GENERALIZED CASIMIR OPERATORS

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Throughout this paper, S will be a ring (not necessarily commutative) with an identity element $1_S \neq 0_S$. We shall use R to denote a second ring, and $\phi: S \to R$ will be a fixed ring homomorphism for which $\phi 1_S = 1_R$.

1. Introduction. In (**7**), Higman generalized the Casimir operator of classical theory and used his generalization to characterize relatively projective and injective modules. As a special case, he obtained a theorem which contains results of Eckmann (**3**) and of Higman himself (**5**), and which also includes Gaschütz's generalization (**4**) of Maschke's theorem. (For a discussion of some of the developments of Maschke's idea of averaging over a finite group, we refer the reader to (**2**, Chapter IX).) In the present paper, we define the Casimir operator of a family of S-homomorphisms of one R-module into another, and we again use this operator to characterize relatively projective and injective modules. In § 4, we give some special cases, the first of which covers the result of Higman (**7**) referred to above.

In § 5, we extend (**7**, Theorem 6) for a special class of pairs R, S. Our result contains a theorem of Popescu (**9**, Proposition 1.3) which in turn generalizes a result of Cartan and Eilenberg (**1**, Chapter IV, Proposition 2.3) on the ring of dual numbers.

2. Relatively projective and injective modules. An abelian group M which is both a left and a right S-module and for which

$$(su)s' = s(us'), \qquad s, s' \in S,\ u \in M,$$

will be referred to as an *S-bimodule.*

A left R-module M may be treated as a left S-module by putting

$$su = (\phi s)u, \qquad s \in S,\ u \in M,$$

and similarly for right modules. In particular, R itself may be regarded as a left or right S-module.

When M is a left S-module and X is an S-bimodule, the tensor product $X \otimes_S M$ may be considered as a left S-module by taking

$$s(x \otimes u) = sx \otimes u, \qquad s \in S, x \in X, u \in M.$$

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Furthermore, the abelian group $\mathrm{Hom}_S(X, M)$ of S-homomorphisms of X into M may be regarded as a left S-module by putting

$$(sf)x = f(xs), \qquad s \in S, f \in \mathrm{Hom}_S(X, M), x \in X.$$

Suppose now that M is a left S-module; it follows from what we said above that $R \otimes_S M$ may be considered as a left S-module. In fact, it may be regarded as a left R-module by taking, in addition,

$$r_1(r_2 \otimes u) = r_1 r_2 \otimes u, \qquad r_1, r_2 \in R, u \in M.$$

When M is a left R-module, the mapping

$$t: R \otimes_S M \to M$$

given by the relation

$$t(r \otimes u) = ru$$

is easily checked to be an R-homomorphism. If $\kappa: M \to R \otimes_S M$ is the S-homomorphism under which $u \to 1_R \otimes u$, then the composition

$$M \xrightarrow{\kappa} R \otimes_S M \xrightarrow{t} M$$

is the identity mapping, which proves that $\ker t$ is an S-direct summand of $R \otimes_S M$.

Definition. The left R-module M will be said to be *ϕ-projective* if $\ker t$ is an R-direct summand of $R \otimes_S M$. Clearly, if M is R-projective, then it is ϕ-projective. Our first result forms part of (**1**, Chapter II, Proposition 6.3).

(1) THEOREM. *For any left S-module M, $R \otimes_S M$ is ϕ-projective.*

If M is a left S-module, then the left S-module $\mathrm{Hom}_S(R, M)$ may be regarded as a left R-module by setting

$$(r_1 f)r_2 = f(r_2 r_1), \qquad r_1, r_2 \in R, f \in \mathrm{Hom}_S(R, M).$$

When M is a left R-module, the mapping $t': M \to \mathrm{Hom}_S(R, M)$ for which $(t'u)r = ru$ is an R-homomorphism, and if $\kappa': \mathrm{Hom}_S(R, M) \to M$ is the S-homomorphism under which $f \to f1_R$, then the composition

$$M \xrightarrow{t'} \mathrm{Hom}_S(R, M) \xrightarrow{\kappa'} M$$

is the identity mapping, which proves that $\mathrm{Im}\, t'$ is an S-direct summand of $\mathrm{Hom}_S(R, M)$.

Definition. The left R-module M is said to be *ϕ-injective* if $\mathrm{Im}\, t'$ is an R-direct summand of $\mathrm{Hom}_S(R, M)$.

If M is R-injective, then it is obviously ϕ-injective. Dual to (1) we have the following result.

(1′) THEOREM. *For any left S-module M,* $\mathrm{Hom}_S(R, M)$ *is ϕ-injective.*

If M is a left R-module and there exists an R-isomorphism $\mathrm{Hom}_S(R, M) \cong R \otimes_S M$, then M is ϕ-projective if and only if it is ϕ-injective.

3. Casimir operators. Throughout this paper, I will denote an indexing set and $\{R_i\}_{i \in I}$, $\{R_i'\}_{i \in I}$ will be families of S-bimodules which are contained in R.

Definition. Let M and H be left S-modules and R-modules, respectively. For each $i \in I$, an S-homomorphism $\bar{\delta}_i$: $R_i \otimes_S M \rightarrow H$ will be said to be *quasi-R-linear* if

$$\bar{\delta}_i(rr' \otimes u) = r\bar{\delta}_i(r' \otimes u) \quad \text{whenever } r \in R,\ r',\ rr' \in R_i, \text{ and } u \in M.$$

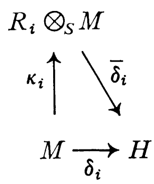
Dually, an S-homomorphism $\tilde{\epsilon}_i$: $H \rightarrow \mathrm{Hom}_S(R_i', M)$ will also be said to be *quasi-R-linear* if

$$(\tilde{\epsilon}_i rh)r' = (\tilde{\epsilon}_i h)(r'r) \quad \text{whenever } r \in R,\ h \in H,\ r',\ r'r \in R_i'.$$

For each $i \in I$, we suppose that to every left S-module M there corresponds an S-homomorphism

$$\kappa_i\colon M \rightarrow R_i \otimes_S M$$

which is such that, if H is a left R-module and δ_i: $M \rightarrow H$ is an S-homomorphism, then there exists a unique quasi-R-linear homomorphism $\bar{\delta}_i$: $R_i \otimes_S M \rightarrow H$ for which $\delta_i = \bar{\delta}_i \kappa_i$, i.e. for which the diagram

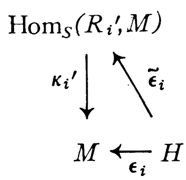


is commutative.

We shall also suppose that, for each $i \in I$, there corresponds to every M an S-homomorphism

$$\kappa_i'\colon \mathrm{Hom}_S(R_i', M) \rightarrow M$$

which is such that, if H is a left R-module and ϵ_i: $H \rightarrow M$ is an S-homomorphism, then there exists a unique quasi-R-linear homomorphism $\tilde{\epsilon}_i$: $H \rightarrow \mathrm{Hom}_S(R_i', M)$ for which $\epsilon_i = \kappa_i' \tilde{\epsilon}_i$, i.e. for which the diagram



is commutative.

Let M be a fixed left R-module, and, for each $i \in I$, let

$$\rho_i: R_i \otimes_S M \to R \otimes_S M, \qquad \rho_i': \mathrm{Hom}_S(R, M) \to \mathrm{Hom}_S(R_i', M)$$

be the S-homomorphisms induced by the inclusion mappings

$$R_i \to R, \qquad R_i' \to R,$$

respectively.

We shall suppose that, for each $i \in I$, there exists an S-homomorphism

$$\lambda_i: \mathrm{Hom}_S(R_i', M) \to R_i \otimes_S M.$$

Definitions. Let M and N be left R-modules and let $\{\alpha_i\}_{i \in I}$ be a family of S-homomorphisms of N into M. If, for each $v \in N$, $t\rho_i\lambda_i\tilde{\alpha}_i v = 0$ for almost all i, then the S-homomorphism

$$\sum_{i \in I} t\rho_i\lambda_i\tilde{\alpha}_i: N \to M,$$

is called a *first Casimir operator* of the family $\{\alpha_i\}_{i \in I}$ and is denoted by $c\{\alpha_i\}$.

Again, let $\{\beta_i\}_{i \in I}$ be a family of S-homomorphisms of M into N; if, for each $u \in M$, $\bar{\beta}_i\lambda_i\rho_i't'u = 0$ for nearly all i, then the S-homomorphism

$$\sum_{i \in I} \bar{\beta}_i\lambda_i\rho_i't': M \to N$$

is called a *second Casimir operator* of $\{\beta_i\}_{i \in I}$ and is denoted by $c'\{\beta_i\}$. (The terminology is that used in (**8**, § 8); for a justification of the use of "Casimir operator", see the Remark following (4) in § 4.)

Note. The sets $\{\alpha_i\}_{i \in I}$ and $\{\beta_i\}_{i \in I}$ possess first and second Casimir operators, respectively, whenever the indexing set I is finite.

(2) THEOREM. *Suppose that, as an S-bimodule, $R = \sum_{i \in I} R_i$ (direct sum) and let M be a left R-module. If*

(a) *M possesses a family $\{\alpha_i\}_{i \in I}$ of S-endomorphisms such that*

$$\sum_{i \in I} \rho_i\lambda_i\tilde{\alpha}_i: M \to R \otimes_S M$$

is an R-homomorphism and $c\{\alpha_i\} = \mathrm{id}_M$, the identity mapping of M,

then

(b) *M is ϕ-projective.*

For each $i \in I$, let σ_i be the S-homomorphism $R \otimes_S M \to R_i \otimes_S M$ induced by the projection mapping $R \to R_i$. If each λ_i is an S-isomorphism and each $\lambda_i^{-1}\sigma_i$ is quasi-R-linear, then (a) *and* (b) *are equivalent.*

Proof. (a) implies (b) at once. Suppose then that each λ_i is an S-isomorphism, that each $\lambda_i^{-1}\sigma_i$ is quasi-R-linear, and that M is ϕ-projective. There exists an R-homomorphism $g: M \to R \otimes_S M$ such that $tg = \mathrm{id}_M$. Let $\alpha_i = \kappa_i'\lambda_i^{-1}\sigma_i g$; since $\lambda_i^{-1}\sigma_i$ is quasi-R-linear, then so is $\lambda_i^{-1}\sigma_i g$, and it follows that $\tilde{\alpha}_i = \lambda_i^{-1}\sigma_i g$.

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Hence

$$\sum_{i\in I} \rho_i\lambda_i\tilde{\alpha}_i = \sum_{i\in I} \rho_i\sigma_i g = g,$$

since $\sum_{i\in I} \rho_i\sigma_i = \mathrm{id}_{R\otimes_S M}$. Also,

$$c\{\alpha_i\} = \sum_{i\in I} t\rho_i\lambda_i\tilde{\alpha}_i = \sum_{i\in I} t\rho_i\sigma_i g = tg = \mathrm{id}_M.$$

Dual to (2), we have the following result.

(2′) THEOREM. *Let the indexing set I be finite. Suppose also that, as an S-bimodule, $R = \sum_{i\in I} R_i'$ (direct sum), and that M is a left R-module. If*

(a′) *M possesses a family $\{\beta_i\}_{i\in I}$ of S-endomorphisms such that*

$$\textstyle\sum_{i\in I} \bar{\beta}_i\lambda_i\rho_i' : \mathrm{Hom}_S(R, M) \to M$$

is an R-homomorphism and $c'\{\beta_i\} = \mathrm{id}_M$,

then

(b′) *M is ϕ-injective.*

For each $i \in I$, let σ_i': $\mathrm{Hom}_S(R_i', M) \to \mathrm{Hom}_S(R, M)$ be the S-homomorphism induced by the projection $R \to R_i'$. If each λ_i is an S-isomorphism and each $\sigma_i'\lambda_i^{-1}$ is quasi-R-linear, then (a′) *and* (b′) *are equivalent.*

4. Examples.

Example 1. We suppose that the indexing set I consists of a single element, and we take $R_i = R_i' = R$. If M is a left S-module, H is a left R-module and δ: $M \to H$, ϵ: $H \to M$ are S-homomorphisms, then there exist unique R-homomorphisms $\bar{\delta}$: $R \otimes_S M \to H$, $\tilde{\epsilon}$: $H \to \mathrm{Hom}_S(R, M)$ such that $\delta = \bar{\delta}\kappa$, $\epsilon = \kappa'\tilde{\epsilon}$, namely the mappings under which $r \otimes u \to r(\delta u)$ and $h \to f$, where $fr = \epsilon(rh)$. We shall assume that, when M is a left R-module, there exists an R-homomorphism

$$\lambda: \mathrm{Hom}_S(R, M) \to R \otimes_S M.$$

From (2) and (2′) we have the following results.

(3) COROLLARY. *Let M be a left R-module. If*

(a) *M possesses an S-endomorphism α such that $c\{\alpha\} = \mathrm{id}_M$,*

then

(b) *M is ϕ-projective.*

If λ is an R-isomorphism, then (a) *and* (b) *are equivalent.*

(3′) COROLLARY. *Let M be a left R-module. If*

(a′) *M possesses an S-endomorphism β such that $c'\{\beta\} = \mathrm{id}_M$,*

then

(b′) *M is ϕ-injective.*

If λ is an R-isomorphism, then (a′) *and* (b′) *are equivalent.*

Note. When λ is an R-isomorphism, it follows from the remark at the end of § 2 that the conditions (a), (b), (a′), (b′) are equivalent.

The results (3) and (3′) above were proved by Higman (**7**, Theorem 5) for a situation similar to the present one. As an application, he considered the situation in which S is a subring of R and R possesses a right S-basis $\{r_1, \ldots, r_n\}$ and a set $\{r_1', \ldots, r_n'\}$ of elements such that

$$\text{(i)} \qquad rr_j = \sum_{k=1}^{n} r_k s_{jk} \ (r \in R, s_{jk} \in S) \text{ implies that } r_j'r = \sum_{k=1}^{n} s_{kj} r_k'.$$

In this case, for any left R-module M, the mapping

$$\lambda\colon \mathrm{Hom}_S(R, M) \to R \otimes_S M,$$

under which

$$f \to \sum_{j=1}^{n} r_j \otimes f r_j',$$

is an R-homomorphism. If N is a second R-module and $\alpha\colon N \to M$ is an S-homomorphism, then it is easily checked that

$$c\{\alpha\} = c'\{\alpha\} = \sum_{j=1}^{n} r_j \alpha r_j'.$$

Furthermore, when $\{r_1', \ldots, r_n'\}$ is a left S-basis of R, λ is an R-isomorphism.

The following result is then an immediate consequence of (3) and (3′).

(4) COROLLARY. *Suppose that S is a subring of R and let $\phi\colon S \to R$ be the inclusion mapping. Let $\{r_1, \ldots, r_n\}$ be a right S-basis of R and let $\{r_1', \ldots, r_n'\}$ be a set of elements of R which satisfy* (i). *Suppose also that M is a left R-module. The condition* (a) *M possesses an S-endomorphism α such that*

$$\sum_{j=1}^{n} r_j \alpha r_j' = \mathrm{id}_M$$

implies (3)(b) *and* (3′)(b′). *If $\{r_1', \ldots, r_n'\}$ is a left S-basis of R, then each of these conditions is equivalent to* (a).

Remark. Let R be a separable algebra over a field S, and suppose that $\{r_1, \ldots, r_n\}$ is a basis of R and that $\{r_1', \ldots, r_n'\}$ is a dual basis of R with respect to some discriminant matrix. If α is a linear transformation of a representation module for R over S, then $c\{\alpha\}$ is the Casimir operator of classical theory; see (**6**).

For applications of (4) to algebras, separable algebras, and groups, the reader is referred to (**7**, Part III).

In § 5 we extend (4) for a special class of pairs R, S.

Example 2. Let J be an indexing set which is partitioned into a family $\{J_i\}_{i \in I}$ of finite subsets. Suppose also that $\{r_j\}_{j \in J}$ is a right S-basis of R and that $\{r_j'\}_{j \in J}$ is a family of elements of R, the members of which are not necessarily distinct, such that

$$r_j(\phi s) = (\phi s) r_j, \quad r_j'(\phi s) = (\phi s) r_j', \qquad j \in J, s \in S.$$

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For each $i \in I$, let R_i be the right S-submodule of R generated by the set $\{r_j\}_{j \in J_i}$; we note that R_i is an S-bimodule and that $R = \sum_{i \in I} R_i$ (direct sum). Also, for each i, let R_i' be an S-bimodule which is contained in R and which contains the set $\{r_j'\}_{j \in J_i}$. Finally, we assume that, for each left S-module M and each $i \in I$, there exists an S-homomorphism

$$\kappa_i': \operatorname{Hom}_S(R_i', M) \to M$$

with the properties specified in § 3. For each $i \in I$, we define an S-homomorphism

$$\lambda_i: \operatorname{Hom}_S(R_i', M) \to R_i \otimes_S M$$

by

$$\lambda_i f = \sum_{j \in J_i} r_j \otimes f r_j'.$$

(5) LEMMA. *Let M be a left R-module and let $\{\alpha_i\}_{i \in I}$ be a family of S-endomorphisms of M such that*

(ii) *for each $u \in M$, $\tilde{\alpha}_i u = 0$ for almost all $i \in I$.*

A necessary and sufficient condition for $\sum_{i \in I} \rho_i \lambda_i \tilde{\alpha}_i: M \to R \otimes_S M$ to be an R-homomorphism is the following:

if $r \in R$ and if, for all $j \in J$, $rr_j = \sum_{k \in J} r_k s_{jk}$, where $s_{jk} \in S$, then, for $k \in J_i$,

(iii) $$\sum_{i \in I} (\tilde{\alpha}_i u)\Big(\sum_{j \in J_i} s_{jk} r_j'\Big) = (\tilde{\alpha}_i r u) r_k', \qquad u \in M.$$

Proof. Suppose that $r \in R$, that $rr_j = \sum_{k \in J} r_k s_{jk}$ for all $j \in J$, and that (iii) holds; then

$$\begin{aligned} r\Big\{\Big(\sum_{i \in I} \rho_i \lambda_i \tilde{\alpha}_i\Big) u\Big\} &= \sum_{i \in I} r \rho_i \lambda_i \tilde{\alpha}_i u = \sum_{i \in I} r\Big\{\sum_{j \in J_i} r_j \otimes (\tilde{\alpha}_i u) r_j'\Big\} \\ &= \sum_{i \in I} \sum_{j \in J_i} \{rr_j \otimes (\tilde{\alpha}_i u) r_j'\} = \sum_{i \in I} \sum_{j \in J_i} \Big\{\Big(\sum_{k \in J} r_k s_{jk}\Big) \otimes (\tilde{\alpha}_i u) r_j'\Big\} \\ &= \sum_{k \in J} \sum_{i \in I} \sum_{j \in J_i} r_k \otimes (\tilde{\alpha}_i u)(s_{jk} r_j') = \sum_{k \in J} \sum_{i \in I} r_k \otimes \Big\{(\tilde{\alpha}_i u)\Big(\sum_{j \in J_i} s_{jk} r_j'\Big)\Big\} \\ &= \sum_{i \in I} \sum_{k \in J_i} r_k \otimes (\tilde{\alpha}_i r u) r_k', \text{ by (iii)}, \\ &= \sum_{i \in I} \rho_i \lambda_i \tilde{\alpha}_i r u = \Big(\sum_{i \in I} \rho_i \lambda_i \tilde{\alpha}_i\Big)(ru), \end{aligned}$$

and thus $\sum_{i \in I} \rho_i \lambda_i \tilde{\alpha}_i$ is an R-homomorphism.

Since $\{r_j\}_{j \in J}$ is a right S-basis of R, each element of $R \otimes_S M$ can be expressed uniquely in the form $\sum_{j \in J} r_j \otimes v_j$, where the v_j belong to M. That (iii) is a necessary condition for $\sum_{i \in I} \rho_i \lambda_i \tilde{\alpha}_i$ to be an R-homomorphism can be seen from the first part of this proof.

The next result follows from (2) and (5).

(6) THEOREM. *Let M be a left R-module. If*

(*a*) *M possesses a family $\{\alpha_i\}_{i\in I}$ of S-endomorphisms which satisfy conditions* (ii) *and* (iii) *and such that $c\{\alpha_i\} = \mathrm{id}_M$,*

then

(b) *M is ϕ-projective.*

For each $i \in I$ let σ_i: $R \otimes_S M \to R_i \otimes_S M$ be the mapping induced by the projection $R \to R_i$. If each λ_i is an isomorphism and each $\lambda_i^{-1}\sigma_i$ is quasi-R-linear, then (a) *and* (b) *are equivalent.*

5. Throughout this section, S will be a subring of R and ϕ: $S \to R$ will be the inclusion mapping. We shall suppose that the elements $r_1, \ldots, r_n, r_1', \ldots, r_n'$ of R commute with every member of S, and that $\{r_1, \ldots, r_n\}$, $\{r_1', \ldots, r_n'\}$ are S-bases of R which satisfy condition (i). We assume also that

$$r_1'r_1 = r_2'r_2 = \ldots = r_n'r_n = a, \quad \text{say,}$$

and that

(iv) $$r_j'r_k = 0 \quad \text{when } j < k.$$

(7) THEOREM. *For any left R-module M, the following conditions are equivalent:*

(a) *M is ϕ-projective;*

(a′) *M is ϕ-injective;*

(b) *M possesses an S-endomorphism α such that*

(v) $$\sum_{j=1}^{n} r_j\alpha r_j' = \mathrm{id}_M;$$

(c) *$M \cong^R R \otimes_S aM$;*

(c′) *$M \cong^R \mathrm{Hom}_S(R, aM)$.*

Proof. The equivalence of (a), (a′), and (b) follows from (4).

(b) $\Rightarrow$ (c). Multiplying both sides of (v) on the left by r_k' and using (iv), we see that

(vi) $$\sum_{j=1}^{k} r_k'r_j\alpha r_j' = r_k'.$$

The relation

$$\psi u = \sum_{j=1}^{n} r_j \otimes a\alpha r_j'u$$

defines a mapping, namely

$$\psi: M \to R \otimes_S aM.$$

If $r \in R$ and $rr_j = \sum_{k=1}^{n} r_k s_{jk}$ $(j = 1, \ldots, n)$, then

$$r(\psi u) = \sum_{j=1}^{n} rr_j \otimes a\alpha r_j'u = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} r_k s_{jk}\right) \otimes a\alpha r_j'u$$

$$= \sum_{k=1}^{n} r_k \otimes a\alpha\left(\sum_{j=1}^{n} s_{jk}r_j'\right)u = \sum_{k=1}^{n} r_k \otimes a\alpha(r_k'r)u,$$

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since condition (i) is satisfied, and hence

$$r(\psi u) = \sum_{j=1}^{n} r_j \otimes a\alpha r_j'(ru) = \psi(ru),$$

thus proving that ψ is an R-homomorphism. If $\psi u = 0$, then, since $\{r_1, \ldots, r_n\}$ is an S-base for R, we can infer that $a\alpha r_j' u = 0$ for each j. Replacing k in (vi) by $1, \ldots, n$ in succession, we see that $r_j' u = 0$ for each j. It follows from (v) that $u = 0$. Thus ψ is injective.

We next show that ψ is surjective. Multiplying both sides of (vi) on the right by r_k and using (iv), we have

(vii) $$a\alpha a = a.$$

Suppose now that $v \in aM$. We can put $v = au$, where $u \in M$, and then

$$a\alpha r_k'(r_k \alpha v) = a\alpha a\alpha a u = au, \quad \text{by (vii)},$$

and hence

$$a\alpha r_k'(r_k \alpha v) = v.$$

In addition, when $j < k$, $a\alpha r_j'(r_k \alpha v) = 0$. Thus, ψ is surjective, and hence is an R-isomorphism.

The implication (b) $\Rightarrow$ (c′) follows at once since $\mathrm{Hom}_S(R, aM) \cong^R R \otimes_S aM$; cf. Example 1.

The implications (c) $\Rightarrow$ (a), (c′) $\Rightarrow$ (a′) were cited in (1) and (1′).

(8) THEOREM. *The R-module M is projective if and only if there exists a projective S-module N such that $M \cong^R R \otimes_S N$. Dually, M is injective if and only if there exists an injective S-module N such that $M \cong^R \mathrm{Hom}_S(R, N)$.*

Proof. If M is R-projective, then it is also ϕ-projective, and hence it follows from (7) that there exists an S-module N such that $M \cong^R R \otimes_S N$. Since R is S-free, it follows that M is S-projective; and thus N, being S-isomorphic to a direct summand of M, is S-projective. The converse follows from (**1**, Chapter II, Proposition 6.1).

6. Examples.

Example 3. Let R be the free left S-module on the set $\{1_S, d, \ldots, d^{n-1}\}$. We make R into a ring by means of the identity

$$(s_0 1_S + s_1 d + \ldots + s_{n-1} d^{n-1})(s_0' 1_S + s_1' d + \ldots + s_{n-1}' d^{n-1})$$
$$= s_0 s_0' 1_S + (s_0 s_1' + s_1 s_0')d + \ldots + (s_0 s_{n-1}' + s_1 s_{n-2}' + \ldots + s_{n-1} s_0')d^{n-1}$$
$$(s_0, \ldots, s_{n-1}, s_0', \ldots, s_{n-1}' \in S),$$

so that $d^n = 0$. We may regard S as a subring of R by identifying s and $s1_S$ for every $s \in S$, in which case d commutes with every member of S. It is clear that if M is a left S-module having an S-endomorphism d for which $d^n = 0$, then M is a left R-module. In (7), we can take

$$r_1 = 1_S, r_2 = d, \ldots, r_n = d^{n-1}, \quad r_1' = d^{n-1}, \quad r_2' = d^{n-2}, \ldots, r_n' = 1_S,$$

the identity in (7) (b) then becomes

$$1_S \alpha d^{n-1} + d\alpha d^{n-2} + \ldots + d^{n-2}\alpha d + d^{n-1}\alpha 1_S = \mathrm{id}_M,$$

and we have (**9**, Proposition 1.3). Taking $n = 2$ yields (**1**, Chapter IV, Proposition 2.3). We remark that, in the former case, $a = d^{n-1}$.

Example 4. Let R be the free left S-module on the set $\{1_S, d_1, d_2, d_1d_2\}$. We make R into a ring by means of the identity

$$\begin{aligned}(s_0 1_S + s_1 d_1 + s_2 d_2 + s_3 d_1 d_2)(s_0' 1_S + s_1' d_1 + s_2' d_2 + s_3' d_1 d_2) \\ = s_0 s_0' 1_S + (s_0 s_1' + s_1 s_0') d_1 + (s_0 s_2' + s_2 s_0') d_2 \\ + (s_0 s_3' + s_1 s_2' + s_2 s_1' + s_3 s_0') d_1 d_2 \qquad (s_0, \ldots, s_3, s_0', \ldots, s_3' \in S),\end{aligned}$$

so that

$$d_1 d_1 = d_2 d_2 = 0 \quad \text{and} \quad d_2 d_1 = d_1 d_2,$$

and, when we identify $s1_S$ and s for each $s \in S$, it follows that

$$d_1 s = s d_1, \qquad d_2 s = s d_2.$$

In (7) we can put

$$r_1 = 1_S, r_2 = d_1, r_3 = d_2, r_4 = d_1 d_2, \qquad r_1' = d_1 d_2, r_2' = d_2, r_3' = d_1, r_4' = 1_S.$$

The identity in (7) (b) then becomes

$$1_S \alpha d_1 d_2 + d_1 \alpha d_2 + d_2 \alpha d_1 + d_1 d_2 \alpha 1_S = \mathrm{id}_M,$$

and $a = d_1 d_2$.

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