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ON A GENERALIZATION OF A THEOREM OF WIENER

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1. Let $V[0, 2\pi]$ denote the class of all normalized functions F of bounded variation in $[0, 2\pi]$ such that $F(x)=2^{-1}\{F(x+0)+F(x-0)\}$ and $F(x+2\pi)-F(x) = F(2\pi)-F(0)$ for all x and let $\{C_n\}$ be the sequence of Fourier–Stieltjes coefficients of F. Wiener [9] (cf. Bari [1, p. 212], Zygmund [10, p. 108]) proved the following theorem.

THEOREM A. *For a function $F \in V[0, 2\pi]$ to be continuous, it is necessary and sufficient that $\{|C_k|^2\}$ or $\{|C_k|\}$ be summable $(C, 1)$ to* 0.

Lozinskiĭ [4] gave the following alternative criterion for continuity of a function of $V[0, 2\pi]$.

THEOREM B. *For a function $F \in V[0, 2\pi]$ to be continuous, it is necessary and sufficient that $\{|C_k|^2\}$ or $\{|C_k|\}$ be summable to zero by logarithmic means.*

Matveev [5] generalized Theorems A and B as follows:

THEOREM C. *For a function $F \in V[0, 2\pi]$ to be continuous, it is necessary and sufficient that $\{|C_k|^2\}$ or $\{|C_k|\}$ be summable $(\bar{N}, p)$ to zero where $(\bar{N}, p)$ is a Riesz method of summability such that either*

(a) $np_n \downarrow 0$ *and* $P_n=p_1+\cdots+p_n \to \infty$ $(n \to \infty)$

or

(b) $p_n>0$, $np_n \uparrow$ *and* $np_n=O(P_n)$ $(n \to \infty)$.

In this paper we first show that Theorem C follows from the following theorem.

THEOREM D. *For a function $F \in V[0, 2\pi]$ to be continuous, it is necessary and sufficient that $\{|C_k|^2\}$ or $\{|C_k|\}$ be summable $(\bar{N}, p)$ to zero where $(\bar{N}, p)$ is a regular Riesz method of summability satisfying the strong regularity condition*

$$(1) \qquad \sum_{k=1}^{n} |\Delta p_k| = o(P_n) \quad (n \to \infty)$$

where $\Delta p_k=p_k-p_{k+1}$ for $k=1,\ldots,n-1$ and $\Delta p_n=p_n$.

Theorem D is contained in the following theorem.

THEOREM E. *For a function $F \in V[0, 2\pi]$ to be continuous, it is necessary and sufficient that $\{|C_k|^2\}$ or $\{|C_k|\}$ be summable $(\bar{N}, p)$ to zero by a regular Riesz method of summability such that*

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$$(2)\qquad \lim_{n\to\infty} P_n^{-1}\sum_{k=1}^{n} p_k \exp(2\pi ikt) = \lim_{n\to\infty} |P_n^{-1}|\sum_{k=1}^{n} |p_k| \exp(2\pi ikt) = 0$$

for all $t\in(0,1)$.

If we consider the method $(\overline{N},p)$ defined by taking $p_1=1$, $p_k=1+[(-1)^k/k]$, then $(\overline{N},p)$ is a strongly regular positive matrix for which $\{np_n\}$ is not monotonic. This shows that Matveev's Theorem C is properly contained in our Theorem D. We next show that condition (1) is not necessary for the validity of Theorem D by constructing an $(\overline{N},p)$ matrix with $p_n\geq 0$ satisfying (2) but not (1).

Finally, we show that although the sufficiency part of Theorem C remains valid the necessity part does not if in Theorem C the condition (b) is replaced by the following condition attributed to Matveev in Bari [1, p. 256]:

(b′) $p_n>0$ and $np_n\uparrow$ but $np_n\leq n^\alpha$ $(n=1,2,\dots)$ for some $\alpha>0$.

Theorem E is a particular case of the following theorem contained in a generalization of Wiener's Theorem A given by the author in [8] (cf. also [7]).

Theorem F. *For a function* $F\in V[0,2\pi]$ *to be continuous, it is necessary and sufficient that* $\{|C_k|^2\}$ *or* $\{|C_k|\}$ *be summable A by a regular matrix* $A=(a_{n,k})$ *for which*

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} a_{n,k}\exp(2\pi ikt) = \lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{n,k}|\exp(2\pi ikt) = 0$$

for all $t\in(0,1)$.

2. In order to prove that Theorem C of Matveev is a particular case of Theorem D, we prove the following theorem.

Theorem 1. A. *If* $\{np_n\}$ *is positive and decreasing, then* $(\overline{N},p)$ *is strongly regular if and only if* $P_n\to\infty$ $(n\to\infty)$.

B. *If* $\{np_n\}$ *is positive and increasing, then the following propositions are equivalent:*

(i) $p_n=o(P_n)(n\to\infty)$,

(ii) $(\overline{N},p)$ *is strongly regular*,

(iii) $\sum_{k=1}^{n} p_k\exp(2\pi ikt)=o(P_n)(n\to\infty)$ *for all* $t\in(0,1)$.

Proof. The proof of the assertion A is trivial since under the hypothesis of A, $\sum_{k=1}^{n}|\Delta p_k|=p_1$. If $\{np_n\}$ is increasing and $p_n=o(P_n)$ $(n\to\infty)$, then

$$\begin{aligned}\sum_{k=1}^{n}|\Delta p_k| &= \sum_{k=1}^{n-1}|kp_k-(k+1)p_{k+1}+p_{k+1}|\frac{1}{k}+p_n\\ &\leq \sum_{k=1}^{n-1}[(k+1)p_{k+1}-kp_k+p_{k+1}]\frac{1}{k}+p_n\\ &\leq 2\sum_{k=1}^{n-1}\frac{p_{k+1}}{k}+2p_n.\end{aligned}$$

Given any $\epsilon>0$, there exists an integer N such that $1/N<\epsilon$ and for all $n\geq N$, $p_n\leq\epsilon P_n$. If we choose $n>2N$, we have

$$\sum_{k=1}^{n-1}\frac{p_{k+1}}{k} = \sum_{k=1}^{N}\frac{p_{k+1}}{k}+\sum_{N+1}^{[n/2]}\frac{p_{k+1}}{k}+\sum_{[n/2]+1}^{n-1}\frac{p_{k+1}}{k}$$

$$\leq \sum_{k=1}^{N}\frac{p_{k+1}}{k}+([n/2]+1)p_{[n/2]+1}\sum_{N+1}^{[n/2]}\frac{1}{k(k+1)}+\epsilon P_n\sum_{[n/2]+1}^{N-1}\frac{1}{k}.$$

Since $\{np_n\}$ is increasing, we have for all $n\geq 1$,

$$P_n = \sum_{1}^{n} p_k \geq \sum_{[n/2]+1}^{n} kp_k\cdot\frac{1}{k} \geq ([n/2]+1)p_{[n/2]+1}\sum_{[n/2]+1}^{n}\frac{1}{k}$$

so that for $n>2N$

$$\sum_{k=1}^{n-1}\frac{p_{k+1}}{k} \leq \sum_{k=1}^{N}\frac{p_{k+1}}{k}+AP_n\sum_{N+1}^{\infty}\frac{1}{k(k+1)}+\epsilon BP_n$$

$$< \sum_{k=1}^{N}\frac{p_{k+1}}{k}+\epsilon CP_n,$$

where A, B, C are positive constants independent of n.

Since

$$P_n = \sum_{1}^{n} kp_k\frac{1}{k} \geq p_1\sum_{1}^{n}\frac{1}{k},$$

it follows that $P_n\to\infty$ as $n\to\infty$ so that, on the one hand $(\overline{N},p)$ is regular and on the other

$$\limsup_{n\to\infty} P_n^{-1}\sum_{k=1}^{n-1}\frac{p_{k+1}}{k} \leq \epsilon.$$

But ϵ being arbitrary, it follows that

$$\lim_{n\to\infty} P_n^{-1}\sum_{k=1}^{n-1}\frac{p_{k+1}}{k} = 0$$

and consequently that

$$\sum_{k=1}^{n} |\Delta p_k| = o(P_n) \quad (n\to\infty).$$

Thus $(\overline{N},p)$ is strongly regular.

Suppose now that $(\overline{N},p)$ is strongly regular, then applying Abel's transformation, we get

$$\sum_{k=1}^{n} p_k \exp(2\pi ikt) = \sum_{k=1}^{n}\Delta p_k\frac{\exp(2\pi it)-\exp(2\pi ikt)}{1-\exp(2\pi it)},$$

from which it follows that B(iii) holds. If B(iii) holds, applying Lebesgue's bounded convergence theorem we get

$$\sum_{1}^{n} p_k^2 = o(P_n^2) \quad (n\to\infty)$$

which implies B(i).

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It is interesting to note that if the hypothesis of monotonicity on the sequence $\{np_n\}$ is dropped, then the method $(\bar{N}, p)$ may neither be strongly regular nor satisfy B(iii) and yet satisfy the condition $p_n=o(P_n)$ $(n\to\infty)$ as can be seen by choosing $p_n=0$ or 1 according as n is even or odd.

3. We now show that there exist $(\bar{N}, p)$ matrices with $p_n\geq 0$ that satisfy condition (2) without being strongly regular. It will follow that strong regularity of $(N, \bar{p})$ is not a necessary condition for the equivalence of the continuity of functions $F\in V[0, 2\pi]$ and the summability $(N, \bar{p})$ to zero of the associated sequences $\{|C_k|^2\}$ or $\{|C_k|\}$ formed by the Fourier–Stieltjes coefficients of F.

The construction of the positive $(\bar{N}, p)$ matrix in question is based on the use of the coefficients of the Rudin–Shapiro polynomials as given in Rudin [6]. These are defined as follows.

We set $P_0(x)=Q_0(x)=x$ and define P_k and Q_k inductively by

$$\left.\begin{aligned} P_{k+1}(x) &= P_k(x)+x^{2^k}Q_k(x) \\ Q_{k+1}(x) &= P_k(x)-x^{2^k}Q_k(x) \end{aligned}\right\} k = 0, 1, 2, \ldots$$

Clearly $P_1(x)=x+x^2$ and $Q_1(x)=x-x^2$. We observe that P_k is a polynomial of degree 2^k and that P_k is a partial sum of P_{k+1}. Hence we can define a sequence $\{\epsilon_n\}$ by setting ϵ_n equal to the nth coefficient of P_k, where $2^k>n$. Clearly $\epsilon_n=1$ or -1. It has been shown by Rudin [6] that

$$\left|\sum_{n=1}^{N} \epsilon_n \exp(2\pi i n\theta)\right| \leq 5\sqrt{N} \quad \text{for } \theta\in[0, 1], \quad N = 1, 2, \ldots \tag{3}$$

Brillhart and Carlitz [2] have shown that if we write

$$n = r_0+r_1\cdot 2+r_2\cdot 2^2+\cdots+r_k\cdot 2^k \quad (k\geq 0), \quad r_i = 0 \quad \text{or} \quad 1,$$

then

$$\epsilon_n = (-1)^{r_0r_1+r_1r_2+\cdots+r_{k-1}r_k}.$$

It follows that the set $\{\epsilon_{4n+1}, \epsilon_{4n+2}, \epsilon_{4n+3}, \epsilon_{4n+4}\}$ consists of either three $+1$'s and one -1 or three -1's and one $+1$. If we put $p_n=\epsilon_n+1$, then for $t\in(0, 1)$

$$\frac{1}{P_n}\sum_1^n p_k \exp(2\pi i k t) = \frac{1}{P_n}\sum_1^n \epsilon_k \exp(2\pi i k t)+\frac{1}{P_n}\sum_1^n \exp(2\pi i k t)$$

tends to zero in view of (3) and the fact that $P_n\geq[n/4]\cdot 2$ so that (2) holds. But $(\bar{N}, p)$ is not strongly regular since

$$\frac{1}{P_n}\sum_{k=1}^{n}\left|\Delta p_k\right| \geq \frac{1}{3[n/4]}([n/4]-1)$$

which does not tend to zero as $n\to\infty$.

4. Passing now to the consideration of Matveev's Theorem C with hypothesis (b) replaced by (b′), we first prove the following theorems.

THEOREM 2. *There exist regular methods of summability* $(\overline{N}, p)$ *for which* $0 < np_n \uparrow$, $np_n \le n^\alpha$ *for* $n=1, 2, \ldots$ *with* $\alpha > 1$ *but* $p_n \ne o(P_n)$ $(n \to \infty)$.

THEOREM 3. *Let* $(\overline{N}, p)$ *be a method of summability such that* $\{p_n\}$ *is positive and* $p_n \ne o(P_n)$ $(n \to \infty)$. *Then there exists a continuous nondecreasing function F in* $V[0, 2\pi]$ *such that* $\{|C_k|^\alpha\}$ *is not summable* $(\overline{N}, p)$ *to zero for any* $\alpha > 0$.

Proof of Theorem 2. Let α be an integer greater than 1. Choose a positive integer $n_1 > 1$ arbitrarily and set $n_k = n_1^{\beta^{k-1}}$ where $\beta = 2\alpha/(\alpha-1)$ and $k=1, 2, \ldots$. Define a sequence $\{p_n\}$ as follows:

$$p_1 = 1, \quad p_2 = \frac{1}{2}, \quad \ldots, \quad p_{n_1-1} = \frac{1}{n_1-1}$$

and

$$p_{n_k} = n_k^{\alpha-1}, \quad p_{n_k+1} = \frac{n_k^\alpha}{n_k+1}, \quad \ldots, \quad p_{n_{k+1}-1} = \frac{n_k^\alpha}{n_{k+1}-1}$$

for $k=1, 2, \ldots$. Clearly the $(\overline{N}, p)$-method defined by the above sequence $\{p_k\}$ is a regular method of summability satisfying the conditions $0 < np_n \uparrow$ and $np_n \le n^\alpha$ for $n=1, 2, \ldots$. Since

$$\begin{aligned} P_{n_k} &= \left(1+\frac{1}{2}+\cdots+\frac{1}{n_1-1}\right)+\left(\frac{1}{n_1}+\cdots+\frac{1}{n_2-1}\right)n_1^\alpha+\cdots \\ &\quad +\left(\frac{1}{n_{k-1}}+\cdots+\frac{1}{n_k-1}\right)n_1^{\alpha\beta^{k-2}}+n_1^{(\alpha-1)\beta^{k-1}} \\ &\le \left(1+\frac{1}{2}+\cdots+\frac{1}{n_k-1}\right)n_1^{\alpha\beta^{k-2}}+n_1^{(\alpha-1)\beta^{k-1}} \\ &\le (1+\beta^{k-1}\log n_1)n_1^{\alpha\beta^{k-2}}+n_1^{(\alpha-1)\beta^{k-1}} \\ &= p_{n_k}\left(1+\frac{1+\beta^{k-1}\log n_1}{n_1^{\alpha\beta^{k-2}}}\right), \end{aligned}$$

it follows that $\lim_{k\to\infty} P_{n_k}^{-1}p_{n_k} = 1$ and consequently that $p_n \ne o(P_n)$ $(n \to \infty)$.

Proof of Theorem 3. Since $p_n \ne o(P_n)$, there exists a $\delta > 0$ and a sequence of positive integers n_ν such that $n_{\nu+1}/n_\nu \ge q > 3$ and $p_{n_\nu} > \delta P_{n_\nu}$.

We form the Riesz product

$$\prod_{\nu=1}^{\infty} (1+\cos n_\nu x).$$

If we set

$$g_k(x) = \prod_{i=1}^{k} (1+\cos n_i x)$$

and

$$F(x)-F(0) = \lim_{k\to\infty} \int_0^x g_k(t)\, dt,$$

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then F is a nondecreasing singular function whose Fourier–Stieltjes coefficients $\{C_k\}$ are such that $C_{n_\nu}=1$ (cf. Zygmund [10, pp. 208–209]). It follows that

$$P_{n_\nu}^{-1}\sum_{1}^{n_\nu} p_k|C_k|^\alpha \geq P_{n_\nu}^{-1}p_{n_\nu} > \delta$$

for all v so that $\{|C_k|^\alpha\}$ is not summable $(\bar{N}, p)$ to zero for any $\alpha>0$.

If a method $(\bar{N}, p)$ satisfies the hypothesis (b′) of Theorem C, then $(\bar{N}, p)\subset(\bar{N}, 1/k)$ (cf. Hardy [3, p. 58]) so that if $\{|C_k^2|\}$ or $\{|C_k|\}$ is summable $(\bar{N}, p)$ to zero, it is also summable $(\bar{N}, 1/k)$ to zero and hence by Theorem D, F is continuous since $(\bar{N}, 1/k)$ is clearly strongly regular. However, if a $(\bar{N}, p)$ matrix satisfying the hypothesis (b′) is not strongly regular (and such matrices do exist in view of Theorem 2), then by Theorem 3, there exist real-valued continuous functions $F\in V[0, 2\pi]$ with Fourier–Stieltjes coefficients $\{C_k\}$ such that $\{|C_k|^2\}$ or $\{|C_k|\}$ is not summable $(\bar{N}, p)$ to zero as $n\to\infty$.

This shows that for $(\bar{N}, p)$ matrices satisfying the hypothesis (b′), the necessity part of Theorem C is not always true.

In connection with Theorem C, it is asserted in [5, pp. 467–68, Remark 4] that in (b) the hypothesis that $np_n=O(P_n)$ cannot be dropped. However the example constructed there merely shows this for condition $p_n=o(P_n)$ which does not always imply $np_n=O(P_n)$ even when $0<np_n\uparrow$.

The above analysis shows that for the validity of Wiener's theorem for summability $(\bar{N}, p)$ with $p_n>0$ and $\{np_n\}$ monotonic the condition (2) is both necessary and sufficient.

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