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# Constructions of Uniformly Convex Functions

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*Abstract.* We give precise conditions under which the composition of a norm with a convex function yields a uniformly convex function on a Banach space. Various applications are given to functions of power type. The results are dualized to study uniform smoothness and several examples are provided.

## 1 Introduction and Preliminary Results

We work in a real Banach space $X$ whose closed unit ball is denoted by $B_X$ and whose unit sphere is denoted by $S_X$. By a *proper function* $f\colon X \to (-\infty, +\infty]$ we mean a function which is somewhere real-valued, in other words, its domain, dom $f$, is not empty. A proper function $f\colon X \to (-\infty, +\infty]$ is *convex* if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)y \quad \text{for all } x, y \in \operatorname{dom} f,\ 0 \le \lambda \le 1.$$

The *conjugate function* of $f\colon X \to (-\infty, +\infty]$ is defined for $x^* \in X^*$ by

$$f^*(x^*) := \sup_{x \in X} \langle x^*, x\rangle - f(x).$$

Relevant background material on convex analysis can be found in various monographs such as [5, 6, 14, 16].

In particular, we will frequently use, without mention, the elementary fact that when $f\colon \mathbb{R} \to (-\infty, +\infty]$ is convex and $t_0$ is in the interior of the domain of $f$, then

$$f'_+(t_0) := \lim_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h}$$

exists and is finite, and satisfies

$$f(t) \ge f(t_0) + f'_+(t_0)(t - t_0)$$

for all $t \in \mathbb{R}$. This is a particular instance of the more general *max formula*; see [5] or [6, Corollary 2.1.3 and Theorem 4.1.10].

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Let $(X, \|\cdot\|)$ be a Banach space. The *modulus of convexity* $\delta_X$ is defined for $0 \leq \varepsilon \leq 2$ by

$$\delta_X(\varepsilon) := \inf\Big\{ 1 - \Big\| \frac{x+y}{2} \Big\| : \|x\| = \|y\| = 1, \text{ and } \|x-y\| \geq \varepsilon \Big\}.$$

In the case $\delta_X(\varepsilon) > 0$ for each $\varepsilon > 0$, we will say $\|\cdot\|$ is a *uniformly convex norm*. If there exist $C > 0$ and $p \geq 2$ such that $\delta_X(\varepsilon) \geq C\varepsilon^p$ for all $0 \leq \varepsilon \leq 2$, then $\delta_X$ is said to be of *power type* $p$. Further information can be found in the excellent books [2, 3, 10], and various equivalent forms of the definition can be found in [9]. Although the terminology is standard, $\delta_X$ is not what is typically called a *modulus*; see [1] for a nice development of relevant terminology.

Analogously, given a proper lower semicontinuous convex function $f\colon X \to (-\infty, +\infty]$, we will say its *modulus of convexity* is the function $\delta_f\colon [0, +\infty) \to [0, +\infty]$ defined by

$$\delta_f(t) := \inf\Big\{ \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\Big(\frac{x+y}{2}\Big) : \|x-y\| \geq t,\ x, y \in \operatorname{dom} f \Big\},$$

where the infimum over the empty set is $+\infty$. We say that $f$ is *uniformly convex* when $\delta_f(t) > 0$ for all $t > 0$, and $f$ has *modulus of convexity of power type* $p$ (or $\delta_f$ *is of power type* $p$) if there exists $C > 0$ so that $\delta_f(t) \geq Ct^p$ for all $t > 0$. In [1, 15, 16], uniformly convex functions are defined using a closely related notion called the *gage of uniform convexity*, and it follows from [15, Remark 2.1] that the definition in those sources is equivalent to the one used here. Some natural confusion may arise with the terminology we use, because a uniformly convex norm is never uniformly convex when considered as a function—it is uniformly convex on its sphere.

A systematic exposition of uniformly convex norms can be found in [10, §IV.4, IV.5], and [16, §3.5] presents a thorough account of uniformly convex functions. However, explicit constructions of such functions, especially those derived from a uniformly convex norm, appear to be somewhat sparse. For example, when $\|\cdot\|$ is a uniformly convex norm on $X$, it is easy to see that $f := \|\cdot\|^r$ with $r > 1$ is *uniformly convex on bounded sets*, that is for each $n \in \mathbb{N}$ and $\varepsilon > 0$,

$$\inf\Big\{ \frac{1}{2} f(x) + \frac{1}{2} f(y) - f\Big(\frac{x+y}{2}\Big) : \|x-y\| \geq \varepsilon,\ x, y \in \operatorname{dom} f \cap nB_X \Big\} > 0;$$

however, $f$ is not necessarily uniformly convex. In fact, [4] shows when $r \geq 2$, that $f$ is uniformly convex *if and only if* $\delta_X$ is of power type $r$. Our goal in this note is provide precise conditions under which $g \circ \|\cdot\|$ is uniformly convex when $g$ is a nondecreasing convex function on $[0, +\infty)$.

In many algorithms, uniform convexity on bounded sets and other weaker forms of convexity suffice for their implementation, as can be seen, for example, in [7, 8]. Nonetheless, beyond their theoretical interest, uniformly convex functions are dual under conjugation to uniformly smooth convex functions [1]. Also, when considered with moduli of power type, there is a tight duality with Hölder continuity conditions on the derivatives (see [1], [16, Theorem 3.5.10, Corollary 3.5.11, Theorem 3.5.12]).

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Because uniformly convex norms, and even those with some power type are (abundantly) available on superreflexive spaces, as is discussed in the monographs [2, 10], we believe it is important to find explicit conditions under which the composition with a norm yields a uniformly convex function (or even better, one with modulus of power type). Inter alia, we adumbrate the somewhat subtle relationship between notions of uniform convexity for norms—based on behavior on the sphere—and those for convex functions.

We will use the following simple examples of uniformly convex functions on the real line recorded in [6, Exercise 5.4.2].

***Fact 1.1*** *Suppose that a function $f$ on $\mathbb{R}$ satisfies $f^{(n)} \geq \alpha > 0$ on $[a, +\infty)$, where $n \geq 2$ is a fixed integer, and that $f^{(k)} \geq 0$ on $[a, +\infty)$ for $k \in \{2, \dots, n+1\}$. Define the function $g$ by $g(x) := f(x)$ for $x \geq a$ and $g(x) := +\infty$ for $x < a$. Then $g$ is uniformly convex with modulus of convexity of power type $n$.*

In particular, for $b > 1$, let $g(x) := b^x$ for $x \geq 0$, and $g(x) := +\infty$ otherwise. Then $g$ is uniformly convex with modulus of convexity of power type $p$ for any $p \geq 2$. Similarly, using Taylor series one can show that for $p \geq 2$ and $g(x) := x^p$ for $x \geq 0$ and $g(x) := +\infty$ otherwise, $g$ is uniformly convex with modulus of convexity of power type $p$.

## 2 Constructions of Uniformly Convex Functions

Our first objective is to determine precisely when a composition with a norm yields a (continuous) uniformly convex function.

***Theorem 2.1*** *Suppose $(X, \|\cdot\|)$ is a Banach space and $f\colon [0, +\infty) \to [0, +\infty)$ is convex and nondecreasing. Then $f \circ \|\cdot\|$ is uniformly convex if and only if $f$ is uniformly convex and $\|\cdot\|$ is a uniformly convex norm while*

$$\liminf_{t\to\infty} f'_+(t) \cdot \delta_X\left(\frac{\varepsilon}{t}\right) \cdot t > 0 \tag{2.1}$$

*for each $\varepsilon > 0$.*

**Proof** ($\Rightarrow$): Clearly $f$ is uniformly convex, because for fixed $x_0 \in S_X$, we have that $f(t) = f(\|tx_0\|)$ and so $f$ is a uniformly convex function. Similarly, $\|\cdot\|$ is a uniformly convex norm. Indeed, suppose $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| \to 2$. Then

$$\frac{1}{2} f(\|x_n\|) + \frac{1}{2} f(\|y_n\|) - f\left(\left\|\frac{x_n + y_n}{2}\right\|\right) \to 0,$$

because $f$ is continuous at 1. The uniform convexity of $f \circ \|\cdot\|$ implies $\|x_n - y_n\| \to 0$; thus $\|\cdot\|$ is a uniformly convex norm.

Thence, suppose for some $\varepsilon > 0$ and $t_n \to \infty$, that $\lim_{n\to\infty} f'_+(t_n) \cdot \delta_X(\frac{\varepsilon}{t_n}) \cdot t_n = 0$. Now choose $u_n, v_n \in S_X$ such that $\|u_n - v_n\| \geq \frac{\varepsilon}{t_n}$ but

$$\left\|\frac{u_n + v_n}{2}\right\| \geq 1 - 2\delta_X\left(\frac{\varepsilon}{t_n}\right).$$

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Let $x_n := t_n u_n$ and $y_n := t_n v_n$. Then $\|x_n - y_n\| \geq \varepsilon$ for all $n$, but

$$f\Big(\Big\|\frac{t_n u_n + t_n v_n}{2}\Big\|\Big) \geq f(\|t_n u_n\|) - 2t_n\delta_X\Big(\frac{\varepsilon}{t_n}\Big)\cdot f'_+(t_n)$$
$$\geq f(\|t_n u_n\|) - 2\varepsilon_n \text{ where } \varepsilon_n = t_n\delta_X\Big(\frac{\varepsilon}{t_n}\Big)\cdot f'_+(t_n) \to 0,$$

which contradicts the uniform convexity of $f \circ \|\cdot\|$.

($\Leftarrow$): Suppose for each $\varepsilon > 0$, $\liminf_{t\to\infty} f'_+(t)\cdot\delta_X(\frac{\varepsilon}{t})\cdot t > 0$, $f$ is uniformly convex and $\|\cdot\|$ is a uniformly convex norm. Suppose $f \circ \|\cdot\|$ is not uniformly convex. Then there exist $(x_n), (y_n) \subset X$ and $\varepsilon > 0$ such that $\|x_n - y_n\| \geq \varepsilon$ for all $n \in \mathbb{N}$, but

$$\frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - f\Big(\Big\|\frac{x_n + y_n}{2}\Big\|\Big) \to 0. \tag{2.2}$$

First suppose $\limsup_{n\to\infty}\big|\,\|x_n\| - \|y_n\|\,\big| > 0$. By switching roles of $x_n$ and $y_n$ as necessary, and passing to a subsequence, we may assume $\|x_n\| - \|y_n\| \geq \eta > 0$ for all $n \in \mathbb{N}$. Thus using the fact $f$ is nondecreasing and uniformly convex, we have

$$\frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - f\Big(\Big\|\frac{x_n + y_n}{2}\Big\|\Big) \geq \frac{1}{2}f(\|x_n\|)$$
$$+\frac{1}{2}f(\|y_n\|) - f\Big(\frac{\|x_n\| + \|y_n\|}{2}\Big)$$
$$\geq \delta_f(\eta) > 0 \quad \text{for all } n \in \mathbb{N}.$$

This is a contradicion with (2.2). Thus, for the rest of the proof we may suppose $(\|x_n\| - \|y_n\|) \to 0$.

*Case* 1: Suppose $(x_n)$ is a bounded sequence. By passing to a subsequence as necessary, we may assume $\|x_n\| \to \alpha$ and $\|y_n\| \to \alpha$ for some $\alpha \geq 0$. Because $\|x_n - y_n\| \geq \varepsilon$, it is clear that $\alpha > 0$, and because $\|\cdot\|$ is a uniformly convex norm, we obtain

$$\limsup_{n\to\infty}\Big\|\frac{x_n + y_n}{2\alpha}\Big\| \leq 1 - \delta_X\Big(\frac{\varepsilon}{\alpha}\Big).$$

Consequently, $\limsup_{n\to\infty}\|\frac{x_n+y_n}{2}\| \leq \alpha[1 - \delta_X(\frac{\varepsilon}{\alpha})]$. Using the fact that $f$ is convex and nondecreasing, we obtain

$$\liminf_{n\to\infty}\frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - f\Big(\Big\|\frac{x_n + y_n}{2}\Big\|\Big)$$
$$\geq \liminf_{n\to\infty} f\Big(\frac{\|x_n\| + \|y_n\|}{2}\Big) - f\Big(\Big\|\frac{x_n + y_n}{2}\Big\|\Big)$$
$$\geq f(\alpha) - f\Big(\alpha - \alpha\delta_X\Big(\frac{\varepsilon}{\alpha}\Big)\Big) > 0.$$

which contradicts (2.2).

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*Case* 2: It remains to consider the situation where $(x_n)$ is unbounded. In fact, any bounded subsequence of $(x_n)$ would yield a contradiction as above, so we let $\alpha_n := \|x_n\|$ and assume $\alpha_n \to \infty$. Further, because we now know that $(\|x_n\| - \|y_n\|) \to 0$, interchanging $x_n$ and $y_n$ as necessary, we write $\|y_n\| = \beta_n$, where $\alpha_n = \beta_n + \eta_n$, $\eta_n \geq 0$ and $\eta_n \to 0$.

Now let $\tilde{x}_n := \frac{1}{\alpha_n} x_n$ and $\tilde{y}_n := \frac{1}{\beta_n} y_n$. Then $\|\tilde{x}_n - \tilde{y}_n\| \geq \frac{\varepsilon - \eta_n}{\alpha_n}$. Fix $N \in \mathbb{N}$ such that $\|\tilde{x}_n - \tilde{y}_n\| \geq \frac{\varepsilon}{2\beta_n}$ for $n \geq N$. Then

$$\Big\| \frac{\tilde{x}_n + \tilde{y}_n}{2} \Big\| \leq 1 - \delta_X\Big(\frac{\varepsilon}{2\beta_n}\Big) \quad \text{for } n \geq N.$$

Let

$$\tilde{\beta}_n := \frac{\beta_n + \alpha_n}{2} - \delta_X\Big(\frac{\varepsilon}{2\beta_n}\Big) \cdot \beta_n.$$

Note that $\|x_n + y_n\| \leq \beta_n \|\tilde{x}_n + \tilde{y}_n\| + \eta_n$, and that $\tilde{\beta}_n / \beta_n \to 1$ (since $\beta_n \to \infty, \eta_n \to 0$). Then, for $n \geq N$, the monotonicity of $f$ ensures that

$$\begin{aligned} f\Big(\Big\|\frac{x_n + y_n}{2}\Big\|\Big) &\leq f\Big(\beta_n \Big\| \frac{\tilde{x}_n + \tilde{y}_n}{2}\Big\| + \frac{\eta_n}{2}\Big) \\ &\leq f\Big(\beta_n - \delta_X\Big(\frac{\varepsilon}{2\beta_n}\Big) \cdot \beta_n + \frac{\eta_n}{2}\Big) \\ &= f(\tilde{\beta}_n). \end{aligned}$$

The convexity of $f$ guarantees that

$$\frac{1}{2} f(\alpha_n) + \frac{1}{2} f(\beta_n) \geq f\Big(\frac{\alpha_n + \beta_n}{2}\Big) \geq f(\tilde{\beta}_n) + \delta_X\Big(\frac{\varepsilon}{2\beta_n}\Big) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n), \quad \text{for } n \geq N.$$

Hence

$$\begin{aligned} f(\tilde{\beta}_n) &\leq \frac{1}{2} f(\alpha_n) + \frac{1}{2} f(\beta_n) - \delta_X\Big(\frac{\varepsilon}{2\beta_n}\Big) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n) \\ &= \frac{1}{2} f(\|x_n\|) + \frac{1}{2} f(\|y_n\|) - \delta_X\Big(\frac{\varepsilon}{2\beta_n}\Big) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n), \quad \text{for } n \geq N. \end{aligned} \tag{2.3}$$

To complete the proof, it remains to verify that

$$\liminf_{n\to\infty} \delta_X\Big(\frac{\varepsilon}{2\beta_n}\Big) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n) > 0, \tag{2.4}$$

and as a consequence it will follow that (2.3) contradicts (2.2). Indeed, since $\tilde{\beta}_n / \beta_n \to 1$, for sufficiently large $n$, $\tilde{\beta}_n \geq \frac{1}{2}\beta_n$ and because $\delta_X$ is nondecreasing on $[0, 2]$, this additionally ensures $\delta_X(\frac{\varepsilon}{2\beta_n}) \geq \delta_X(\frac{\frac{\varepsilon}{4}}{\tilde{\beta}_n})$ for such $n$. Consequently,

$$\delta_X\Big(\frac{\varepsilon}{2\beta_n}\Big) \cdot \beta_n \cdot f'_+(\tilde{\beta}_n) \geq \frac{1}{2} \delta_X\Big(\frac{\varepsilon/4}{\tilde{\beta}_n}\Big) \cdot \tilde{\beta}_n \cdot f'_+(\tilde{\beta}_n) \quad \text{for sufficiently large } n.$$

Applying (2.1) with $\varepsilon/4$ replacing $\varepsilon$ to the right-hand side of the previous inequality, one deduces (2.4) as desired. ■

We next construct continuous uniformly convex functions using any uniformly convex norm on a superreflexive Banach space.

***Example 2.2*** Let $X$ be a Banach space with uniformly convex norm $\|\cdot\|$. We define $f(t) := t^2$ for $0 \le t \le 1$ while

$$f(t) := t^2 + \int_1^t \frac{1}{\delta_X(u^{-2})}\,du \quad \text{when } t > 1.$$

We may apply Theorem 2.1 to show $f \circ \|\cdot\|$ is uniformly convex. We recall that $\delta_X$ is continuous, positive, and nondecreasing on $(0, 2]$ (see [12]), and so $f'$ is positive and increasing on $[0, +\infty)$. Thus $f$ is convex and increasing on $[0, +\infty)$. Moreover, $t \mapsto t^2$ is uniformly convex (hence so is its sum with another convex function), and so $f$ is uniformly convex. Now, for $t > 1$, $f'(t) = 2t + 1/\delta_X(t^{-2})$. For fixed $\varepsilon > 0$ when $t > \varepsilon^{-1}$, we then have

$$f'_+(t)\cdot\delta_X\Big(\frac{\varepsilon}{t}\Big)\cdot t > \frac{1}{\delta_X(t^{-2})}\cdot\delta_X\Big(\frac{\varepsilon}{t}\Big)\cdot t \ge t$$

and so (2.1) holds.

Further examples will be given after the following more quantitative result concerning moduli of power type.

***Theorem 2.3*** *Let $(X, \|\cdot\|)$ be a Banach space, let $f\colon [0, +\infty) \to [0, +\infty)$ be a convex nondecreasing function and let $p \ge 2$.*

(i) *Suppose $\delta_f$ and $\delta_X$ are both of power type $p$ and $f'_+(t) \ge Ct^{p-1}$ for some $C > 0$ and for all $t > 0$. Then $f \circ \|\cdot\|$ has modulus of convexity of power type $p$.*

(ii) *Conversely, if $f \circ \|\cdot\|$ has modulus of convexity of power type $p$, then $\delta_f$ and $\delta_X$ are both of power type $p$. In the case that $\delta_X$ additionally satisfies*

$$0 < \liminf_{\varepsilon\to 0^+} \frac{\delta_X(\varepsilon)}{\varepsilon^p} < \infty, \tag{2.5}$$

i.e., *the modulus $\delta_X$ is no better than power type $p$, then there exists $C > 0$ such that*

$$f'_+(t) \ge Ct^{p-1} \quad \textit{for all } t > 0.$$

**Proof** (i) Let $A$, $B$, and $C$ be positive constants such that

$$\delta_f(\varepsilon) \ge A\varepsilon^p \text{ for all } \varepsilon > 0, \qquad \delta_X(\varepsilon) \ge B\varepsilon^p \text{ for all } 0 \le \varepsilon \le 2,$$
$$f'_+(t) \ge Ct^{p-1} \text{ for all } t > 0.$$

Let $\varepsilon > 0$ be fixed, and suppose $x, y \in X$ satisfy $\|x - y\| \ge \varepsilon$. We may assume $\|y\| \le \|x\|$. Suppose first that $\|y\| + \varepsilon/2 \le \|x\|$. Using the modulus of convexity of

$f$, we obtain

$$\frac{1}{2}f(\|x\|)+\frac{1}{2}f(\|y\|)-f\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \geq \frac{1}{2}f(\|x\|)+\frac{1}{2}f(\|y\|)-f\Big(\frac{\|x\|+\|y\|}{2}\Big) \geq A\Big(\frac{\varepsilon}{2}\Big)^p. \tag{2.6}$$

Thus, for the remainder of the proof we will assume $\|y\|+\varepsilon/2>\|x\|$.

Let $a:=\|y\|$ and $\tilde{x}:=x/\|x\|$, $\tilde{y}:=y/\|y\|$. Then $\|y-a\tilde{x}\|>\varepsilon/2$. Consequently, $\|\tilde{y}-\tilde{x}\|>\frac{\varepsilon}{2a}$. Because $\delta_X(t)\geq Bt^p$ for $0\leq t\leq 2$, we deduce that

$$\Big\|\frac{\tilde{x}+\tilde{y}}{2}\Big\|\leq 1-B\Big(\frac{\varepsilon}{2a}\Big)^p$$

and so

$$\Big\|\frac{x+y}{2}\Big\|\leq a\Big(\Big\|\frac{\tilde{x}+\tilde{y}}{2}\Big\|\Big)+\frac{\|x\|-a}{2}\leq\frac{1}{2}\|x\|+\frac{1}{2}\|y\|-Ba\Big(\frac{\varepsilon}{2a}\Big)^p. \tag{2.7}$$

*Case* 1: We suppose $Ba(\frac{\varepsilon}{2a})^p\geq a/2$. Recalling that $\|x\|+\|y\|\geq\|x-y\|\geq\varepsilon$, we have $\|y\|\geq\varepsilon/4$ since $\|y\|\geq\|x\|-\varepsilon/2$. Because $a=\|y\|$, it follows that $a/2\geq\varepsilon/8$. Thus, letting $t_0:=(\|x\|+\|y\|)/2-a/2$, we have $t_0\geq a/2$, and the nondecreasing property of $f$ ensures

$$f\Big(\Big\|\frac{x+y}{2}\Big\|\Big)\leq f(t_0).$$

Now we use this with the convexity of $f$ to compute,

$$\begin{aligned}\frac{1}{2}f(\|x\|)+\frac{1}{2}f(\|y\|)\geq f\Big(\frac{\|x\|+\|y\|}{2}\Big) &\geq f(t_0)+f'_+(t_0)\cdot(a/2)\\ &\geq f(t_0)+f'_+(a/2)\cdot(a/2)\geq f(t_0)+f'_+(\varepsilon/8)\cdot(\varepsilon/8)\\ &\geq f\Big(\Big\|\frac{x+y}{2}\Big\|\Big)+C\Big(\frac{\varepsilon}{8}\Big)^p.\end{aligned} \tag{2.8}$$

*Case* 2: It remains to address the situation when $Ba(\frac{\varepsilon}{2a})^p\leq a/2$. Then the right-hand side of (2.7) is at least $a/2$. Now use the fact $f'_+(t)\geq C(a/2)^{p-1}$ when $t\geq a/2$ to compute

$$\begin{aligned}f\Big(\Big\|\frac{x+y}{2}\Big\|\Big)&\leq f\Big(\frac{1}{2}\|x\|+\frac{1}{2}\|y\|\Big)-Ba\Big(\frac{\varepsilon}{2a}\Big)^p\cdot C\Big(\frac{a}{2}\Big)^{p-1}\\ &\leq\frac{1}{2}f(\|x\|)+\frac{1}{2}f(\|y\|)-BC\Big(\frac{\varepsilon}{4}\Big)^p.\end{aligned} \tag{2.9}$$

Putting (2.6), (2.8), and (2.9) together, we see that $f\circ\|\cdot\|$ has modulus of convexity of power type $p$ as desired.

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(ii) Because $f \circ \|\cdot\|$ has modulus of convexity of power type $p$, one need only fix $x_0 \in S_X$ and consider $f(t) = f(\|tx_0\|)$ for $t \geq 0$ to see that $f$ has modulus of convexity of power type $p$.

Also, let $\beta := f'_+(1)$ and let $A > 0$ be such that $\delta_{f\circ\|\cdot\|}(\varepsilon) \geq A\varepsilon^p$ when $\varepsilon > 0$. Fix $\varepsilon \in (0,2]$ and choose $x, y \in S_X$ with $\|x - y\| \geq \varepsilon$ and $\|\frac{x+y}{2}\| \geq 1 - 2\delta_X(\varepsilon)$. Then

$$f(1) - A\varepsilon^p = f\Big(\frac{\|x\| + \|y\|}{2}\Big) - A\varepsilon^p \geq f\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \geq f(1) - 2\beta\delta_X(\varepsilon)$$

and it follows $\delta_X(\varepsilon) \geq \frac{A}{2\beta}\varepsilon^p$. Thus $\delta_X$ is of power type $p$ as desired.

It remains to verify $f'_+(t) \geq Ct^{p-1}$ for some $C > 0$ and all $t > 0$ when (2.5) is satisfied. Indeed, in this case, we find $(u_n), (v_n) \subset S_X$ and $M > 0$ such that

$$\varepsilon_n := \|u_n - v_n\| \to 0^+ \quad \text{and} \quad \Big\|\frac{u_n + v_n}{2}\Big\| \geq 1 - M\varepsilon_n^p.$$

Now fix $t > 0$, and let $x_n := tu_n$ and $y_n := tv_n$. Then

$$\Big\|\frac{x_n + y_n}{2}\Big\| \geq t(1 - M\varepsilon_n^p). \tag{2.10}$$

Then $\|x_n - y_n\| = t\varepsilon_n$ and the modulus of convexity of $f \circ \|\cdot\|$ ensures that

$$f\Big(\Big\|\frac{x_n + y_n}{2}\Big\|\Big) \leq \frac{1}{2}f(\|x_n\|) + \frac{1}{2}f(\|y_n\|) - A(t\varepsilon_n)^p = f(t) - At^p\varepsilon_n^p. \tag{2.11}$$

The convexity of $f$ implies that $f(t - tM\varepsilon_n^p) \geq f(t) - f'_+(t)(tM\varepsilon_n^p)$. Using this along with (2.10) and the fact $f$ is nondecreasing, we obtain

$$f\Big(\Big\|\frac{x_n + y_n}{2}\Big\|\Big) \geq f(t - tM\varepsilon_n^p) \geq f(t) - f'_+(t)(tM\varepsilon_n^p). \tag{2.12}$$

Combining (2.11) and (2.12) implies $f'_+(t) \geq \frac{A}{M}t^{p-1}$, and so $C := \frac{A}{M} > 0$ is as desired. ∎

The following corollary recovers a result from [4] whose proof proceeded via establishing uniform smoothness and invoking duality results from [1].

***Corollary 2.4*** ([4, Theorem 2.3]) *Let $(X, \|\cdot\|)$ be a Banach space, and suppose $f := \|\cdot\|^p$, where $p \geq 2$. Then the following are equivalent:*

(i) *$f$ is uniformly convex;*
(ii) *$\delta_X$ is of power type $p$;*
(iii) *$f$ has modulus of convexity of power type $p$.*

**Proof** (i) ⇒ (ii): Suppose $f$ is uniformly convex, then (2.1) holds with $\varepsilon = 1$. Consequently,

$$\liminf_{t\to\infty} pt^p\ \delta_X(t^{-1}) > 0,$$

and so there exist $C > 0$ and $t_0 > 0$ such that $p\,t^p\,\delta_X(t^{-1}) > C$ when $t > t_0$. In particular, for $0 < \varepsilon < 1/t_0$, we have $\delta_X(\varepsilon) > K\varepsilon^p$ where $K := Cp^{-1}$.

(ii) ⇒ (iii): This follows from Theorem 2.3(i) because the function $t \mapsto |t|^p$ has modulus of convexity of power type $p$ (see the paragraph after Fact 1.1).

(iii) ⇒ (i): This is trivial. ∎

***Example 2.5*** Let $(X, \|\cdot\|)$ be a Banach space and $b > 1$. Suppose $\delta_X$ is of power type $p$, where $p \geq 2$. Then $f := b^{\|\cdot\|}$ is uniformly convex with modulus of convexity of power type $p$. However, even on $\mathbb{R}^2$ there are uniformly convex norms $|||\cdot|||$ so that $h := b^{|||\cdot|||}$ is not uniformly convex.

**Proof** Let $g(t) := b^t$ for $t \geq 0$. Then $g'(t) \geq Ct^p$ for some $C > 0$ and all $t \geq 0$, and $g$ has modulus of convexity of power type $p$ by Fact 1.1. According to Theorem 2.3, $f$ has modulus of convexity of power type $p$.

For the claim concerning $h$, we appeal to [11, Theorem 2.8] to obtain a uniformly convex norm $|||\cdot|||$ on $\mathbb{R}^2$ so that when $Y := (\mathbb{R}^2, |||\cdot|||)$, we have

$$\liminf_{t\to\infty} t\, b^t \log(b)\delta_Y(t^{-1}) = 0.$$

Then (2.1) fails, and so Theorem 2.1 ensures $h$ is not uniformly convex. ∎

One may view the above conditions dually. For this, let $(X, \|\cdot\|)$ be a Banach space. Then the *modulus of smoothness*, $\rho_X$, is defined for $\tau > 0$ by

$$\rho_X(\tau) := \sup\Big\{ \frac{\|x + \tau h\| + \|x - \tau h\| - 2}{2} : \|x\| = \|h\| = 1 \Big\}.$$

Given $1 < q \leq 2$, we will say $\rho_X$ *is of power type* $q$ if there exists $C > 0$ so that $\rho_{\|\cdot\|}(\tau) \leq C\tau^q$ for $\tau > 0$; see [2, 3, 10] for further information. Analogously, we define *the modulus of smoothness of a convex function* $f$ for $\tau > 0$ by

$$\rho_f(\tau) := \sup\Big\{ \frac{1}{2} f(x + \tau h) + \frac{1}{2} f(x - \tau h) - f(x) : x \in X,\ \|h\| = 1 \Big\};$$

and when $\rho_f(\tau) \leq C\tau^q$ for some $C > 0$ and all $\tau > 0$, we will say $\rho_f$ is *of power type* $q$. See [1], [16, p. 204ff] or [6, §5.4] for further information on this and related topics. We note also that given $h := f \circ \|\cdot\|$, then the conjugate is given by

$$h^*(\phi) = \sup_{x\in X} \phi(x) - f(\|x\|) = \sup_{x\in X} \|\phi\|\|x\| - f(\|x\|) = f^*(\|\phi\|).$$

We may now present the following as a sample dual version of Theorem 2.3.

***Corollary 2.6*** *Let $(X, \|\cdot\|)$ be a Banach space, $1 < q \leq 2$ and $f\colon \mathbb{R} \to \mathbb{R}$ be convex, nondecreasing with $[0, +\infty) \subset \operatorname{dom} f^*$.*

(i) *Suppose $\rho_X$ is of power type $q$, $f$ has modulus of smoothness of power type $q$ while $f'(t) \leq Ct^{q-1}$ for some $C > 0$ and all $t \geq 0$. Then $f \circ \|\cdot\|$ has modulus of smoothness of power type $q$.*

(ii) *Conversely, suppose $f \circ \|\cdot\|$ has modulus of smoothness of power type $q$. Then $\rho_X$ is of power type $q$, $f$ has modulus of smoothness of power type $q$, and if $\rho_X$ is not better than power type $q$, then $f'(t) \leq Ct^{q-1}$ for some $C > 0$ and all $t \geq 0$.*

**Proof** (i) We may shift $f$ vertically so that $f(0) = 0$. Because $f$ is convex, nondecreasing, and $f'(t) \leq Ct^{q-1}$ for $t \geq 0$, it follows that $f'(0) = 0$, and also $f(t) = 0$ for $t < 0$. Consequently, $f^*$ is nonnegative and nondecreasing on $[0, +\infty)$, which is its

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domain. Now let $h := f \circ \|\cdot\|$ as above. According to [10, Proposition IV.1.12], the dual norm $\|\cdot\|$ has modulus of convexity of power type $p$, where $p$ is the conjugate index of $q$, that is, $p^{-1} + q^{-1} = 1$. Now let $t \in \partial f^*(y)$, where $y \geq 0$. Then $t \geq 0$, $y \in \partial f(t)$, and so $y \leq Ct^{q-1}$. Thus $t \geq Ky^{1/(q-1)}$, where $K := C^{\frac{1}{1-q}}$, or equivalently $t \geq Ky^{p-1}$. This implies $(f^*)'+(y) \geq Ky^{p-1}$ for all $y \geq 0$. Moreover, because $\rho_f$ is of power type $q$, it follows that $f^*$ has modulus of convexity of power type $p$ (see [16, Corollary 3.5.11]). Thus we may appeal to Theorem 2.3(i) to deduce $h^*$ has modulus of convexity of power type $p$. Applying [16, Corollary 3.5.11] once again, we conclude $h$ has modulus of smoothness of power type $q$.

(ii) The details are analogous to (i). This part follows from Theorem 2.3(ii), again by invoking duality results of [16, Corollary 3.5.11] and [10, Proposition IV.1.12]. ■

***Remark 2.7*** (1) Although our primary focus centers on continuous uniformly convex functions, one can deduce certain restricted domain cases from the observation that with $f$ as in Theorem 2.1 or 2.3, one has $\delta_{h_1} \leq \delta_{h_2}$, with $h_1 := f \circ \|\cdot\|$ and $h_2 := g \circ \|\cdot\|$, where for fixed $a \geq 0$ we define $g(t) := f(t)$ for $t \leq a$ and $g(t) := +\infty$ otherwise. In the converse directions one additionally needs $[0, a] \subset \operatorname{dom} f$ for some $a > 0$ to deduce properties about $\delta_X$ from those of $\delta_{f \circ \|\cdot\|}$.

(2) It follows from part (1) that the requirement $[0, +\infty) \subset \operatorname{dom} f^*$ is not necessary in Corollary 2.6(i). However, Corollary 2.6(ii) can fail when $f$ is a constant function, so some growth requirement on $f$ is needed (for example, to ensure $[0, a] \subset \operatorname{dom} f^*$ for some $a > 0$).

We used the definition for $\delta_X$ as in [3, 10]. However, if one defines $\delta(\varepsilon) := \delta_X(2\varepsilon)$ for $0 \leq \varepsilon \leq 1$ (see [1, p. 724]), duality formulas such as [3, Proposition A.3(ii)] for $\rho_{X^*}$ can then be naturally expressed in terms of the conjugate of $\delta$. Along this line, several neat duality relations for both norms and convex functions are derived in [1]. Because of the power of duality, it is often a matter of taste whether one prefers to start, for example, with Corollary 2.6 or Theorem 2.3 and derive the other through conjugation. As a final application of duality, we illustrate the restrictiveness of obtaining functions that are simultaneously uniformly convex and uniformly smooth.

***Proposition 2.8*** *Suppose $(X, \|\cdot\|)$ is a Banach space and $f: X \to \mathbb{R}$ is both uniformly convex and uniformly smooth. Then $X$ is isomorphic to a Hilbert space. Moreover, $g := \|\cdot\|^p$ is simultaneously uniformly convex and uniformly smooth if and only if $p = 2$ and both $\delta_X$ and $\rho_X$ are of power type 2.*

**Proof** Let $f$ be as given. Because $f$ is uniformly convex, [16, Proposition 3.5.8] implies that

$$\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|^2} > 0.$$

Because continuous convex functions are bounded below on bounded sets, we have $f \geq 4a\|\cdot\|^2 + b$ for some $a > 0$ and $b \in \mathbb{R}$. Thus by replacing $f$ with $f - b$, we may assume $f \geq 4a\|\cdot\|^2$. Then $f^* \leq a\|\cdot\|^2$. Additionally, $f^*$ is uniformly convex because $f$ is uniformly smooth [1], [16, Theorem 3.5.12]. According to [4, Theorem 3.7], $X^*$ admits a norm with modulus of convexity of power type 2.

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Proceeding similarly with $f^*$, one can show that $f - B \leq A\|\cdot\|^2$ for some $A > 0$ and constant $B$. Applying [4, Theorem 3.7] shows that $X$ admits a norm with modulus of convexity of power type 2. It follows from [10, Propositions IV.1.12, IV.5.10, IV.5.12] that $X$ has type 2 and cotype 2, and so $X$ is isomorphic to a Hilbert space by Kwapien's theorem [13].

For the "moreover" assertion, we note that the "only if" claim follows from Corollaries 2.4 and 2.6. For the "if" assertion, as in the previous paragraph, the duality results just cited imply that $f$ and $f^*$ are both uniformly convex and hence [16, Proposition 3.5.8] implies that both $p \geq 2$ and its conjugate index $q \geq 2$; consequently, $p = 2$ as claimed. ■

## References


[1] D. Azé, D. and J.-P. Penot, *Uniformly convex and uniformly smooth convex functions,*. Ann. Fac. Sci. Toulouse Math. **4**(1995), no. 4, 705–730.

[2] B. Beauzamy, *Introduction to Banach Spaces and Their Geometry*. North Holland Mathematical Studies 68. North-Holland, Amsterdam, 1982.

[3] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*. American Mathematical Society Colloquium Publications 48. American Mathematical Society, Providence, RI, 2000.

[4] J. M. Borwein, A. Guirao, P. Hájek, and J. Vanderwerff, *Uniformly convex functions on Banach spaces*. Proc. Amer. Math. Soc., **137**(2009), no. 3, 1081–1091. http://dx.doi.org/10.1090/S0002-9939-08-09630-5

[5] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*. CMS Books in Mathematics 3. Springer, New York, 2005.

[6] J. M. Borwein and J. Vanderwerff, *Convex Functions: Constructions, Characterizations and Counterexamples*. Encyclopedia of Mathematics and Applications 109. Cambridge University Press, Cambridge, 2010.

[7] D. Butnariu and A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*. Applied Optimization 40. Kluwer, Dordrecht, 2000.

[8] D. Butnariu and E. Resmerita, *Bregman distances, totally convex functions, and a method for solving operator equations in Banach spaces.* Abstr. Appl. Anal. **2006**, Art. ID 84919.

[9] J. Daneš, *On local and global moduli of convexity*. Comment. Math. Univ. Carolinae **17**(1976), 413–420.

[10] R. Deville, G. Godefroy, V. Zizler, *Smoothness and Renormings in Banach spaces*. Pitman Monographs and Surveys in Pure and Applied Mathematics 64. Longman Scientific & Technical, Harlow, 1993.

[11] A. Guirao. and P. Hájek, *On the moduli of convexity*. Proc. Amer. Math. Soc. **135**(2007), no. 10, 3233–3240. http://dx.doi.org/10.1090/S0002-9939-07-09030-2

[12] V. I. Gurariĭ, *Differential properties of the convexity moduli of Banach spaces*. Mat. Issled. **2**(1967), vyp. 1, 141–148.

[13] S. Kwapien, *Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients*. Studia Math. **44**(1972), 583–595.

[14] R. T. Rockafellar, *Convex Analysis*. Princeton Mathematical Series 28. Princeton University Press, Princeton, New Jersey, 1970.

[15] C. Zălinescu, *On uniformly convex functions*. J. Math. Anal. Appl. **95**(1983), no. 2, 344–374. http://dx.doi.org/10.1016/0022-247X(83)90112-9

[16] ———, *Convex Analysis in General Vector Spaces*. World Scientific Publishing, River Edge, NJ, 2002.



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