MULTIPLICATION ON SPACES WITH COMULTIPLICATION*

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Let A be an H-space and K a space. It is well known that [K, A] is a loop. Suppose A has a comultiplication as well, that is, cat $A < 2$. Then we shall prove that [K, A] is a Moufang loop. This generalises a result of C. W. Norman who proved this for the case where A is the circle, the 3-sphere or the 7-sphere. It also improves the known result that [K, A] is a diassociative loop if A has a comultiplication as well, since Moufang loops are diassociative.

1. We shall work in the category of spaces with base points and having the homotopy type of countable CW-complexes. All maps and homotopies are to respect base points which we shall usually denote by the symbol * . For simplicity, we shall frequently use the same symbol for a map and its homotopy class. Given spaces X, Y, we denote the set of homotopy classes of maps from X to Y by [X, Y]. The symbol Σ shall stand for the suspension functor.

We recall briefly that a loop is a set M together with a binary operation (which we shall denote by + even if the operation is not commutative) satisfying the following axioms: (1) there is an identity 0 in M satisfying $0 + a = a = a + 0$ for all a in M ; (2) the equations $x + a = b$, $a + y = b$ admit a unique pair of solutions x, y in M where a, b are elements of M. We observe that an associative loop is a group. Given elements a, b, c of a loop M, we define the commutator [a, b] and associator [a, b, c] by the equations $a + b = (b + a) + [a, b]$ and $(a + b) + c = \{a + (b + c)\} + [a, b, c]$.

We now recall some loop-theoretic notions. According to the axioms above, every element of a loop has a unique left inverse and a unique right inverse. A loop is called inversive if for every element of the loop its left inverse coincides with its right inverse. We denote the inverse of an element a of an inversive loop by -a . An inversive loop is called power associative if for every element a, we have

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$(a + a) + a = a + (a + a)$, $-a + (a + a) = a = (a + a) - a$, that is, $[a, a, a] = 0$, $[a, a, -a] = 0$, $[-a, a, a] = 0$ for every element a. An inversive loop is called diassociative if for every two elements a, b of the loop, we have $[a, a, b] = 0$, $[a, b, a] = 0$, $[a, b, b] = 0$, $[-a, a, b] = 0$, $[a, b, -b] = 0$ and $[-a, b, a] = 0$. Thus, an inversive loop is power associative if the subloop generated by an element is a cyclic group, and an inversive loop is diassociative if the subloop generated by any two elements is a group. It is clear that a diassociative loop is power associative. Finally, we say that a loop is Moufang if it satisfies the identity $(a + b) + (c + a) = \{a + (b + c)\} + a$ for all elements a, b, c of the loop. It is known that a Moufang loop is diassociative (see Moufang's Theorem [2, page 117]).

For the sake of completeness, we briefly recall the definitions of the conilpotency, the category and the weak category of a space. Let X be a topological space and let ΣX be its suspension. Let $\phi : \Sigma X \to \Sigma X \vee \Sigma X$ be the suspension comultiplication and $\mu : \Sigma X \to \Sigma X$ the inverse. We define the basic cocommutator map $c : \Sigma X \to \Sigma X \vee \Sigma X$ by $c = \nabla(1 \vee 1 \vee \mu \vee \mu)(\phi \vee \phi)\phi$ where $\nabla : \Sigma X \vee \Sigma X \vee \Sigma X \vee \Sigma X \to \Sigma X \vee \Sigma X$ is the folding map. The cocommutator map c_1 of weight 1 is the identity. Suppose the cocommutator map $c_k : \Sigma X \to \Sigma X \vee \ldots \vee \Sigma X$ (k terms) of weight k has been defined. We then define the cocommutator map of weight $(k+1)$ by $c_{k+1} = (c_k \vee 1)c : \Sigma X \to \Sigma X \vee \ldots \vee \Sigma X$ ($k+1$ terms). The conilpotency class of X, conil X, is the least integer $k \geq 0$ such that $c_{k+1} \simeq *$. If no such integer exists, we put conil $X = \infty$.

Now for each integer $n \geq 1$, let X^n denote the cartesian product of n copies of X and let $T_1(X^n)$ be the subspace of X^n consisting of all points with at least one coordinate at the base point $*$ of X. Let $j : T_1(X^n) \to X^n$ be the inclusion and let $\Delta : X \to X^n$ be the diagonal map. Then we say that the category of X is less than n, $\text{cat}\, X < n$, if there is a map $\phi : X \to T_1(X^n)$ such that $j\phi \simeq \Delta$. The weak category of X, $\text{wcat}\, X$, is the least integer $k \geq 0$ such that $q\Delta \simeq *$ where $\Delta : X \to X^{k+1}$ is the diagonal map and $q : X^{k+1} \to X^{(k+1)}$ is the projection of the cartesian product onto the smashed product.

2. We now recall some results from [8]. Let A be an H-space with multiplication $\phi : A \times A \to A$. Let K be a space. Then we have a loop $[K, A]$. For each integer $n \geq 1$, let $T_1(K^n)$ denote the subset of K^n consisting of points with at least one coordinate equal to the base point $*$ of K. Let $j_n : T_1(K^n) \to K^n$ be the inclusion, and $q_n : K^n \to K^{(n)}$ be the projection where $K^{(n)}$ denotes the smashed product of n copies

of K. Then we have a cofibration $T_1(K^n) \xrightarrow{j_n} K^n \xrightarrow{q_n} K^{(n)}$. Let $\Delta_n : K \to K^n$ be the diagonal map. We have homomorphisms $j_n^{\#} : [K^n, A] \to [T_1(K^n), A]$, $\Delta_n^{\#} : [K^n, A] \to [K, A]$. Then it is shown in [8] that $\Delta_n^{\#}(\ker j_n^{\#})$ is a normal subloop of $[K, A]$. Let us denote $\Delta_n^{\#}(\ker j_n^{\#})$ by G_n for each $n \geq 1$. Then $G_1 = [K, A]$ and $G_{n+1} \subset G_n$ (see [8]). It is also shown in [8] that $G_n/G_{n+1} \subset Z(G_1/G_{n+1})$ where $Z(G_1/G_{n+1})$ is the centre of the loop G_1/G_{n+1}.

Suppose conil $K < n$. Then we claim that $G_n = 0$. For suppose $f \in G_n$. Then we have $f = f_1 \Delta_n$ where $f_1 \in \operatorname{Ker} j_n^{\#}$, that is, $f_1 j_n \simeq * : T_1(K^n) \to A$. From the cofibration $T_1(K^n) \xrightarrow{j_n} K^n \xrightarrow{q_n} K^{(n)}$ it follows that $f_1 \simeq f_2 q_n$ where $f_2 : K^{(n)} \to A$. Hence we have $f = f_2 q_n \Delta_n$, and hence $\Sigma f = \Sigma f_2\, \Sigma(q_n \Delta_n)$. Since conil $K < n$, we have that $\Sigma(q_n \Delta_n) \simeq *$ by [3; 4; 6]. Hence $\Sigma f \simeq *$. Since A is an H-space, $\Sigma : [K, A] \to [\Sigma K, \Sigma A]$ is one-to-one. Hence $f \simeq *$. Thus we see that if conil $K < n$, we have an ascending chain of normal subloops $0 = G_n \subset G_{n-1} \subset \ldots \subset G_1 = [K, A]$ such that $G_i/G_{i+1} \subset Z(G_1/G_{i+1})$. Hence as in [8], we have that $[K, A]$ is a loop which is centrally nilpotent of class $\leq n - 1$ and nuclearly nilpotent of class $\leq [\frac{1}{2}n]$ where $[x]$ denotes the integral part of x. This improves [8, Theorem 1.1] by replacing the condition w cat $K < n$ there by conil $K < n$. We have normalised w cat and cat in this paper so that our value is one less than in [8], and we observe that with the normalisation, conil $K \leq$ w cat $K \leq$ cat K (see [3]). Strict inequalities can occur (see [3]). We state our result formally as the following theorem.

THEOREM 1. *Let* A *be an H-space and* K *a space. If* conil $K < n$, *then* $[K, A]$ *is a loop which is centrally nilpotent of class* $\leq n - 1$ *and nuclearly nilpotent of class* $\leq [\frac{1}{2}n]$.

We now consider the normal subloops G_i of $[K, A]$. With the same assumptions as in Theorem 1, we have the following result.

THEOREM 2. *Let* $f \in G_r$, $g \in G_s$, $h \in G_t$. *Then the associator* $[f, g, h] \in G_{r+s+t}$ *and hence* $(f + g) + h = f + (g + h)$ *if* $r + s + t \geq n$.

COROLLARY. *If* conil $K < n$, *then* $G_{[\frac{n+2}{3}]}$ *is a group*. *Hence if* conil $K < 3$, *then* $[K, A]$ *is a group*.

The corollary follows immediately from Theorem 2.

Proof of Theorem 2. Let $\phi : A \times A \to A$ be the H-structure on A. Let $j_3 : T_1(A^3) \to A^3$ be the inclusion. Then it can be checked that $\phi(\phi \times 1)j_3 \simeq \phi(1 \times \phi)j_3$. From the cofibration $T_1(A^3) \xrightarrow{j_3} A^3 \xrightarrow{q_3} A^{(3)}$ it follows that we can write $\phi(\phi \times 1) = \phi(1 \times \phi) + \psi q_3$ where $\psi : A^{(3)} \to A$. We have the map $(f \times g \times h)\Delta_3 : K \to A^3$ and hence we have $(f + g) + h = \{f + (g + h)\} + \psi q_3(f \times g \times h)\Delta_3$. We observe that $\psi q_3(f \times g \times h)\Delta_3$ is the associator $[f, g, h]$. Since $f \in G_r$, $g \in G_s$, $h \in G_t$ we can write $f = f_1 q_r \Delta_r$, $g = g_1 q_s \Delta_s$, $h = h_1 q_t \Delta_t$. Hence the associator $[f, g, h] = \psi q_3(f \times g \times h)\Delta_3$
$= \psi q_3(f_1 \times g_1 \times h_1)(q_r \times q_s \times q_t)\Delta_{r+s+t}$
$= \psi(f_1 \wedge g_1 \wedge h_1) q_3(q_r \times q_s \times q_t)\Delta_{r+s+t}$
$= \psi(f_1 \wedge g_1 \wedge h_1) q_{r+s+t} \Delta_{r+s+t}$. Thus $[f, g, h] \in G_{r+s+t}$. If conil $K < n$ and $r + s + t \geq n$ we have $\Sigma(q_{r+s+t}\Delta_{r+s+t}) = 0$. Hence $\Sigma[f, g, h] = 0$. Since A is an H-space, Σ is one-to-one, and hence $[f, g, h] = 0$. Thus $(f + g) + h = f + (g + h)$. This proves Theorem 2.

THEOREM 3. *Let* $f \in G_r$, $g \in G_s$. *Then the commutator* $[f, g] \in G_{r+s}$.

Proof. We have the H-structure $\phi : A \times A \to A$. Let $\phi_1 : A \times A \to A$ be any other H-structure on A. Then $\phi j \simeq \nabla$ and $\phi_1 j \simeq \nabla$ where $j : A \vee A \to A \times A$ is the inclusion and $\nabla : A \vee A \to A$ is the folding map. Hence we can write $\phi = \phi_1 + \psi q_2$ where $q_2 : A^2 \to A^{(2)}$ is the projection and $\psi : A^{(2)} \to A$ is a map and $+$ is the operation in $[A^2, A]$ induced by ϕ. Composing with $(f \times g)\Delta$ we have $f + g = (f \oplus g) + \psi q_2(f \times g)\Delta$ where $\oplus$ is the operation in $[K, A]$ induced by ϕ_1. Now we can write $f = f_1 q_r \Delta_r$, $g = g_1 q_s \Delta_s$ and hence $\psi q_2(f \times g)\Delta = \psi q_2(f_1 \times g_1)(q_r \times q_s)\Delta_{r+s}$
$= \psi(f_1 \wedge g_1) q_{r+s} \Delta_{r+s}$. Thus $f + g = (f \oplus g) + \psi(f_1 \wedge g_1)q_{r+s}\Delta_{r+s}$.

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Clearly $\psi(f_1 \wedge g_1)\, q_{r+s} \Delta_{r+s} \in G_{r+s}$. If we take ϕT for ϕ_1 where $T : A \times A \to A \times A$ is the switching map, then $f \oplus g$ is just $g + f$, and hence we see that $[f, g] \in G_{r+s}$. This proves Theorem 3.

We observe that in the general case if $r + s \geq n$, then our usual arguments will show that $\psi\, q_2(f \times g)\, \Delta = 0$ and hence $f + g = f \oplus g$. This gives us the following corollary.

COROLLARY. _Let_ $f \in G_r$, $g \in G_s$. _Then if_ $r + s \geq n$, $f + g = f \oplus g$ _where_ $+$, $\oplus$ _are induced by H-structures_ ϕ, ϕ_1 _on_ A. _Thus the loop structure on_ $G_{[\frac{1}{2}(n+1)]}$ _is independent of the H-structure on_ A _and hence_ $G_{[\frac{1}{2}(n+1)]}$ _is an abelian group. In particular, if_ conil $K < 2$, _then the loop structure of_ $[K, A]$ _is independent of the H-structure on_ A _and hence_ $[K, A]$ _is an abelian group_.

3. We now consider conilpotency and category conditions on the H-space.

THEOREM 4. _Let_ (A, ϕ) _be an H-space such that_ conil $A < 3$. _Then for any space_ K, $[K, A]$ _is a power associative loop_.

Proof. Since conil $A < 3$, by the above we see that $[A, A]$ is a group. Hence there is an element -1 in $[A, A]$ such that $1 - 1 = 0 = -1 + 1$. Let f be an element of $[K, A]$. Then $0 = (1 - 1)f = f + (-1)f$ and $0 = (-1 + 1)f = (-1)f + f$. Thus $[K, A]$ is inversive. Also $(f + f) + f = (1 + 1)f + f = \{(1 + 1) + 1\} f = \{1 + (1 + 1)\} f = f + (f + f)$, $(-f) + (f + f) = (-1)f + (1 + 1)f = \{(-1) + (1 + 1)\} f = f = \{(1 + 1) - 1\} f = (f + f) - f$. Thus $[K, A]$ is a power associative loop.

We now state a result from [8] that we shall need.

THEOREM 5 (Norman [8]). _Let_ (A, ϕ) _be an H-space and_ K _a space such that_ cat $K < 4$. _Then the associator in_ $[K, A]$ _satisfies the expansions_

$$[a + a_1, b, c] = [a, b, c] + [a_1, b, c]$$

$$[a, b + b_1, c] = [a, b, c] + [a, b_1, c]$$

$$[a, b, c + c_1] = [a, b, c] + [a, b, c_1].$$

Finally, we recall another known result.

THEOREM 6 (O'Neill [9]). Let (A, ϕ) be an H-space such that cat $A < 2$. Then for any space K, $[K, A]$ is a diassociative loop.

We are now ready to prove our result. Suppose (A, ϕ) is an H-space. Let K be a space such that cat $K < 4$. Then by the above results, we see that $[K, A]$ is a loop of central nilpotency ≤ 3, and that all associators in $[K, A]$ lie in the centre of the loop. Since cat $K < 4$, the associators expand according to Theorem 5. It is then easily checked that $[K, A]$ is Moufang if and only if the associators satisfy the rule $[a, b, c] = [a, c, a] + [b, c, a]$ for all a, b, c in $[K, A]$. Now, suppose further that cat $A < 2$. Then, by Theorem 6, $[K, A]$ is diassociative so that $[a, c, a] = 0$ for all a, c. Thus in this case, $[K, A]$ is Moufang if and only if $[a, b, c] = [b, c, a]$ for all a, b, c in $[K, A]$.

THEOREM 7. Let (A, ϕ) be an H-space such that cat $A < 2$. If K is a space such that cat $K < 4$, then $[K, A]$ is a Moufang loop.

Proof. Let a, b, c be elements of $[K, A]$. We need to show that $[a, b, c] = [b, c, a]$. Now we can write $b + c = (c + b) + [b, c]$. Hence $[a, b + c, c + b] = [a, (c + b) + [b, c], c + b] = [a, c + b, c + b] + [a, [b, c], c + b]$. Since cat $A < 2$, we have that $[a, c + b, c + b] = 0$. On the other hand, since conil $K \leq$ cat $K < 4$, our results, Theorems 2 and 3 on associators and commutators, show that $[a, [b, c], c + b] = 0$. Hence $[a, b + c, c + b] = 0$. Now expanding this associator according to Theorem 5 and using Theorem 6, we see that $[a, b + c, c + b] = [a, b, c] + [a, c, b]$ since all the other terms vanish. Thus $[a, b, c] + [a, c, b] = 0$. If we now apply the same process to the associator $[a + b, c, b + a]$, we obtain the equation $[a, c, b] + [b, c, a] = 0$. Since these associators all lie in the centre of $[K, A]$, we obtain the equation $[a, b, c] = [b, c, a]$. Thus $[K, A]$ is Moufang.

Our main theorem now follows as a corollary of the above.

THEOREM 8. Let (A, ϕ) be an H-space such that cat $A < 2$. Then for any space K, $[K, A]$ is a Moufang loop.

Proof. Since cat $A < 2$, it follows that cat $A^3 < 4$. Hence $[A^3, A]$ is a Moufang loop. Let $\pi_1, \pi_2, \pi_3 : A^3 \rightarrow A$ be the projections onto the factors. Then $(\pi_1 + \pi_2) + (\pi_3 + \pi_1) = \{\pi_1 + (\pi_2 + \pi_3)\} + \pi_1$. Let f, g, h be elements of $[K, A]$. Composing this equation on the right with $(f \times g \times h)\Delta_3$, we obtain the equation $(f + g) + (h + f) = \{f + (g + h)\} + f$. Thus $[K, A]$ is Moufang.

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