1, \ldots, Y_n\}$ of spectral maximal subspaces of T such that $\mathbf{H} = Y_1 + \cdots + Y_n$ and $\sigma(T|_{Y_i}) \subset G_i$ for every $1 \le i \le n$. As one of the generalized concepts of decomposability, we define the following; an operator $T \in \mathcal{L}(\mathbf{H})$ is *quasidecomposable* if T has Dunford's property (C) and satisfies the condition that for every finite open covering $\{G_1, \ldots, G_n\}$ of $\mathbf{C}$ there corresponds a system $\{Y_1, \ldots, Y_n\}$ of T-invariant subspaces such that $\mathbf{H} = \vee_{i=1}^{n} Y_i$ and $\sigma(T|_{Y_i}) \subset G_i$ for every $1 \le i \le n$. As an application of Theorem 1.7 we have the following corollary.

Corollary 1.13. *Let T with polar decomposition $U|T|$ be a quasiaffinity in $\mathcal{L}(\mathbf{H})$. If $\tilde{T}$ is decomposable, then T is quasidecomposable.*

Proof. If $\tilde{T}$ is decomposable, it has Dunford's property (C) from [**8**]. Then T has Dunford's property (C), by Theorem 1.12. Since $TU|T|^{\frac{1}{2}} = U|T|^{\frac{1}{2}}\tilde{T}$, Corollary 1.3 implies that

$$U|T|^{\frac{1}{2}}H_{\tilde{T}}(F) \subset H_T(F),$$

for each closed F. Let $\{G_1, \ldots, G_n\}$ be an open cover of $\mathbf{C}$. Then

$$\mathbf{H} = H_{\tilde{T}}(\bar{G}_1) + \cdots + H_{\tilde{T}}(\bar{G}_n).$$

Since $\overline{U|T|^{\frac{1}{2}}\mathbf{H}} = \mathbf{H}$, we have

$$U|T|^{\frac{1}{2}}H_{\tilde{T}}(\bar{G}_1) + \cdots + U|T|^{\frac{1}{2}}H_{\tilde{T}}(\bar{G}_n) \subset H_T(\bar{G}_1) + \cdots + H_T(\bar{G}_n).$$

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Hence

$$\mathbf{H} = \overline{U|T|^{\frac{1}{2}}\mathbf{H}} = \overline{U|T|^{\frac{1}{2}}[H_{\tilde{T}}(\bar{G}_1) + \cdots + H_{\tilde{T}}(\bar{G}_n)]}$$
$$\subset \overline{H_T(\bar{G}_1) + \cdots + H_T(\bar{G}_n)}.$$

Thus

$$\mathbf{H} = \vee_{i=1}^{n} H_T(\bar{G}_i).$$

Since T has Dunford's property (C), by [**2**, Proposition 3.8]

$$\sigma\left(T|_{H_T(\bar{G}_i)}\right) \subset \bar{G}_i,$$

for each i, so that T is quasidecomposable. □

An operator $T \in \mathcal{L}(\mathbf{H})$ is said to satisfy the *property* (β) if for every open subset G of $\mathbf{C}$ and every sequence $f_n : G \longrightarrow \mathbf{H}$ of $\mathbf{H}$-valued analytic functions such that $(T - \lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of $G, f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G.

The following theorem shows that Aluthge transforms preserve the property (β).

THEOREM 1.14. *An operator T with polar decomposition $U|T|$ satisfies the property (β) if and only if an operator $\tilde{T}$ does.*

Proof. Assume T satisfies the property (β). Let $f_n \in \mathcal{O}(V, \mathbf{H})$ be such that $(\tilde{T} - \lambda)f_n(\lambda)$ converges uniformly to 0 on compact subsets G of V. Since $T(U|T|^{\frac{1}{2}}) = (U|T|^{\frac{1}{2}})\tilde{T}$, $(T - \lambda)U|T|^{\frac{1}{2}}f_n(\lambda)$ converges uniformly to 0 for all $\lambda \in G$. Since T satisfies the property (β), $U|T|^{\frac{1}{2}}f_n(\lambda)$ converges uniformly to 0 for all $\lambda \in G$. Since $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, $\tilde{T}f_n(\lambda)$ converges uniformly to 0 for all $\lambda \in G$. Hence $\lambda f_n(\lambda)$ converges uniformly to 0 for all $\lambda \in G$. Since 0 is hyponormal and hyponormal operators satisfy the property (β), $f_n(\lambda)$ converges uniformly to 0 for all $\lambda \in G$. Hence $\tilde{T}$ satisfies the property (β).

The proof of the converse is similar. □

COROLLARY 1.15. *If $\tilde{T}$ is algebraic (i.e., $p(\tilde{T}) = 0$ for some nonzero polynomial p), then $T = U|T|$ (polar decomposition) satisfies the property (β).*

Proof. If $\tilde{T}$ is algebraic, then it satisfies the property (β) by [**6**]. Hence, by Theorem 1.14, T satisfies the property (β). □

As an application of Theorem 1.14, we have the following corollary.

COROLLARY 1.16. *If T is p-hyponormal, then it satisfies the property (β).*

Proof. Since $\tilde{\tilde{T}}$ is hyponormal by [**1**], it satisfies the property (β). Hence from two applications of Theorem 1.14, T satisfies the property (β). □

COROLLARY 1.17. *Suppose that T is p-hyponormal and S satisfies the property (β). If S and T are quasisimilar, then S satisfies Weyl's theorem (i.e., $\sigma(T) - \omega(T) = \pi_{00}(T)$, where $\pi_{00}(T)$ denotes the set of all eigenvalues of finite multiplicity of T and $\omega(T)$ denotes the Weyl spectrum of T).*

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Proof. Since T satisfies the property (β), by Corollary 1.16, [**10**] implies that S satisfies Weyl's theorem. □

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